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Energy variability approach in plasma oscillations modelled by a modified Duffing equation

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Abstract

In many nonlinear systems, regular and chaotic behaviours are strongly linked to the energy variability of the system. Energy variability plays a major role in chaotic systems. The Melnikov function provides a measure of the distance between a stable and unstable manifold. If the two manifolds intersect, chaos is possible. The concept of energy variability is introduced in the work through the Melnikov integral. In the paper, we apply the energy variability approach to plasma oscillations modelled by the modified Duffing equation. Due to the energy variability approach, the plasma oscillations show very interesting results during the evolution shown by the works. We observed periodic, quasiperiodic and chaotic oscillations in the system by adjusting the amplitude (f) of the external excitation, energy variability parameter (ϵ), quadratic (β) and cubic (δ) nonlinear parameters. Control of chaos is also observed in some parameter values. The numerical results are demonstrated by a bifurcation diagram, phase plot, Poincaré map and time series graph.

Keywords: Plasma oscillation, Periodic force, Energy variability, Quasiperiodic, Chaos, Melnikov integral
2020 MSC: 34C55,37M20,37G35,74H10,70K30,70K50

1 Introduction

Chaos is a characteristic of a complicated deterministic dynamical system with completely unpredictable behaviour without perfect information. The theories of chaotic dynamical systems are applied to many fields such as sociology [27], economics [30], physics [19], chemistry [14], biology [26] and engineering [28]. Over the past century, chaotic behaviour has become more evident in a variety of active topics. Furthermore, chaos theory is very surprising because chaos is found within almost trivial systems [16]. We have been researching nonlinear systems with constant energy for a long time. But in studying the actual phenomenon, energy is considered a variable. The concept of energy variability in nonlinear dynamics was introduced by Ali [1] and he applied the concept to the Duffing-vander Pol oscillator. Saha et al. [24] studied the energy variability in chaotic dynamical systems and Bharti et al. analyzed the concept of transient chaos and energy variability in double-well Duffing-van der Pol oscillator [7] and Ueda oscillator [8]. Energy variability analysis and methodology are very helpful in understanding the dynamics and evolutionary behavior of nonlinear dynamical systems. Therefore, one can proceed with variable energy to enhance the investigation of various dynamical behaviors in some nonlinear dynamical systems.

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The analytical technique like Melnikov method can be used to investigate various nonlinear behaviours in some nonlinear systems driven by external perturbations. The Melnikov method is an effective analytical technique to determine the threshold condition for onset of chaos near homoclinic or heteroclinic orbit of a system with deterministic or random perturbation. Recently, this method has been applied to certain nonlinear systems. In particular, Sanjuan [25] analysed the effect of nonlinear damping on the universal escape oscillator. Awerjeewicz et al. [6] applied Melnikov's method in the presence of dry friction for a stick-slip oscillator. Bikdesh et al. [10] discussed the Melnikov analysis for a ship with general roll-damping model. Using Melnikov method, Trueba et al. [29] studied the effect of nonlinear damping on certain nonlinear oscillators. Borowice et al. [11] applied the Melnikov criterion to examine global homoclinic bifurcation and transition to chaos in a case of the Duffing system with nonlinear fractional damping and external excitation. Energy variability is a concept where Hamiltonian of the system has been derived in association with Melnikov function.

2 Melnikov integral and Energy variable

A Hamiltonian system generally refers to a conservative autonomous system in which force can be obtained from energy function. For a Hamiltonian system, certain periodic oscillations persist when small nonlinearities are added to the linear part of the system. We consider a two-dimensional system of form

$$\dot{x} = \frac{\partial H}{\partial y}(x,y) + \epsilon g_1(x,y,t) \tag{2.1a}$$

$$\dot{y} = -\frac{\partial H}{\partial x}(x,y) + \epsilon g_2(x,y,t), \tag{2.1b}$$

