Int. J. Nonlinear Anal. Appl. 14 (2023) 10, 293–302
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.28086.3797

# Best proximity point theorems for generalized weakly contractive mappings in metric spaces

Yohannes Gebru Aemro*, Fayise Adare Demu

*Department of Mathematics, College of Natural Science and Computational, Wolkite University, Wolkite, Ethiopia*

*(Communicated by Abasalt Bodaghi)*

**Abstract**

The aim of this paper is to establish certain new classes of proximal contraction mappings and establish some best proximity point theorems for such kinds of mapping, thereby we extend some fixed point theorems for generalized weakly contractive mappings in metric spaces to the case of non-self mapping.

Keywords: Contractive Mapping, Weakly Contractive Mapping, Best Proximity Point, Generalized Proximal Weakly Contractive Mapping.
2020 MSC: 47H10, 54H25

## 1 Introduction

In nonlinear functional analysis, fixed point theory and best proximity point theory play an important role in the existence of certain differential and integral equations. As a consequence fixed point theory is very useful for various quantitative sciences that involve such equations. The most remarkable paper in this field was reported by Banach in 1922 [3]. In his paper Banach proved that every contraction in a complete metric space has a unique fixed point. Following this paper many researchers have extended and generalized this remarkable fixed point theorem of Banach by changing either the conditions of the mappings or the construction of the space. In 1977, Alber [1] generalized Banach's contraction principle by introducing the concept of weak contraction mappings in Hilbert spaces. The Weak contraction principle states that every weak contraction mapping on a complete Hilbert space has a unique fixed point. Rhoades [13] extended weak contraction principle in Hilbert spaces to metric spaces. Since then, many authors obtained generalizations and extensions of the weak contraction principle. Khan [7] obtained fixed point theorems in metric spaces by introducing the concept of altering distance functions. In particular, Choudhury [4] obtained a generalization of the weak contraction principle in metric spaces by using altering distance functions.

**Definition 1.1.** [7] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

i) $\psi$ is monotone increasing and continuous;
ii) $\psi(t) = 0$ if and only if $t = 0$.

Further, let $\Phi$ denote the class of all functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

*Corresponding author
*Email addresses:* yohannesgebru2005@gmail.com (Yohannes Gebru Aemro), fayiseadare@gmail.com (Fayise Adare Demu)

*Received:* August 2022 *Accepted:* June 2023

i) $\phi$ is a lower semi-continuous function;
ii) $\phi(t) = 0 \iff t = 0$.

**Definition 1.2.** [6] Let (X,d) be a metric space. The mapping $T : X \to X$ is said to be a contractive mapping if

$$d(Tx, Ty) < d(x, y) \; for \; all \; x, y \in X \; with \; x \neq y.$$

**Definition 1.3.** [13] Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. $T$ is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \; for \; all \; x, y \in X,$$

where $\phi \in \Phi$

**Lemma 1.4.** [2] If a sequence $\{x_n\} \subset X$ is not Cauchy, then there exist $\epsilon > 0$ and two sub-sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which $m(k) > n(k) > k$, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)-1}, x_{n(k)}) < \epsilon$. Moreover, suppose that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Then we have:

(i) $\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$;

(ii) $\lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon$;

(iii) $\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \epsilon$;

(iv) $\lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$.

**Definition 1.5.** [5] A function $T : X \to [0, \infty)$, where $X$ is a metric space, is called lower semi-continuous if, for all $x \in X$ and $\{x_n\} \subset X$ with, $\lim_{n\to\infty} x_n = x$, we have $T(x) \leq \liminf_{n\to\infty} T(x_n)$.

**Definition 1.6.** [5] Let $X$ be a metric space with metric d, let $T : X \to X$, and let $\varphi : X \to [0, \infty)$ be a lower semi-continuous function. Then $T$ is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, t, \varphi)) \; for \; all \; x, y \in X, \tag{1.1}$$

where, $\psi \in \Psi, \phi \in \Phi$,

$$m(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty), 1/2\{d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)\}\}$$

and

$$l(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty)\}.$$

Cho has proved the following fixed point for generalized weakly contractive mappings in metric spaces:

**Theorem 1.7.** [5] Let $X$ be complete. If $T$ is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.

## 2 Preliminaries

Fixed point theory is essential for solving various equations of the form $Tx = x$ for self-mappings $T$ defined on subsets of metric spaces or other spaces. Given non-empty subsets $A$ and $B$ of a metric space and a non-self-mapping $T : A \to B$, the equation $Tx = x$ does not necessarily has a solution, which is known as a fixed point of the mapping $T$. However, in such conditions, it may be considered to determine an element $x$ for which the error $d(x, Tx)$ is minimum, in which case $x$ and $Tx$ are in close proximity to each other. It has been noted that the best proximity point theory is relevant to this end.

