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A new subclass of analytic functions and some results related to Booth lemniscate

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Abstract

In this paper, we introduce the subclass $\mathcal{KS}(\alpha)$ of univalent functions in $\mathcal{A}$ and study some properties of this class. We apply matters of differential subordinations, to investigate some results concerning the subclasses $\mathcal{KS}(\alpha)$ and $\mathcal{BS}(\alpha)$ of $\mathcal{A}$, where $\alpha \in [0,1)$.

Keywords: Starlike functions, Convex functions, Differential subordination, Booth lemniscate
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1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions f on the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathbb{U}). \tag{1.1}$$

and let S denote the subclass of $\mathcal{A}$ consisting of all univalent functions. For a real number γ with $0 \leqslant \gamma < 1$, a function $f \in \mathcal{A}$ is called starlike of order γ if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \gamma, \qquad (z \in \mathbb{U}),$$

and f is called convex of order γ if

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > \gamma, \qquad (z \in \mathbb{U}).$$

we denote by $\mathcal{S}^*(\gamma)$ and $K(\gamma)$ the classes of starlike and convex functions of order γ, respectively. In particular we set $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $K(0) \equiv K$. It is clear that a function $f \in \mathcal{A}$ belongs to the class K if and only if $zf'(z)$ belongs to the class $\mathcal{S}^*$. Suppose f be an analytic function in $\mathbb{U}$ with $f'(0) \neq 0$, then the function f is called close-to-convex if there exists a convex function g such that:

$$\operatorname{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > 0, \qquad (z \in \mathbb{U}).$$

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We denote by $\mathcal{C}$ the class of all close-to-convex functions in $\mathbb{U}$. Refer to [2, 8, 9, 10, 11] for various published papers dealing with mentioned classes.

Suppose that $\mathcal{H} = \mathcal{H}(\mathbb{U})$ be the class of all analytic functions in $\mathbb{U}$ and n be a positive integer number and $a \in \mathbb{C}$. We set:

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}, f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}.$$

Suppose that $\alpha \in [0, 1)$, in this paper (as seen in Piejko and Sokol [7]), we apply a family of univalent functions in $\mathbb{U}$ as follows:

$$F_\alpha(z) = \frac{z}{1 - \alpha z^2} = z + \sum_{n=1}^{\infty} \alpha^n z^{2n+1}, \qquad (z \in \mathbb{U}). \tag{1.2}$$

Note that for $\alpha \in [0, 1)$:

$$\operatorname{Re}\left\{\frac{zF_\alpha'(z)}{F_\alpha}\right\} = \operatorname{Re}\left\{\frac{1 + \alpha z^2}{1 - \alpha z^2}\right\} > 0, \qquad (z \in \mathbb{U}),$$

then $F_\alpha(z)$ is starlike in $\mathbb{U}$. Also $F_\alpha(\mathbb{U}) = D(\alpha)$, where

$$D(\alpha) = \left\{x + iy \in \mathbb{C} \mid (x^2 + y^2)^2 - \frac{x^2}{(1-\alpha)^2} - \frac{y^2}{(1+\alpha^2)} < 0\right\},$$

when $\alpha \in [0, 1)$ and

$$D(1) = \left\{x + iy \in \mathbb{C} \mid x + iy \neq it, \qquad \forall t \in (-\infty, 1/2] \cup [1/2, \infty)\right\}.$$

The curve

$$(x^2 + y^2)^2 - \frac{x^2}{(1-\alpha)^2} - \frac{y^2}{(1+\alpha)^2} = 0, \qquad \big((x, y) \neq 0\big),$$

is called the Booth lemniscate of elliptic type. See [4] for more explanations. Let f and g belong to $\mathcal{H}$. The function f is subordinate to g, denoted by $f \prec g$, if there exists an analytic function w in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| \leqslant |z| < 1$ such that $f(z) = g(w(z))$. Moreover if g is a univalent function in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Now we recall from (Kargar et al. 2017 [4]) the following definition.

Definition 1.1. Let $f \in \mathcal{A}$ and $\alpha \in [0, 1)$. We say $f \in \mathcal{BS}(\alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} - 1 \prec F_\alpha(z),$$

where $F_\alpha(z)$ is given by (1.2).

Furthermore, we mention from [4] a main lemma as follows.

Lemma 1.2. If F_α is given by (1.2), then we have:

$$\frac{1}{\alpha - 1} < \operatorname{Re}\{F_\alpha(z)\} < \frac{1}{1 - \alpha}, \qquad (z \in \mathbb{U}),$$

where $\alpha \in [0, 1)$.