where $(x, y) \in \mathbb{R}^2$ and the perturbation function g is periodic in time with period $T = \frac{2\pi}{\omega}$, where ω is the phase frequency. We can rewrite these equations in vector form, defining $q = (x, y)$, $DH = (-\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y})$, $g = (g_1, g_2)$. Now if $0 < \epsilon << 1$, Eq.(2.1) is considered to be a perturbed form of the original Hamiltonian. Taking $\epsilon = 0$, the unperturbed system is given by

$$\dot{x} = \frac{\partial H}{\partial y}(x,y) \tag{2.2a}$$

$$\dot{y} = -\frac{\partial H}{\partial x}(x,y), \tag{2.2b}$$

This system is Hamiltonian. A Hamiltonian system has the property that there exists a function $H = H(x, y)$ is called the Hamiltonian. Here x, y is the unperturbed homoclinic orbit solutions starting from the saddle point of the original Hamiltonian system. The Hamiltonian is an integral of motion, namely,

$$H(x,y) = C, \tag{2.3}$$

where C is an energy constant. $H(x, y) = 0$ represents the separatrix in the phase plane. Since the system is perturbed system, the energy level C becomes a variable throughout the motion. So Eq.(2.3) is an integral of system (Eq.2.1) and differentiating Eq.(2.3) with respect to time and substituting from Eq.(2.1), we obtain [1, 7, 8, 24]:

$$\frac{dC}{dt}(x,y,t) = \epsilon\left(DH(x,y) \wedge g(x,y,t)\right). \tag{2.4}$$

Here the wedge product is defined by $f \wedge g = f_0 g_1 - f_1 g_0$. Energy variable for a perturbed Hamiltonian system be defined in association with the Melnikov function. The Melnikov function provides the measure of the distance between the stable and unstable manifolds of the saddle equilibrium point of the Poincaré map of sections near the separatrix. The Melnikov theory states that chaos is possible if these two manifolds intersect, which corresponds to that Melnikov function has a simple zero. As in [16], the Melnikov function is given by

$$M(t_0) = \int_{-\infty}^{+\infty} DH(q_0(t-t_0)) \wedge g(q_0(t-t_0), t)dt, \tag{2.5}$$

where $q_0(t - t_0)$ is the separatrix of the Hamiltonian system (Eq.2.3). Ignoring a multiplication by constant and substituting from Eq.(2.4), we find,

$$M(t_0) = \int_{-\infty}^{+\infty} \frac{dC}{dt}(q_0(t-t_0), t). \tag{2.6}$$

Eq.(2.6) gives the physical meaning of the Melnikov integral. This integral is nothing but the total variation of energy variable C with respect to time over the homoclinic orbit for certain given initial conditions.

3 Energy Variability Approach in Plasma Oscillations

Plasma is a gas that has undergone significant ionization, thus containing its elements, ions, electrons, and neutrals. The presence of electromagnetic field influences the motion of the plasma through interaction with the charged particles. The model most often used to describe the interaction between the plasma and the magnetic field is the magneto-hydrodynamic (MHD) model. The fluid-model describes the ions and electrons as two inter-penetrating fluids with Maxwell equations describing the electric and magnetic fields. The equations that make up the two fluid model are the Euler equations with additional source terms and Maxwell equations of electromagnetism [18]. This model has been a source of growing interest to researchers over the years. Nowadays it is an interesting task because of its potential applications. In particular, the nonlinear description of plasma oscillations is of interest as a result of its importance in the semiconductor industry [22, 23]. Arshad et al. studied the flow of a viscous, incompressible, hybrid nanofluid over a stretched, rotating, permeable plate with a heat source and chemical reaction under the influence of a magnetic field [3], heat transfer rate for conventional and hybrid nanofluids, incorporating the Hall Effect over a stretchable surface [5], the convective magneto-hydrodynamic single-phase flow for comparative analysis of two different base fluids above an exponentially stretchable porous surface under the effect of the chemical reaction [4] and a numerical study on the hybrid nanofluid flow between a permeable rotating system [2]. Marc et al. [21] investigated the precision calibration of the Duffing oscillator with phase control. Experiments suggest that some plasma behavior is approximately described by anharmonic oscillations. The following equation governs the model