A best proximity point theorem provides sufficient conditions that confirm the existence of an optimal solution to the problem of globally minimizing the error $d(x, Tx)$, and hence the existence of an approximate solution to the equation $Tx = x$. In fact, with respect to the fact that $d(x, Tx) \geq d(A, B)$ for all $x$, a best proximity point theorem requires the global minimum of the error $d(x, Tx)$ to be the least possible value $d(A, B)$. Eventually, a best proximity point theorem offers sufficient conditions for the existence of an element x, called a best proximity point of the mapping T, satisfying the condition $d(x, Tx) = d(A, B)$. Moreover, it is interesting to observe that best proximity theorems also appear as a natural generalization of fixed point theorems, i.e., a best proximity point reduces to a fixed point if the mapping under consideration is a self-mapping. see[11, 12, 10, 8, 14]

**Definition 2.1.** Let $(X, d)$ be a metric space and $A$ and $B$ be nonempty subsets of the metric space $X$. A mapping $T : A \to B$ is called a k-contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for any $x, y \in A$.

It is clear that a k-contraction coincides with the celebrated Banach contraction if one takes $A = B$ where $A$ is a complete subset of $X$. [15] Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. We denote by $A_0$ and $B_0$ the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B),\ for\ some\ y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B),\ for\ some\ x \in A\},$$

where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

**Definition 2.2.** [9] Let $A, B$ be non-empty subsets of metric space $(X, d)$. Given a non-self mapping $T : A \to B$, then an element $x^* \in A$ is called a best proximity point of the mapping $T$ if this condition satisfied:

$$d(x^*, Tx^*) = d(A, B).$$

## 3 Main result

In this section, we introduce a new class of mappings, which we call them "Generalized proximal weakly contractive mappings" and state and prove the existence and uniqueness of the best proximity point for such kinds.

**Definition 3.1.** Let $(X, d)$ be a metric space and A and B be two non-empty subsets of metric space $(X, d)$. A map $T : A \to B$ is said to be a generalized proximal weakly contractive mapping, if for all $x, y, s, r \in A$

$$d(s, Tx) = d(A, B) \quad \text{and} \quad d(r, Ty) = d(A, B).$$

Then

$$\psi(d(s, r) + \varphi(s) + \varphi(r)) \leq \psi(m_r(x, y, s, r, d, \varphi)) - \phi(l_r(x, y, s, r, d, \varphi)),$$

where

$$m_r(x, y, s, r, d, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(x, s) + \varphi(x) + \varphi(s), d(y, r) + \varphi(y) + \varphi(r),$$
$$1/2[d(x, r) + \varphi(x) + \varphi(r) + d(y, s) + \varphi(y) + \varphi(s)]\}$$

and

$$l_r(x, y, s, r, d, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, r) + \varphi(y) + \varphi(r)\}$$

and $\psi \in \Psi, \phi \in \Phi$ and $\varphi$ is a lower semi-continuous function.

**Theorem 3.2.** Let $(A, B)$ be a pair of non-empty closed subsets of a complete metric space $(X, d)$. Consider a mapping $T : A \to B$ satisfying the following conditions:

i) T is a generalized proximal weakly contractive mapping;
ii) $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$;
iii) T is a continuous mapping.

Then there exist a point $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$. Furthermore, $T$ has a unique best proximity point.

