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Further results about the transcendental meromorphic solution of a special Fermat-type equation

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Abstract

In this paper, we mainly investigate the finite order transcendental meromorphic solutions of Fermat-type equations and also we consider here linear difference operator of meromorphic function. In addition, we extend some recent result obtained in [1]. The example is exhibited to validate certain claims of the main result.

Keywords: Nevanlinna theory, Linear difference operator, Finite order, Entire and Meromorphic solutions, Fermat type equation
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1 Introduction

For a meromorphic function f in the complex plane $\mathbb{C}$, we shall use the standard notations, definitions and basic results of Nevalinna theory of meromorphic functions (see [6],[9]. The notation $S(r, f)$, is defined to be any quantity logarithmic measure. The order of f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

In this article, we define shift and difference operators of $f(z)$ by $f(z+c)$ and $\Delta_c f(z) = f(z+c) - f(z)$, respectively. Note that $\Delta_c^n f(z) = \Delta_c^{n-1}\Delta_c f(z)$, where c is a non-zero complex number and $n \geq 2$ is a positive integer.

For further generalization of $\Delta_c f(z)$, we now define the linear difference operator of an entire(meromorphic) function f as $\mathcal{L}_c(f) = f(z+c) + c_0 f(z)$, where c_0 is a finite complex constant. Clearly, for the particular choice of the constant $c_0 = -1$, we get $\mathcal{L}_c(f) = \Delta_c f$.

2 Preliminaries and Main result

For the existence of solutions of non-linear q-shift equation, in 2011, Qi [11] obtained the following theorems:
Theorem A. Let $q(z), p(z)$ be polynomials and let n, m be distinct positive integers. Then the equation

$$f^m(z) + q(z)f(z+c)^n = p(z) \tag{2.1}$$

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has no transcendental entire solutions of finite order.

In 2015, Qi-Liu-Yang [10] obtained the meromorphic variant of Theorem A and improved this as follows:
Theorem B. [10] Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq n+4, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r,f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r,q(z)) = S(r,f)$. Then, $f(z)$ is not a solution of equation

$$f^m(z) + q(z)f(z+c)^n = p(z) \tag{2.2}$$

Theorem C. [10] Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq n+2, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r,f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r,q(z)) = S(r,f)$. Then, $f(z)$ is not a solution of equation 2.2.

In 2021, A. Banerjee and T. Biswas [1] investigated the following result.
Theorem D. [1] Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq (\tau+1)(n+2)+2, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r,f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r,q(z)) = S(r,f)$. Then, $f(z)$ is not a solution of the non-linear c-shift equation

$$f^m(z) + q(z)(L_c(z,f))^n = p(z). \tag{2.3}$$

In this article we extend Theorem-D at the expense of replacing $(L_c(z,f))^n$ by $[f^n(f-1)^s\mathcal{L}_c(f)]^{(k)}$.

Theorem 2.1. Let $f(z)$ be a transcendental meromorphic function with finite order, m, n, s and k be a positive integers such that $m \geq (s+1)(nk+k+sk+4)+2, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r,f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r,q(z)) = S(r,f)$. Then, $f(z)$ is not a solution of the linear difference operator

$$f^m(z) + q(z)[f^n(f-1)^s\mathcal{L}_c(f)]^{(k)} = p(z). \tag{2.4}$$

Corollary 2.2. Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq n+2, p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r,f)$ and $q(z)$ be nonzero meromorphic function satisfying that $T(r,q(z)) = S(r,f)$. Then, $f(z)$ is not a solution of equation (2.4).

The next example show that if the condition $m \geq n+2$ is omitted then the equation (2.4) can admit a transcendental entire solution. Considering $n=1, k=1$ and $m=1$ we have the following example.

Example 2.3. The function $f(z) = ze^{\frac{\pi i z}{c}}$ satisfies the equation $f(z) + \frac{1}{z+1}[\mathcal{L}_c(f)] = \frac{z(z+2)}{z+1}e^{\frac{\pi i z}{c}}$, where the coefficients of $\mathcal{L}_c(f)$ is chosen such that they satisfy simultaneously the equations

$$\begin{cases} a_0 - a_1 + a_2 - ... + (-1)^k a_k = 1, \\ -a_1 + 2a_2 - 3a_3 - ... + k(-1)^k a_k = 0. \end{cases}$$

To proceed further we require the following lemmas:

Lemma 2.4. [4] Let $f(z)$ be a finite order meromorphic function and $\varepsilon > 0$, then $T(r, f(z+c)) = T(r, f(z)) + o(r^{\sigma-1+\varepsilon}) + O(\log r)$ and $\sigma(f(z+c)) = \sigma(f(z))$. Thus, if $f(z)$ is a transcendental meromorphic function with finite order, then we know $T(r, f(z+c)) = T(r,f) + S(r,f)$.