$$\ddot{x} + \omega_0^2 x + \beta x^2 + \delta x^3 = f \sin \omega t, \tag{3.1}$$

where ω_0^2 and ω are respectively the natural and external frequencies, f stands for the amplitude of the external excitation, β and δ are the quadratic and cubic nonlinearities parameters. Eq.(3.1) describes in plasma physics, the electron beam surfaces, the Tonks-Datter resonances of mercury vapour and low frequency ion sound waves. Many researchers have studied various nonlinear behaviors in Eq.(3.1). For example, Mahaffey [20] presented unforced system of Eq.(3.1) for plasma oscillations having features of shift frequency of oscillation and asymmetric of amplitude of oscillations and changes in the resonance response curve. Enjieu-kadji et al. modified this model to study the passive aerodynamics control of plasma instabilities [13], nonlinear dynamics of plasma oscillations modulated by an anharmonic oscillator [15] and regular and chaotic behaviours of plasma oscillations modelled by a modified Duffing equation [12]. Recently, Bhuvaneshwari et al. [9] investigated the enhanced vibrational resonance by an amplitude modulated signal in a nonlinear dissipation two-fluid model. Jayaprakash et al. [17] used the above model (Eq.3.1) to describe floating potential fluctuations (FPF) in a dc glow discharge plasma system with constricted anode geometry. In the present work, we wish to analyse the occurrence of regular and chaotic behaviours in a perturbed plasma oscillations using energy variability through Melnikov integral.

4 Numerical Simulations

The plasma oscillations defined in Eq.(3.1) are constant energy oscillations. By using the energy variability approach, these oscillations can be transformed into a new form where the energy acts as a variable and can be written as

$$\dot{x} = y, \tag{4.1a}$$
$$\dot{y} = -\omega_0^2 x - \beta x^2 - \delta x^3 + \epsilon f \sin \omega t, \tag{4.1b}$$

where the perturbation term ϵ is very small (ie. $0 < \epsilon \leq 1$). The presence of ϵ indicates that we are dealing with variable energy. For this system, using Eq.(2.1), we can write Eq.(4.1) in the form

$$DH \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -\omega_0^2 x - \beta x^2 - \delta x^3 \end{bmatrix}. \tag{4.2}$$

and

$$g \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ f \sin \omega t \end{bmatrix}. \tag{4.3}$$

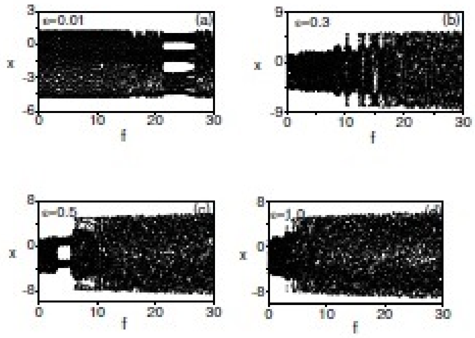

Figure 1: Bifurcation diagrams of the plasma oscillations modelled by a modified Duffing equation (Eq.4.1) for four values of $\epsilon = 0.01, 0.3, 0.5$ and 1.0. Other parameters are fixed at $\omega_0^2 = 1.0, \beta = 2.05, \delta = 1.5$ and $\omega = 3.0$.

The dynamics of the system is generally sensitive to the parameter of the system (Eq.3.1). Since the exact analytical solution of the nonlinear system is not known, we want to perform numerical simulations for some choices of the parameters of the system. Let us now analyse the occurrence of regular and chaotic behaviours of the system (Eq.4.1) with variable energy.