**Proof .** We prove the existence of best proximity point. Since $A_0$ is non-empty set, $A_0$ contains at least one element, say $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$, there exists $x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$. Then by the definition of $A_0$ we have that $x_1 \in A_0$. Similarly, since $Tx_1 \in T(A_0) \subset B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Continuing this process in a similar fashion, we obtain the sequence $\{x_n\} \subset A_0$ such that

$$d(x_n, Tx_{n-1}) = d(A, B),\ \text{for all}\ n \in \mathbb{N}. \tag{3.1}$$

Now, suppose that there exist an $n_0 \in N$ with $x_{n_0} = x_{n_0+1}$. From (3.1) we have

$$d(x_{n_0+1}, Tx_{n_0}) = d(A, B).$$

Therefore, $d(x_{n_0}, Tx_{n_0}) = d(A, B)$. So $x_{n_0}$ is a best proximity point of $T$. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Claim: $d(x_n, x_{n+1}) \to 0$. Since $T$ is a generalized proximal weakly contractive mapping and by using (3.1) we have

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{and} \quad d(x_{n+2}, Tx_{n+1}) = d(A, B).$$

Then,

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) \leq& \psi(m_r(x_n, x_{n+1}, x_{n+2}, d, \varphi)) - \phi(l_r(x_n, x_{n+1}, x_{n+2}, d, \varphi)) \\ =& \psi(\max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), \\ & d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}), 1/2[d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2}) \\ & + d(x_{n+1}, x_{n+1}) + \varphi(x_{n+1}) + \varphi(x_{n+1})]\}) - \phi(\max\{d(x_n, x_{n+1}) + \varphi(x_n) \\ & + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\}). \end{aligned} \tag{3.2}$$

We have

$$\begin{aligned} & 1/2(d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2}) + d(x_{n+1}, x_{n+1})) + \varphi(x_{n+1}) + \varphi(x_{n+1})) \\ =& 1/2(d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+1})). \end{aligned}$$

By using triangular inequality,

$$d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}).$$

Hence

$$\begin{aligned} & 1/2(d(x_n, x_{n+2}) + \varphi(x_n) + \varphi(x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+1})) \\ \leq& 1/2(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) + d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1})) + \varphi(x_{n+2})) \\ \leq& \max(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})). \end{aligned}$$

Then from (3.2) we obtain

$$\begin{aligned} & \psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1})) + \varphi(x_{n+2})) \\ & \leq \psi(\max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\}) \\ & - \phi(\max\{d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\}). \end{aligned} \tag{3.3}$$

Suppose that,

$$d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) < d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}),$$

for some positive integer $n$. Then from (3.3) we get

$$\psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) \leq \psi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) - \phi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}))$$

that is,

$$\phi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) \leq 0,$$

which implies that,

$$\phi(d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})) = 0.$$

From the properties of $\phi$,

$$d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}) = 0.$$

$$x_{n+1} = x_{n+2} \quad \text{and} \quad \varphi(x_{n+1}) = \varphi(x_{n+2}) = 0,$$

which is a contradiction. Therefore,

$$d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2}) < d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \quad \text{for any} \quad n \in \mathbb{N}. \tag{3.4}$$

So

$$\begin{aligned}\psi(d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))\le\psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))\\-\phi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))<\psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})).\end{aligned}\tag{3.5}$$

So,

$$\psi(d(x_{n+1},x_{n+2})+\varphi(x_{n+1})+\varphi(x_{n+2}))<\psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})).$$

It follows from (3.4) that the sequence

$$\{d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}$$

is decreasing and bounded below, hence there exists $r\ge 0$ such that

$$\lim_{n\to\infty}\{d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}\to r.$$

We claim that $r=0$. Assume $r>0$. Taking the limsup in both sides as $n\to\infty$ in (3.5), by using the continuity of $\psi$ and the lower semi-continuity of $\phi$ it follow that,

$$\psi(r)\le\psi(r)-\liminf_{n\to\infty}\phi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))\le\psi(r)-\phi(r).$$

Since $r>0$, so, $\phi(r)>0$. Hence

$$\psi(r)\le\psi(r)-\phi(r)<\psi(r),$$

a contradiction. Thus $\phi(r)=0$, i.e,

$$\lim_{n\to\infty}(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))=0,$$

which implies

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0\quad\text{and}\tag{3.6}$$

$$\lim_{n\to\infty}\varphi(x_n)=0.\tag{3.7}$$

Now, we prove that the sequence $\{x_n\}$ is Cauchy. If $\{x_n\}$ is not Cauchy, then there exist $\epsilon>0$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that for all positive integers k, $n(k)>m(k)>k$, $d(x_{m(k)},x_{n(k)})\ge\epsilon$ and $d(x_{m(k)},x_{n(k)-1})<\epsilon$. Since T is a generalized proximal weakly contractive mapping and from (3.1) we have

$$d(x_{m(k)+1},Tx_{m(k)})=d(A,B)\quad\text{and}\quad d(x_{n(k)+1},Tx_{n(k)})=d(A,B).$$