From Lemma 1.2, if $f \in \mathcal{BS}(\alpha)$ then:

$$\frac{\alpha}{\alpha - 1} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{2 - \alpha}{1 - \alpha}, \qquad (z \in \mathbb{U}). \tag{1.3}$$

Therefore in particular, $\mathcal{BS}(0) \subset \mathcal{S}^*$. Now, we are interested to produce a new subclass of $\mathcal{A}$ as follows.

Definition 1.3. Let $f \in \mathcal{A}$, $\alpha \in [0,1)$ and $F_\alpha(z)$ is given by (1.2). Then $f \in \mathcal{KS}(\alpha)$ if and only if

$$\frac{zf''(z)}{f'(z)} \prec F_\alpha(z). \tag{1.4}$$

Remark 1.4. From the Definition 1.1 and the Definition 1.3, $f \in \mathcal{KS}(\alpha)$ if and only if $zf' \in \mathcal{BS}(\alpha)$.

Remark 1.5. By Lemma 1.2, if $f \in \mathcal{KS}(\alpha)$ then

$$\frac{1}{\alpha-1} < \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} < \frac{1}{1-\alpha}, \qquad (z \in \mathbb{U}). \tag{1.5}$$

Therefore in particular, $\mathcal{KS}(0) \subset K$.

Corollary 1.6. Let $f \in \mathcal{A}$ and $\alpha \in [0,1)$, then $f \in \mathcal{KS}(\alpha)$ if and only if there exists an analytic function w in $\mathbb{U}$, with $w(0)=0$ and $|w(z)|<1$, such that

$$f'(z) = \exp \int_0^z \frac{F_\alpha\big(w(t)\big)}{t} dt, \qquad (z \in \mathbb{U}). \tag{1.6}$$

Proof . Let $f \in \mathcal{KS}(\alpha)$. So there exists an analytic function $w(z)$ in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ such that:

$$\frac{zf''(z)}{f'(z)} = F_\alpha\big(w(z)\big), \qquad (z \in \mathbb{U}).$$

Equivalently

$$\frac{d}{dz}\log f'(z) = \frac{F_\alpha\big(w(z)\big)}{z}, \qquad (z \in \mathbb{U}),$$

then we have

$$f'(z) = \exp \int_0^z \frac{F_\alpha\big(w(t)\big)}{t} dt, \qquad (z \in \mathbb{U}).$$

Now, if a function f satisfies the condition (1.6), it is easy considering that $f \in \mathcal{KS}(\alpha)$. $\square$

As example with setting $w(z)=z$ in (1.6), we conclude that

$$f(z) = \int_0^z \left(\frac{1+\sqrt{\alpha}t}{1-\sqrt{\alpha}t}\right)^{\frac{1}{2\sqrt{\alpha}}} dt,$$

belongs to the class $\mathcal{KS}(\alpha)$. For proving main results, we require to express some lemmas.

Lemma 1.7 (See [6]). Let h be convex in $\mathbb{U}$ with $h(0)=a$, $\gamma \neq 0$ and $\operatorname{Re}\gamma \geqslant 0$. If $p \in \mathcal{H}[a,n]$ and

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{(\frac{\gamma}{n})-1} dt.$$

The function q is convex and is the best (a,n)–dominant.

Lemma 1.8 (Miller and Mocanu [6]). Let h be starlike in $\mathbb{U}$, with $h(0) = 0$ and $a \neq 0$. If $p \in \mathcal{H}[a, n]$ satisfies

$$\frac{zp'(z)}{p(z)} \prec h(z),$$

then

$$p(z) \prec q(z) = a \exp\Big[n^{-1} \int_0^z h(t)t^{-1}dt\Big],$$

and q is the best (a, n)–dominant.

Lemma 1.9 (See [6]). Let h be convex, with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$. If $p \in \mathcal{H}[1, n]$ satisfies

$$p^2(z) + 2p(z) \cdot zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z) = \sqrt{Q(z)}$$

where

$$Q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{(\frac{1}{n})-1}dt,$$

and the function q is the best $(1, n)$–dominant.