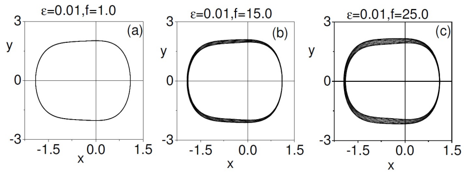

Figure 2: Phase plots of the system (Eq.4.1) for three values of $f = 1.0, 15.0, 25.0$ chosen in the Fig.1(a) with $\epsilon = 0.01$. Other parameters are fixed at $\omega_0^2 = 1.0, \beta = 2.05, \delta = 1.5$ and $\omega = 3.0$.

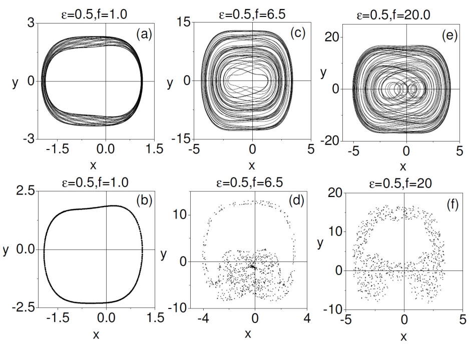

Figure 3: Phase plots of the system (Eq.4.1) for three values of $f = 1.0, 6.5, 20.0$ chosen in the Fig.1(c) with $\epsilon = 0.5$. Other parameters are fixed at $\omega_0^2 = 1.0, \beta = 2.05, \delta = 1.5$ and $\omega = 3.0$.

For this purpose, the system (Eq.4.1) is solved by the Fourth-order Runge kutta method with time step size $\Delta t = (2\pi/\omega)/200$. Initial condition that we used in the simulation is $(x(0), \dot{x}(0)) = (1, 1)$. Numerical solutions corresponding to 500 first drive cycles are left as transient. We analysed the system Eq(.8) by varying the amplitude (f) of the external excitation, perturbation parameter (ϵ), quadratic (β) and cubic nonlinearities (δ) parameters of the system. The numerical results are demonstrated through a bifurcation diagram, phase portrait, Poincaré map and time series graph. First, we analyse the effect of amplitude (f) of the external excitation for four values of $\epsilon = 0.01, 0.3, 0.5$ and 1.0. Other parameters values are fixed as $\omega_0^2 = 1, \beta = 2.05, \delta = 1.5$ and $\omega = 3.0$. Figure 1(a) shows the bifurcation diagram of f versus x with $\epsilon = 0.01$ (ie. energy part contributes very little). In this diagram, the ordinate represents $x(t)$ values collected at time t equal to each integral multiple of $2\pi/\omega$ (Poincaré points) after leaving the sufficiently transient evolution. For the above set of parameters, as shown in the bifurcation diagram (Fig.1(a)), when the amplitude of the force (f) varies, the system (Eq.4.1) can change from periodic to quasiperiodic oscillations. Therefore if the energy tends to zero in the system the motion is periodic as well as quasiperiodic and not chaotic. Now we analyse the effects of $\epsilon = 0.3, 0.5$ and 1.0 (ie. the energy part contributes significantly but is a variable and not a constant) in the system (Eq.4.1). The corresponding bifurcation diagrams are shown in Figs.1(b), 1(c) and 1(d). From these bifurcation diagrams, the system shows transition from periodic to quasiperiodic oscillations and finally chaotic states when the amplitude f of the force varies from a small value. Hence the energy part plays a very significant role in the enhancement or suppression of chaos. To illustrate such situations, we have represented by phase portraits and Poincaré maps using the parameters of the bifurcation diagram, for which periodic and quasiperiodic oscillations can be seen in Figs.2 and 3. The phase portraits of the system (Eq.4.1) for $\epsilon = 0.01$ with f=1.0,15.0 and 25.0 selected in Fig.1(a) are shown in Fig.2. The system shows a transition from periodic to quasiperiodic oscillations for $\epsilon = 0.01$, which is clearly confirmed in Fig.2.