Then,

$$\begin{aligned}&\psi(d(x_{m(k)+1},x_{n(k)+1})+\varphi(x_{m(k)+1})+\varphi(x_{n(k)+1}))\\&\le\psi(m_r(x_{m(k)},x_{n(k)},x_{m(k)+1},x_{n(k)+1},d,\varphi))-\phi(l_r(x_{m(k)},x_{n(k)},x_{m(k)+1},x_{n(k)+1},d,\varphi))\\&=\psi(\max(d(x_{m(k)},x_{n(k)})+\varphi(x_{m(k)})+\varphi(x_{n(k)}),d(x_{m(k)},x_{m(k)+1})+\varphi(x_{m(k)})\\&+\varphi(x_{m(k)+1}),d(x_{n(k)},x_{n(k)+1})+\varphi(x_{n(k)})+\varphi(x_{n(k+1)}),1/2\{d(x_{m(k)},x_{n(k)+1})\\&+\varphi(x_{m(k)})+\varphi(x_{n(k)+1})+d(x_{n(k)},x_{m(k)+1})+\varphi(x_{n(k)})+\varphi(x_{m(k)+1})\}))\\&-\phi(\max(d(x_{m(k)},x_{n(k)})+\varphi(x_{m(k)})+\varphi(x_{n(k)}),\\&d(x_{n(k)},x_{n(k)+1})+\varphi(x_{n(k)})+\varphi(x_{n(k)+1}))).\end{aligned}\tag{3.8}$$

Taking limsup as $k\to\infty$ in (3.8), by using the continuity of $\psi$ and the lower semi-continuity of $\phi$ and applying Lemma (1.4), (3.6) and (3.7) it follow that,

$$\psi(\epsilon)\le\psi(\epsilon)-\liminf_{n\to\infty}\phi(d(x_{m(k)},x_{n(k)})+\varphi(x_{m(k)})+\varphi(x_{n(k)}))\le\psi(\epsilon)-\phi(\epsilon),$$

which implies that $\phi(\epsilon) = 0$. From the properties of $\phi$, $\epsilon = 0$. This contradict the fact that $\epsilon > 0$. So $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\} \subset A$ and A is a closed subset of the complete metric space $(X, d)$, there exists $x^* \in A$ such that $\lim_{k\to\infty} x_n = x^*$ and since $\varphi$ is lower semi-continuous,

$$\varphi(x^*) \leq \liminf_{n\to\infty} \varphi(x_n) \leq \lim_{n\to\infty} \varphi(x_n) = 0,$$

this implies that, $\varphi(x^*) = 0$. Since $T$ is continuous, we have $\lim_{n\to\infty} Tx_n = Tx^*$ and $d(x_{n+1}, Tx_n) \to d(x^*, Tx^*)$. So $d(x^*, Tx^*) = d(A, B)$. Hence $x^*$ is a best proximity point of $T$. We prove that the best proximity point of $T$ is unique. Let $p$ and $q$ be two best proximity points of $T$ and $p \neq q$. Therefore,

$$d(p, Tp) = d(A, B) \quad \text{and} \quad d(q, Tq) = d(A, B).$$

Since $T$ is a generalized proximal weakly contractive mapping, we have

$$\begin{aligned}
\psi(d(p,q) + \varphi(p) + \varphi(q)) \leq &\psi(m_r(p,q,p,q,d,\varphi)) - \phi(l_r(p,q,p,q,d,\varphi)) \\
= &\psi(\max\{d(p,q) + \varphi(p) + \varphi(q), d(p,p) + \varphi(p) + \varphi(p), d(q,q) + \varphi(q) + \varphi(q), \\
&1/2[d(p,q) + \varphi(p) + \varphi(q) + d(q,p) + \varphi(q) + \varphi(p)]\}) \\
&- \phi(\max\{d(p,q) + \varphi(p) + \varphi(q), d(q,q) + \varphi(q) + \varphi(q)\}) \\
= &\psi(\max\{d(p,q) + \varphi(p) + \varphi(q), d(p,q) + \varphi(p) + \varphi(q)\}) \\
&- \phi(\max\{d(p,q) + \varphi(p) + \varphi(q) d(q,q) + \varphi(q) + \varphi(q)\}) \\
= &\psi(d(p,q) + \varphi(p) + \varphi(q)) - \phi(d(p,q) + \varphi(p) + \varphi(q)).
\end{aligned}$$