Lemma 1.10 (Miller and Mocanu [6]). Let Q denote the set of functions q that are analytic and injective on $\overline{\mathbb{U}} \backslash E(q)$ where

$$E(q) := \big\{\xi \in \partial\mathbb{U} : \lim_{z \to \xi} q(z) = \infty\big\},$$

and $q'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \backslash E(q)$. Let $q \in Q$ with $q(0) = a$, and let

$$p(z) = a + a_n z^n + \dots$$

be analytic in $\mathbb{U}$ with $p(z) \not\equiv a$ and $n \geqslant 1$. If $p \not\prec q$, then there exist $m \geqslant n \geqslant 1$ and points $z_0 \in \mathbb{U}$, $\xi_0 \in \partial\mathbb{U} \backslash E(q)$ so that $p\big(|z| < |z_0|\big) \subset q(\mathbb{U})$, $p(z_0) = q(\xi_0)$ and $z_0 p'(z_0) = m\xi_0 q'(\xi_0)$, and

$$\operatorname{Re}\Big\{\frac{z_0 p''(z_0)}{p'(z_0)} + 1\Big\} \geqslant m \operatorname{Re}\Big\{\frac{z_0 q''(z_0)}{q'(z_0)} + 1\Big\}.$$

Also, we require generalization of the Nunokawas lemma as following.

Lemma 1.11 (See [7]). Let p be an analytic function in $\mathbb{U}$ with $p(z) \neq 0$ and

$$p(z) = 1 + \sum_{n=m\geqslant 1}^{\infty} c_n z^n, \qquad (c_m \neq 0).$$

If there exists $z_0 \in \mathbb{U}$ such that

$$|\arg\{p(z)\}| < \frac{\pi\beta}{2} \qquad \text{for} \qquad |z| < |z_0|,$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\beta}{2}$$

for some $\beta > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell\beta,$$

where

$$\ell \geqslant \frac{m}{2}\big(a+\frac{1}{a}\big) \geqslant m \quad \text{when} \quad \arg\{p(z_0)\} = \frac{\pi\beta}{2}$$

and

$$\ell \leqslant -\frac{m}{2}\big(a+\frac{1}{a}\big) \leqslant -m \quad \text{when} \quad \arg\{p(z_0)\} = -\frac{\pi\beta}{2},$$

where

$$\big\{p(z_0)\big\}^{\frac{1}{\beta}} = \pm ia \quad \text{with} \quad a>0.$$

Lemma 1.12 (See [6]). Let $\Omega \subset \mathbb{C}$ and $p \in \mathcal{H}[a,n]$ with $\operatorname{Re} a > 0$. If a function $\Psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ satisfies the condition

$$\Psi\big(\rho i, \sigma, \mu, \nu; z\big) \notin \Omega, \qquad (z \in \mathbb{U}),$$

for all ρ, σ, μ, $\nu \in \mathbb{R}$, $\sigma \leqslant -\frac{n}{2}\frac{|a-i\rho|^2}{\operatorname{Re} a}$, $\sigma+\mu \leqslant 0$, then

$$\Psi\big(p(z), zp'(z), z^2p''(z); z\big) \in \Omega \Longrightarrow \operatorname{Re} p(z) > 0.$$

2 Main results

In the beginning, we prove one of main results in this section.

Lemma 2.1. Let $f \in \mathcal{KS}(\alpha)$. If $0<\alpha<1$, then

$$f'(z) \prec q(z) = \Big(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\Big)^{\frac{1}{2\sqrt{\alpha}}}, \tag{2.1}$$

and q is the best $(1,1)$–dominant and if $\alpha = 0$, then

$$f'(z) \prec \exp(z), \tag{2.2}$$

and $\exp(z)$ is the best $(1,1)$–dominant. Furthermore for $\alpha \in (0,1)$, we have

$$|\arg\{f'(z)\}| < \frac{1}{2\sqrt{\alpha}} \arctan\Big\{\frac{2\sqrt{\alpha}}{1-\alpha}\Big\}. \tag{2.3}$$

Proof . Let $p(z) = f'(z)$. Therefore $p \in \mathcal{H}[1,1]$ and

$$\frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)} \prec F_\alpha(z).$$

Because of the starlikeness of F_α, by supposing $n=a=1$ in Lemma 1.8, we conclude that

$$f'(z) \prec q(z) = \exp\Big[\int_0^z \frac{F_\alpha(t)}{t}dt\Big] = \Big(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\Big)^{\frac{1}{2\sqrt{\alpha}}},$$

and $q(z)$ is the best $(1,1)$–dominant. Also in the case $\alpha = 0$, for considering the relation (2.2), we perform the procedure of the case $\alpha \in (0,1)$ with $F_\alpha(z) = z$. If $\alpha \in (0,1)$ and $w(z) = \frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}$ $(z \in \mathbb{U})$, we can easily conclude that w maps the open disk $\mathbb{U}$ onto the disk with the center $C = \frac{1+\alpha}{1-\alpha}$ and the radius $R = \frac{2\sqrt{\alpha}}{1-\alpha}$. Equivalently we have

$$\Big|w(z) - \frac{1+\alpha}{1-\alpha}\Big| < \frac{2\sqrt{\alpha}}{1-\alpha}, \qquad (z \in \mathbb{U}),$$

thus with a simple calculation, we follow

$$0 < \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} < \operatorname{Re} w < \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}$$

and therefore

$$\left|\arg\left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{\frac{1}{2\sqrt{\alpha}}}\right| < \frac{1}{2\sqrt{\alpha}}\arctan\frac{2\sqrt{\alpha}}{1-\alpha}, \tag{2.4}$$

and the result is obtained. □

Theorem 2.2. For $0 \leqslant \alpha < 1$, $\mathcal{KS}(\alpha) \subset S$.