To justify the appearance of chaotic oscillations, the phase plots are presented in Fig.3 along with the respective Poincaré diagrams for some values of f selected in Fig.1(c). Phase portraits of the system for $f = 1.0, 6.5$ and 20.0 with $\epsilon = 0.5$ are given in Figs.3(a), 3(c) and 3(e) and the corresponding Poincaré maps are presented in Figs.3(b),3(d) and 3(f). From Fig.3, we see clearly that the system displays transition from quasiperiodic oscillations to chaotic oscillations. Time series graphs of $x(t)$ and $y(t)$ for the time duration from $t = 0$ to $t = 200$ for exactly the same parameters values as in case of Fig.3 are given in Fig.4. The time series graphs shown in Fig.4 shows quasiperiodic oscillations for $f = 1.0$, whereas chaotic oscillations for $f = 6.5$ and 20 with $\epsilon = 0.5$ for time duration from $t = 0$ to $t = 200$.

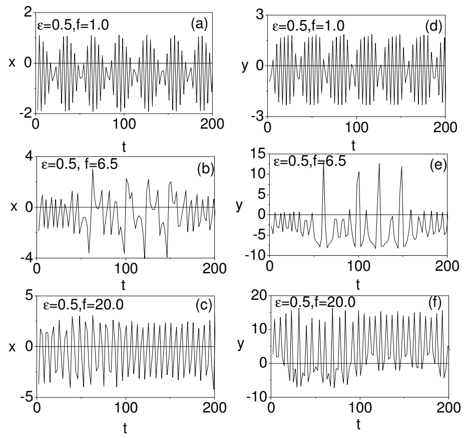

Figure 4: The time series graphs of x and y for three values of $f = 1.0, 6.5, 20.0$ chosen in the Fig,1(c) with $\epsilon = 0.5$. Other parameters are fixed at $\omega_0^2 = 1.0, \beta = 2.05, \delta = 1.5$ and $\omega = 3.0$.

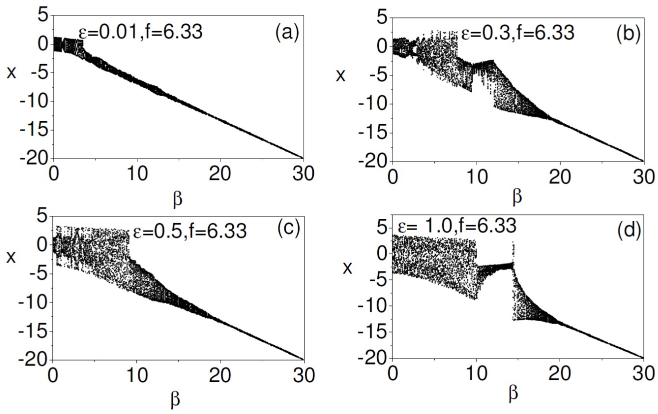

Figure 5: Bifurcation diagrams of the plasma oscillations modelled by a modified Duffing equation (Eq.4.1) for four values of $\epsilon = 0.01, 0.3, 0.5$ and 1.0. Other parameters are fixed at $\omega_0^2 = 1.0, f = 6.33, \delta = 1.5$ and $\omega = 3.0$.