Lemma 2.5. [5] Let $f(z)$ be a meromorphic function with finite order, and let $c \in \mathbb{C}$ and $\delta \in (0,1)$. Then $m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^\delta}\right) = S(r,f)$.

Lemma 2.6. [7] Let f be a non-constant meromorphic function with finite order and $c \in \mathbb{C}$. Then

$$N(r,\infty; f(z+c)) \le N(r,\infty; f(z)) + S(r,f),$$
$$N(r,\infty; f(z+c)) \le N(r,\infty; f) + S(r,f).$$

Proof of Theorem 2.1. Suppose by contradiction that $f(z)$ is a transcendental meromorphic function with finite order satisfying equation (2.4). If $T(r,p(z)) = S(r,f)$, then applying Lemma 2.4 to equation (2.4), we have

$$\begin{aligned} m.T(r,f) &= T(r,f^m) \\ &= T(r, p(z) - q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)}) \\ &= T(r, [f^n(f-1)^s \mathcal{L}_c(f)]^{(k)}) + S(r,f) \\ &\le (nk + sk + (s+1)k)T(r,f) + S(r,f), \end{aligned}$$

which contradicts the assumption that $m \ge (s+1)(nk+k+sk+4)+2$. If $T(r,p(z)) = S(r,f)$, differentiating equation (2.4), we get

$$(f^m)' + (q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)})' = p'(z). \tag{2.5}$$

Next dividing (2.5) by (2.4) we have

$$p'(z)[f^m(z) + q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)}] = p(z)[(f^m)' + (q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)})']$$

$$f^m(z) = \frac{\frac{p'(z)}{p(z)} q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)} - (q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)})'}{\frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}}. \tag{2.6}$$

First observe that $\frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}$ cannot vanish identically. Indeed, if $\frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)} \equiv 0$, then we get $p(z) = \beta f^m(z)$, where β is a non-zero constant. Substituting the above equality to equation (2.4), we have $q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)} = (\beta - 2) f^m(z)$. From Lemma 2.4 and above equation, we immediately see as above that $mT(r,f) \le (nk + sk + (s+1)k)T(r,f) + S(r,f)$, which is a contradiction to $m \ge (s+1)(nk+k+sk+4)+2$. From equation (2.6), we know

$$\begin{aligned} mT(r,f) =& T(r,f^m) \\ \le& m(r, q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)}) + m\left(r, \frac{p'(z)}{p(z)} - \frac{([f^n(f-1)^s \mathcal{L}_c(f)]^{(k)})'}{[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)}}\right) \\ &+ N\left(r, \frac{p'(z)}{p(z)} q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)} - (q(z)([f^n(f-1)^s \mathcal{L}_c(f)]^{(k)}))'\right) \\ &+ m\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}\right) + N\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}\right) + S(r,f). \end{aligned} \tag{2.7}$$

As Lemma 2.4 together with equation (2.6) implies that

$$(m - nk - sk - (s+1)k)T(r,f) + S(r,f) \le T(r,p(z)) \le (m + nk + sk + (s+1)k)T(r,f) + S(r,f),$$

we conclude that

$$S(r,p(z)) = S(r,f). \tag{2.8}$$

Applying Lemmas 2.4, 2.5 and (2.8) to equation (2.7), we obtain that

$$\begin{aligned} mT(r,f) \le& k(s+n+2)m(r,f) + N\left(r, \frac{p'(z)}{p(z)} q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)} - (q(z)([f^n(f-1)^s \mathcal{L}_c(f)]^{(k)}))'\right) \\ &+ N\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}\right) + S(r,f). \end{aligned} \tag{2.9}$$

Let

$$\alpha(z) = \frac{p'(z)}{p(z)} q(z)[f^n(f-1)^s \mathcal{L}_c(f)]^{(k)} - (q(z)([f^n(f-1)^s \mathcal{L}_c(f)]^{(k)}))' \tag{2.10}$$

and

$$\beta(z) = \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}. \tag{2.11}$$

First of all, we deal with $N(r, \alpha(z))$. From (2.4) and (2.10), we know the poles of $\alpha(z)$ are at the zeros of $p(z)$ and at the poles of $f(z), f(z+jc), (j = 1, 2, ..., s)$ and $q(z)$. Poles of $p(z)$ will not contribute towards the poles of $\alpha(z)$ as from the equation (2.4) we know that the poles of $p(z)$ should be at the poles of $f(z), f(z+jc), (j = 1, 2, ..., s)$ and $q(z)$. We note that $T(r, q(z)) = S(r, f)$.