That is, $\psi(d(p,q)) \leq \psi(d(p,q)) - \phi(d(p,q))$. This means that $\phi(d(p,q)) = 0$ and so $d(p,q) = 0$. Which is a contradiction to $p \neq q$. Hence the best proximity point is unique. □

**Corollary 3.3.** Let $(A, B)$ be a pair of non-empty closed subsets of a complete metric space $(X, d)$. Consider a mapping $T : A \to B$ satisfying the following conditions:

(i) For all $x, y, s, r \in A$

$$d(s, Tx) = d(A, B), \qquad d(r, Ty) = d(A, B),$$

then,

$$\psi(d(s,r) + \varphi(s) + \varphi(r)) \leq \psi(1/2[d(x,r) + \varphi(x) + \varphi(r) + d(y,s) + \varphi(y) + \varphi(s)]) - \phi(l_r(x,y,s,r,d,\varphi)),$$

where

$$l_r(x, y.s, r, d, \varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(y,r) + \varphi(y) + \varphi(r)\}$$

$\psi \in \Psi$ and $\phi \in \Phi$ and $\varphi$ is a lower semi-continuous function.

(ii) $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$;

(iii) $T$ is a continuous mapping.

Then there exists a unique $x^* \in A$ such that $d(x^*, Tx^*) = d(A, B)$.

**Proof .** First, notice that

$$\begin{aligned}
1/2[d(x,r) + \varphi(x) + \varphi(r) + d(y,s) + \varphi(y) + \varphi(s)] \leq &\max\{d(x,y) + \varphi(x) + \varphi(y), d(x,s) + \varphi(x) + \varphi(s), d(y,r) + \varphi(y) \\
&+ \varphi(r), 1/2[d(x,r) + \varphi(x) + \varphi(r) + d(y,s) + \varphi(y) + \varphi(s)]\} \\
= &m_r(x,y,s,r,d,\varphi).
\end{aligned}$$

Since $\psi$ is non-decreasing for all $x, y, s, r \in A$, we have

$$\psi(d(s,r) + \varphi(s) + \varphi(r)) \leq \psi(1/2[d(x,r) + \varphi(x) + \varphi(r) + d(y,s) + \varphi(y) + \varphi(s)])$$

$$-\phi(l_r(x,y,s,r,d,\varphi)) \leq \psi(m_r(x,y,s,r,d,\varphi)) - \phi(l_r(x,y,s,r,d,\varphi)).$$

The desired result is obtained by applying Theorem 3.2. □

**Remark 3.4.** Let $A = B$. Then Theorem (3.2 ) reduces to corollary (3.5).

**Corollary 3.5.** Let $A$ be a nonempty closed subsets of a complete metric space $(X, d)$. Consider a mapping $T : A \to A$ satisfying the following condition:

i) Let for all $x, y \in A$,

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \le \psi(m_r(x, y, Tx, Ty, d, \varphi)) - \phi(l_r(x, y, Tx, Ty, d, \varphi)),$$

where

$$m_r(x, y, Tx, Ty, d, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(x, Tx) + \varphi(x) + \varphi(Tx), d(y, Ty) + \varphi(y) + \varphi(Ty),$$
$$1/2[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)]\}$$

and

$$l_r(x, y, Tx, Ty, d, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y), d(y, Ty) + \varphi(y) + \varphi(Ty)\}$$

and $\psi \in \Psi, \phi \in \Phi$ and $\varphi$ is a lower semi-continuous function.

Then there exists a unique $x^* \in A$ such that $Tx^* = x^*$ and $\varphi(x^*) = 0$.