A bifurcation diagram, phase portrait, Poincaré map and time series graph are also plotted (Eq.4.1) to show how the quadratic (β) and cubic (δ) nonlinear parameters can affect the dynamics of the system. In our numerical simulations, we started the calculations from the same parameters values and initial conditions. The system (Eq.4.1) transitions from quasiperiodic oscillations to chaotic oscillations and from chaotic oscillations to quasiperiodic oscillations when the quadratic and cubic nonlinearities vary as shown in Fig.(5-12). Fig.5 shows the bifurcation diagrams for four values of $\epsilon = 0.01, 0.3, 0.5$ and 1.0 respectively, where we have plotted the values of x against the quadratic nonlinearity (β). Other values of the parameters are fixed as $\omega_0^2 = 1, \delta = 1.5, \omega = 3.0$ and $f = 6.33$ (at this value chaotic oscillations occur in the unperturbed system (Eq.(3.1)). As ϵ is increased from small values, a transition from chaotic motion to quasiperiodic motion and expansion of the chaotic region appear in the system. To illustrate such situations, we have represented the phase plots and time series graphs in Figs.6 and 7 using the parameters of the bifurcation diagram given in Fig.5(c). The phase plots and the corresponding Poincaré maps of the system for four values of β, namely $\beta = 5.0, 10.0$ and 20.0 respectively are given in Fig.6. From these phase plots and Poincaré maps, we see clearly that chaotic oscillations become quasiperiodic oscillations as β is increased from small values. The time series graphs of $x(t)$ and $y(t)$ for the time duration from $t = 0$ to $t = 200$ for exactly the same parameters values as in the case of Fig.6 are given in Fig.7. It shows the chaotic motion for $\beta = 5.0$, 10 and $\beta = 20$. However, when we increase the time duration e.g. from $t = 1900$ to $t = 2100$, the behaviour of the time series, as given in Fig.8 shows the chaotic motion for $\beta = 5$ and 10 but quasiperiodic motion for $\beta = 20$.

Finally, the effect of the cubic nonlinear parameter (δ) on the system (Eq.4.1) through phase plots and time series graphs for the values of the parameters $\omega_0^2 = 1, \beta = 2.05, f = 6.33$, $\omega = 3.0$, $\epsilon = 0.1$ and $\epsilon = 0.5$ respectively are presented in Fig.(9-12). First we take $\epsilon = 0.1$(ie. energy part contributes very little) and any value of δ, for example, here we have taken $\delta = 2.5, 15.0$ and 30.0. The phase portraits of the system (Eq.4.1) for these parameter values are given in Fig.9. From this plot, the system shows transition from quaisperiodic oscillations to chaotic and finally settles to quasiperiodic oscillations. Time series graphs of $x(t)$ and $y(t)$ for time durations from $t = 0$ to $t = 200$ and $t = 1900$ to $t = 2100$ are presented in Fig.10 for exactly the same parameter values as in the case of Fig.9. Fig.10 shows quasiperiodic motion for $\delta = 2.5$ and 30.0 and chaotic motion for $\delta = 15.0$. Now we take another value of $\epsilon = 0.5$ (ie. energy part contributes significantly). For $\epsilon = 0.5$, we observe a transition from chaotic oscillations to quasiperiodic oscillations. For example, taking $\epsilon = 0.5$, $\delta = 2.5, 15.0$ and 30.0, the phase plots and the corresponding Poincaré maps of the system (Eq.4.1) are given in Fig.11. From this we see clearly the transition from chaotic oscillations to quasiperiodic oscillations. The time series graphs of $x(t)$ for the time duration from $t = 0$ to $t = 200$ for exactly the same parameters values as in Fig.11 are given in Figs.12(a-c) show chaotic oscillations for $\delta = 2.5$ and 15.0 and quasiperiodic oscillations for $\delta = 30.0$. However, when we increase the time duration, for example, from $t = 1900$ to $t = 2100$ the same behaviours are observed as in the Figs.12(a-c), which is presented in Figs.12(d-f). From the above observations, we conclude that evolutionary behaviours of the system can be periodic, quasiperiodic and chaotic motions depending upon the parameters values of the external excitation and the system parameters β and δ. Also, the energy variability plays an important role in displaying periodic, quasiperiodic and chaos of the plasma oscillations.