If z_0 is a zero of $p(z)$ then by (2.10), z_0 is at most a simple pole of $\alpha(z)$. If z_0 is a pole of $f(z)$ of multiplicity t but not a pole of $f(z+jc), (j = 1, 2, ..., s)$, then z_0 will be a pole of $\alpha(z)$ of multiplicity at most $tnk+1$. Next suppose z_1 be any pole of $f(z)$ of multiplicity t_0 and a pole of at least one $f(z+jc), (j = 1, 2, ..., s)$, of multiplicity $t_i \geq 0$. Then z_1 may or may not be a pole of $[f^n(f-1)^s \mathcal{L}_c(f)]$. From the above arguments and our assumption, we conclude that

$$N(r, \alpha) \leq \overline{N}\left(r, \frac{1}{p(z)}\right) + kN(r, f^n) + kN(r, (f-1)^s) + kN(r, \mathcal{L}_c(f)) + \overline{N}(r, f) + \overline{N}(r, \mathcal{L}_c(f)) + S(r, f)$$

$$N(r, \alpha) \leq nkN(r, f) + skN(r, (f-1)) + kN(r, \mathcal{L}_c(f)) + ((s+1)+1)\overline{N}(r, f) + S(r, f). \tag{2.12}$$

Next, we turn our attention towards the poles of $\beta(z)$ are at the zeros of $p(z)$ and $f(z)$ and at the poles of $f(z), f(z+jc), (j = 1, 2, ..., s)$. If z_0 is a zero of $p(z)$, zero of $f(z)$, or pole of $f(z), f(z+jc), (j = 1, 2, ..., s)$, then by (2.11) we know z_0 will be at most a simple pole of $\beta(z)$. If z_0 is a pole of $f(z)$ but not a pole of $f(z), f(z+jc), (j = 1, 2, ..., s)$, then by the Laurent expansion of $\beta(z)$ at z_0, we obtain that $\beta(z)$ is analytic at z_0. Therefore, from our assumption and the discussions above, we know

$$N(r, \beta) \leq \overline{N}\left(r, \frac{1}{p(z)}\right) + \overline{N}(r, f) + \overline{N}(r, (f-1)) + \overline{N}(r, \mathcal{L}_c(f)) + \overline{N}(r, \frac{1}{f}) + S(r, f)$$

$$N(r, \beta) \leq \overline{N}(r, f) + \overline{N}(r, (f-1)) + \overline{N}(r, \mathcal{L}_c(f)) + \overline{N}(r, \frac{1}{f}) + S(r, f). \tag{2.13}$$

Using Lemma 2.6, from equations (2.9), (2.12) and (2.13) we have

$$\begin{aligned} mT(r, f) &\leq (nk+k)m(r, f) + nkN(r, f) + skN(r, f) + kN(r, \mathcal{L}_c(f)) + ((s+1)+1)\overline{N}(r, f) \\ &+ \overline{N}(r, f) + \overline{N}(r, (f-1)) + \overline{N}(r, \mathcal{L}_c(f)) + \overline{N}(r, \frac{1}{f}) + S(r, f) \\ &\leq (nk+k)m(r, f) + nkN(r, f) + skN(r, f) + k(s+1)N(r, f) + ((s+1)+1)\overline{N}(r, f) \\ &+ \overline{N}(r, f) + (2s+1)\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + S(r, f) \\ &\leq (nk+k)m(r, f) + ((s+1)k + nk + sk)N(r, f) + ((s+1)+1)\overline{N}(r, f) \\ &+ 2(s+1)\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + S(r, f) \\ &\leq \{(s+1)(nk+k+sk+4)+1\}T(r, f) + S(r, f), \end{aligned}$$

which contradicts the assumption that $m \geq (s+1)(nk+k+sk+4)+2$. This completes the proof of the theorem.

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