**Proof .** Using Theorem (1.7) when $A = B$, the desired result follows. □

**Example 3.6.** Let $X = R^2$ and $d : X \times X \to [0, \infty)$ be define by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|, \quad \text{for all} \quad (x_1, x_2), (y_1, y_2) \in X.$$

Then $(X, d)$ is a complete metric space. Suppose the closed subsets:

$$A = \{(x, 0) : 0 \le x \le 1\},$$
$$B = \{(x, 1) : 0 \le x \le 1\}.$$

Let $T : A \to B$ be the mapping defined by $T(x, 0) = \left(\frac{x^2}{2(1+x)}, 1\right)$, $\psi(t) = \dfrac{3t}{2}$, for all $t \ge 0$ and

$$\varphi(t) = \begin{cases} \text{t/2}, & 0 \le t \le 1; \\ \text{t/2+1/2}, & 1 < t \le 2; \\ \text{t}, & t > 2. \end{cases}$$

Then $\varphi$ is lower semi-continuous and

$$t/2 \le \varphi(t) \le t \quad \text{for all } t \ge 0.$$

Assume that a function $\phi : [0, \infty) \to [0, \infty)$ is define by

$$\phi(t) = \frac{3t}{4 + 2t}.$$

$$d(A, B) = \inf\{d((x, 0), (x, 1)) : (x, 0) \in A, (x, 1) \in B\} = \inf\{|x - x| + |0 - 1|\} = 1.$$

We know that

$$A_0 = A, B_0 = B \quad \text{and} \quad T(A_0) \subseteq B_0.$$

Now, we check that T is a generalized proximal weakly contraction. In fact, for $(x, 0), (y, 0), (s, 0), (r, 0) \in A$, we have

$$d((s, 0), T(x, 0)) = d(A, B) \text{ implies that } d((s, 0), (\frac{x^2}{2(1 + x)}, 1)) = 1.$$

So $s = \frac{x^2}{2(1+x)}$ and $d((r,0), T(y,0)) = d(A,B)$ implies

$$d((r,0), (\frac{y^2}{2(1+y)}, 1)) \text{ which implies that } r = \frac{y^2}{2(1+y)}.$$

We shows that

$$\psi(d((s,0),(r,0)) + \varphi((s,0)) + \varphi((r,0))) \leq \psi(m_r((x,0),(y,0)(s,0),(r,0),d,\varphi)) - \phi(l_r(x,0),(y,0),(s,0),(r,0),d,\varphi)).$$

Suppose that $x \geq y$ (the same argument works for $y \geq x$). Then we have,

$$\begin{aligned} m_r((x,0),(y,0),(s,0),(r,0),d,\varphi) = \max\{ & d((x,0),(y,0)) + \varphi(x,0) + \varphi(y,0), d((x,0),(s,0)) + \varphi(x,0) + \varphi((s,0)),\\ & d((y,0),(r,0)) + \varphi(y,0) + \varphi((r,0)), 1/2\{d((x,0),(r,0)) + \varphi(x,0) + \varphi((r,0))\\ & + d((y,0),(s,0)) + \varphi(y,0) + \varphi((s,0))\}\}, \end{aligned}$$

$$\begin{aligned} & 1/2\{d((x,0),(r,0)) + \varphi(x,0) + \varphi((r,0)) + d((y,0),(s,0)) + \varphi(y,0) + \varphi((s,0))\}\}\\ \geq & 1/2\{d((x,0),(r,0)) + \frac{(x,0)}{2} + \frac{(r,0)}{2} + d((y,0),(s,0)) + \frac{(s,0)}{2} + \frac{(s,0)}{2}\}\\ \geq & 1/2\{1/2\{d((x,0),(r,0)) + (x,0) + (r,0) + d((y,0),(s,0)) + (y,0) + (s,0)\}\}\\ = & 1/2\left\{d((x,0),(\frac{y^2}{2(1+y)},0)) + (x,0) + (\frac{y^2}{2(1+y)},0) + d((y,0),(\frac{x^2}{2(1+x)},0)) + (y,0) + (\frac{x^2}{2(1+x)},0)\right\}\\ = & 1/4\left\{(|x - \frac{y^2}{2(1+y)}| + |0-0|) + (x,0) + (\frac{y^2}{2(1+y)},0) + (|y - \frac{x^2}{2(1+x)}| + |0-0|) + (y,0) + (\frac{x^2}{2(1+x)},0)\right\}\\ = & \begin{cases} 1/2((x,0) + (\frac{x^2}{2(1+x)},0)), & y \leq (\frac{x^2}{2(1+x)},0);\\ 1/2((x,0) + (y,0)), & \text{otherwise.} \end{cases}\\ > & (\frac{x}{2},0). \end{aligned}$$