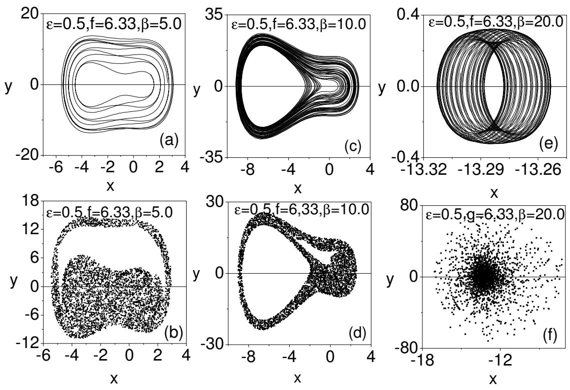

Figure 6: Phase plots and the corresponding Poincaré maps of the system (Eq.4.1) for three values of $\beta = 5.0, 10., 20.0$ chosen in the Fig.5(c) with $\epsilon = 0.5$. Other parameters are fixed at $\omega_0^2 = 1.0, f = 6.33, \delta = 1.5$ and $\omega = 3.0$.

5 Conclusion

In the paper we have numerically studied the phenomenon of periodic, quasiperiodic and chaotic oscillations in the plasma oscillations governed by modified Duffing equation using the energy variability approach. From the numerical results, we observed that this type of approach is very helpful to understand the various dynamical behaviours of plasma oscillations. And these behaviors are very sensitive to the energy variation parameter ϵ in plasma oscillations. We discussed the influence of the amplitude (f) of the excitation, energy variability parameter ϵ, quadratic (β) and cubic (δ) nonlinear parameters of the system and these parameters significantly alter the dynamics of the system. With the appropriate choices of the parameters of the system we observed the transition from quasiperiodic oscillations to chaotic oscillations and from chaotic oscillations to quasiperiodic oscillations. The quadratic nonlinearity βx^2 term modifies the Duffing equation, a strong chaotic as well as periodic behaviours exhibited by the system is reported as β varies. Also, we have analyzed the influence of β in accordance with the amplitude f of the forcing term and cubic nonlinearity δ parameters of the system. Due to the presence of these terms, more complex dynamical behaviours are observed in the modified Duffing system than that of the original Duffing system.

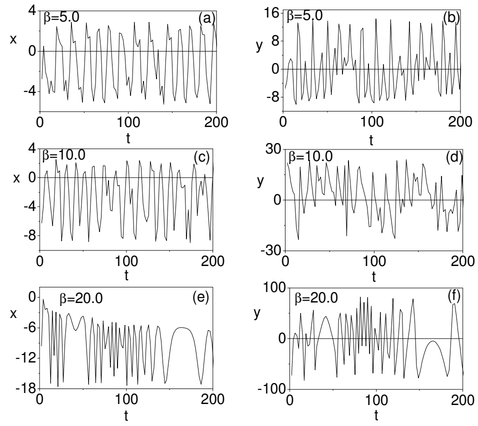

Figure 7: The time series graphs for $t = 0$ to $t = 200$ of x and y of the system (Eq.4.1) for three values of $\beta = 5.0, 10.0, 20.0$ chosen in the Fig.5(c) with $\epsilon = 0.5$. Other parameters are fixed at $\omega_0^2 = 1.0, f = 6.33, \delta = 1.5$ and $\omega = 3.0$.

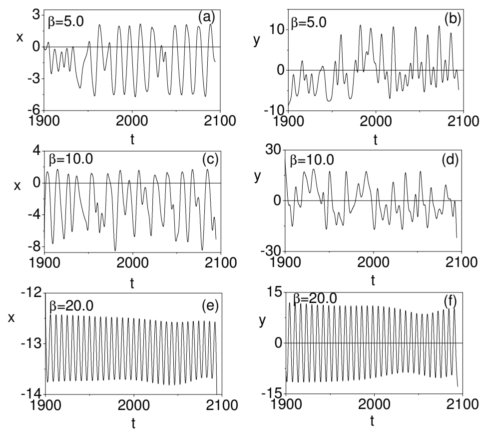

Figure 8: The time series graphs for $t = 1900$ to $t = 2100$ of x and y of the system (Eq.4.1) for three values of $\beta = 5.0, 10.0, 20.0$ chosen in the Fig.5(c) with $\epsilon = 0.5$. Other parameters are fixed at $\omega_0^2 = 1.0, f = 6.33, \delta = 1.5$ and $\omega = 3.0$.

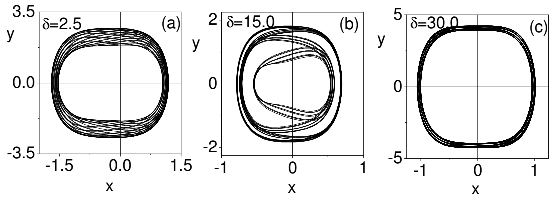

Figure 9: Phase plots and the corresponding Poincaré maps of the system (Eq.4.1) for three values of $\delta = 5.0, 10., 20.0$ with $\epsilon = 0.1$. Other parameters are fixed at $\omega_0^2 = 1.0, f = 6.33, \beta = 2.05$ and $\omega = 3.0$.

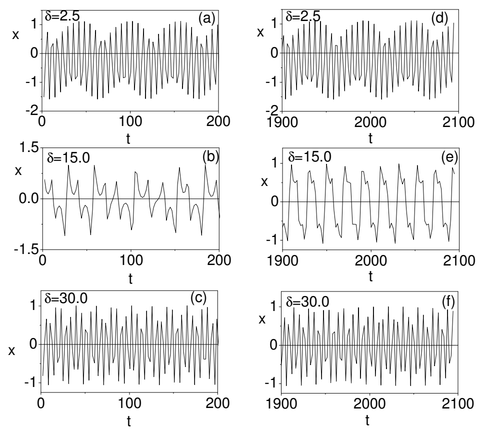

Figure 10: The time series graphs for $t = 0$ to $t = 200$ of x of the system (Eq.4.1) for three values of $\delta = 5.0, 10.0, 20.0$ with $\epsilon = 0.1$. Other parameters are fixed at $\omega_0^2 = 1.0, f = 6.33, \beta = 2.05$ and $\omega = 3.0$.

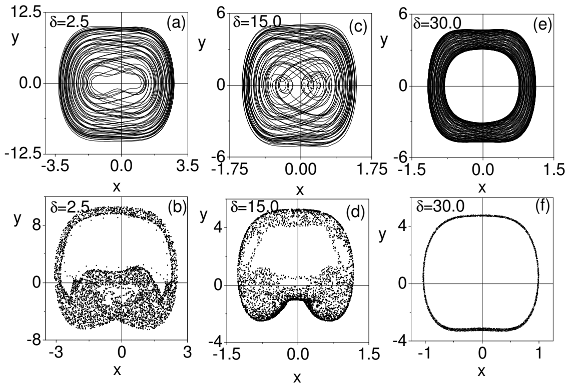

Figure 11: Phase plots and the corresponding Poincaré maps of the system (Eq.4.1) for three values of $\delta = 5.0, 10., 20.0$ with $\epsilon = 0.5$. Other parameters are fixed at $\omega_0^2 = 1.0, f = 6.33, \beta = 2.05$ and $\omega = 3.0$.

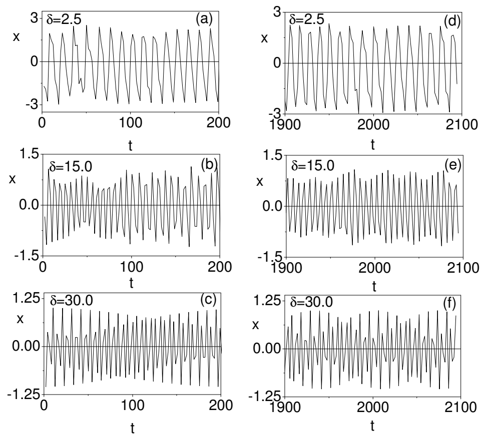

Figure 12: The time series graphs for $t = 0$ to $t = 200$ of x of the system (Eq.4.1) for three values of $\delta = 5.0, 10.0, 20.0$ with $\epsilon = 0.5$. Other parameters are fixed at $\omega_0^2 = 1.0, f = 6.33, \beta = 2.05$ and $\omega = 3.0$.

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