Thus we have,

$$\begin{aligned} m_r((x,0),(y,0),(s,0),(r,0),d,\varphi) = & \max\{d((x,0),(y,0)) + \varphi(x,0) + \varphi(y,0), d((x,0),(s,0)) + \varphi(x,0) + \varphi(s,0),\\ & d((y,0),(r,0)) + \varphi(y,0) + \varphi(r,0), 1/2\{d((x,0),(r,0)) + \varphi(x,0) + \varphi(r,0) +\\ & d((y,0),(s,0)) + \varphi(y,0) + \varphi(s,0)\}\}\\ \geq & \max\{d((x,0),(y,0)) + (x,0)/2 + (y,0)/2, d((x,0),(s,0)) + (x,0) + (s,0)/2,\\ & d((y,0),(r,0)) + (y,0) + (y,0)/2, 1/2\{d((x,0),(r,0)) + (x,0)/2 + (y,0)/2\\ & + d((y,0),(s,0)) + (y,0)/2 + (s,0)\}\}\\ \geq & 1/2\{\max\{d((x,0),(y,0)) + (x,0) + (y,0), d((x,0),(s,0)) + (x,0) + (s,0),\\ & d((y,0),(r,0)) + (y,0) + (r,0)), 1/2\{d((x,0),(r,0)) + (x,0) + (r,0)\\ & + d((y,0),(s,0)) + (y,0) + (s,0))\}\}\}\\ = & 1/2\left\{\max\left\{(|x-y| + |0-0) + (x,0) + (y,0), (|x - \frac{x^2}{2(1+x))}| + |0-0|) + (x,0)\right.\right.\\ & \left.\left. + \frac{x^2}{2(1+x)},0), (|y - \frac{y^2}{2(1+y)})| + |0-0|) + (y,0) + (\frac{y^2}{2(1+y)},0), (\frac{x}{2},0)\right\}\right\}\\ = & 1/2\left\{\max\left\{2(x,0), 2(x,0), 2(y,0), (\frac{x}{2},0)\right\}\right\} = (x,0). \end{aligned}$$

Then

$$\psi(m_r((x,0),(y,0),(s,0),(r,0),d,\varphi)) = \frac{3 * (x,0)}{2} = \frac{3(x,0)}{2},$$

$$\begin{aligned}
l_r\{(x,0),(y,0),(s,0),(r,0),d,\varphi\} =& \max\{d((x,0),(y,0))+\varphi(x,0)+\varphi(y,0), d((y,0),(r,0))+\varphi(y,0)+\varphi(r,0)\}\\
\leq& \max\{d((x,0),(y,0))+(x,0)+(y,0), d((y,0),(r,0))+(y,0)+(r,0))\}\\
=& \max\left\{(|x-y|+|0-0|)+(x,0)+(y,0),(|y-\frac{y^2}{2(1+x)})|+|0-0|),(y,0)\right.\\
&\left.+(\frac{y^2}{2(1+x)},0)\right\}\\
=& \max\{2(x,0),2(y,0)\}=2(x,0).
\end{aligned}$$

Then

$$\begin{aligned}
\phi\{l_r(x,0),(y,0),(s,0),(r,0),d,\varphi\} =& \frac{3*2(x,0)}{4+2*2(x,0)} = \frac{6(x,0)}{4+4(x,0)}\\
=& \frac{3(x,0)}{2(1+(x,0))},
\end{aligned}$$

$$\begin{aligned}
\psi(d((s,0),(r,0))+\varphi(s,0)+\varphi(r0)) \leq& \psi(d((s,0),(r,0))+(s,0)+(r,0))\\
=& \psi(d((\frac{x^2}{2(1+x)},0),(\frac{y^2}{2(1+y)},0))+(\frac{x^2}{2(1+x)},0)+(\frac{y^2}{2(1+y,0)},0))\\
=& \psi((|(\frac{x^2}{2(1+x)})-(\frac{y^2}{2(1+y)})|+|0-0|)+(\frac{x^2}{2(1+x)},0)+(\frac{y^2}{2(1+y,0)},0))\\
=& \psi(2(\frac{x^2}{2(1+x)},0)) = \psi(\frac{x^2}{(1+x)},0) = (\frac{3x^2}{2(1+x)},0).
\end{aligned}$$

Thus

$$\psi(d((s,0),(r,0))+\varphi(s,0)+\varphi(y,0)) \leq (\frac{3x^2}{2(1+x)},0).$$

Hence

$$\begin{aligned}
\psi(m_r((x,0),(y,0),(s,0),(r,0),d,\varphi))-\phi(l_r(x,0),(y,0),(s,0),(r,0),d,\varphi)) =& \frac{3(x,0)}{2} - \frac{3(x,0)}{2(1+(x,0))}\\
=& \frac{6(x,0)+(6(x,0)^2)-6(x,0)}{4(1+(x,0))} = \frac{3(x,0)^2}{2(1+x)}\\
\geq& \psi(d((s,0),(r,0))+\varphi(s,0))+\varphi(r,0)),
\end{aligned}$$

which is implies that

$$\psi(d((s,0),(r,0))+\varphi(s,0)+\varphi(r,0)) \leq \psi(m_r((x,0),(y,0),(s,0),(r,0),d,\varphi))-\phi(l_r(x,0),(y,0),(s,0),(r,0),d,\varphi)).$$

Therefore T is a generalized proximal weakly contractive mapping. Hence, all the hypotheses of Theorem 4.0.1 are satisfied. Thus T has a unique best proximity point, then there exist $(x^*,0)\in A$ such that

$$d((x^*,0),T(x^*,0)) = d(A,B) = 1.$$

This implies that

$$d((x^*,0),T(x^*,0)) = d((x^*,0),(\frac{(x^*)^2}{2(1+x^*)},1)) = 1,$$

which implies that

$$|x^* - \frac{(x^*)^2}{2(1+x^*)}| + |0-1| = 1.$$

From this we get

$$x^* - \frac{(x^*)^2}{2(1+x^*)} = 0$$

and

$$x^* = 0 \text{ and } x^* = -2.$$

But $-2 \notin [0,1]$. The point

$$(x^*, 0) \in A \text{ is } (0,0) \in A.$$

Therefore $(x^*, 0) = (0,0)$ is the best proximity point of $T$.

## References


[1] Ya.I. Alber and S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert spaces*, New Results in Operator Theory and Its Applications (Basel) (I. Gohberg and Yu. Lyubich, eds.), Birkhäuser Basel 1997, pp. 7–22.

[2] J. Bae and S.H. Cho, *Fixed points of weak $\alpha$-contraction type maps*, Fixed Point Theory Appl. **2014** (2014), no. 1, 175.

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), no. 1, 133–181.

[4] S. Binayak, Choudhury, P. Konar, B.E. Rhoades, and N. Metiya, *Fixed point theorems for generalized weakly contractive mappings*, Nonlinear Anal. **74** (2011), no. 6, 2116–2126.

[5] S. Cho, *Fixed point theorems for generalized weakly contractive mappings in metric spaces with applications*, J. Fixed Point Theory Appl. **2018** (2018), no. 3, 1687–1812.

[6] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. Lond. Math. Soc. **1** (1962), no. 1, 74–79.

[7] M.S Khan, M Swaleh, and S Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1984), no. 1, 1–9.

[8] Z. Mitrović, A. Dinmohammadi, V. Parvaneh, and E. Jakupović, *Best approximation for multivalued mappings in para normed spaces*, J. Interdiscip. Math. **24** (2021), no. 6, 1617–1625.

[9] S. Mondal and L.K. Dey, *Some common best proximity point theorems in a complete metric space*, Afr. Mat. **28** (2017), 85–97.

[10] V. Parvaneh M.R. Haddadi, and M. Moursalian, *Global optimal approximate solutions of best proximity points*, Filomat **35** (2021), no. 5, 1555–1564.

[11] V. Parvaneh, B. Mohammadi, and H. Aydi, *On extended interpolative Ćirić–Reich–Rus type F-contractions and an application*, J. Inequal. Appl. **2019** (2019), 1–11.

[12] A. Razanii, M. Abbas, and V. Parvaneh, *Periodic points of t-Ciric generalized contraction mappings in ordered metric spaces*, Georgian Math. J. **19** (2012), no. 4, 597–610.

[13] Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47** (2001) no. 4 2683–2693.

[14] J.R. Roshan, Z. Kadelburg, Z. Mustafa, V. Parvaneh, *$b_2$-metric spaces and some fixed point theorems*, J. Fixed Point Theory Appl. **2014** (2014), no. 1, 1–23.

[15] S. Sadiq Basha, *Best proximity point theorems generalizing the contraction principle*, Nonlinear Anal. **74** (2011) no. 17, 5844–5850.