Proof . Suppose $f \in \mathcal{KS}(\alpha)$. Through the relation (2.3), it is clear that for $\alpha \in [\frac{1}{4}, 1)$:

$$|\arg\{f'(z)\}| < \frac{\pi}{2}. \tag{2.5}$$

Thus $\operatorname{Re} f'(z) > 0$ and by Noshiro–Warshawski theorem [1] f is univalent in $\mathbb{U}$. Moreover from (1.5) for $\alpha \in [0, \frac{1}{3}]$ we have

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \frac{\alpha}{\alpha - 1} \geqslant -\frac{1}{2}, \qquad (z \in \mathbb{U}), \tag{2.6}$$

and by Kaplan [3], we conclude that f is close-to-convex. Then f is univalent in $\mathbb{U}$. □

Theorem 2.3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{KS}(\alpha)$,

a) If $\alpha \in (0, \frac{1}{4}]$, then

$$\frac{f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z \left(\frac{1+\sqrt{\alpha}t}{1-\sqrt{\alpha}t}\right)^{\frac{1}{2\sqrt{\alpha}}} dt, \tag{2.7}$$

and $q(z)$ is convex and the best $(1,1)$–dominant. Also if $\alpha = 0$, then

$$\frac{f(z)}{z} \prec \frac{1}{z}(e^z - 1), \tag{2.8}$$

and $\frac{1}{z}(e^z - 1)$ is convex and the best $(1,1)$–dominant.

b) If $\alpha \in [\frac{1}{4}, 1)$, then

$$\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\frac{1}{2\sqrt{\alpha}}}. \tag{2.9}$$

Therefore for $\alpha \in [\frac{1}{4}, 1)$, we have $\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > 0$.

c) If $\alpha \in [\frac{1}{4}, 1)$, then

$$\sqrt{\frac{f(z)}{z}} \prec \sqrt{\frac{2}{z}\ln(1+z) - 1}. \tag{2.10}$$

Therefore $\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z}}\right\} > \sqrt{2\ln 2 - 1}$.

Proof .

a) Let $p(z) = \frac{f(z)}{z}$, then $p \in \mathcal{H}[1,1]$ and $p(0) = 1$. Since $f'(z) = p(z) + zp'(z)$, by using the relation (2.1) we obtain

$$p(z) + zp'(z) \prec \left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{\frac{1}{2\sqrt{\alpha}}}.$$

Suppose $h(z) = \left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{\frac{1}{2\sqrt{\alpha}}}$. By [5] it is showed that for $\alpha \in (0, \frac{1}{4}]$, the function $h(z)$ is convex and $h(0) = 1$, then by taking $\gamma = 1$ and $n = 1$ in Lemma 1.7, the relation (2.6) is obtained. Moreover $q(z) = \frac{1}{z}\int_0^z \left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{\frac{1}{2\sqrt{\alpha}}} dt$ is convex and the best $(1,1)$–dominant. For the case $\alpha = 0$ similar to the past case, by applying the relation (2.2) and Lemma 1.7, the relation (2.7) is obtained.

b) Let $p(z) = \frac{f(z)}{z}$ be the form:

$$p(z) = 1 + a_2 z + a_3 z^2 + \ldots = 1 + \sum_{n=1}^{\infty} a_{n+1} z^n, \qquad (z \in \mathbb{U}).$$

We want to show $p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta} = q(z)$, where $\beta = \frac{1}{2\sqrt{\alpha}}$. Suppose $p \not\prec q$. From Lemma 1.10, there exist points $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \backslash E(q)$ such that

$$|\arg\{p(z_0)\}| = \frac{\pi\beta}{2}$$

and

$$|\arg\{p(z)\}| < \frac{\pi\beta}{2}, \qquad (|z| < |z_0|).$$

Then by Lemma 1.11, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell\beta, \tag{2.11}$$

where

$$\ell \geqslant \frac{1}{2}\big(a + \frac{1}{a}\big) \geqslant 1 \quad \text{when} \quad \arg\{p(z_0)\} = \frac{\pi\beta}{2}, \tag{2.12}$$

and

$$\ell \leqslant -\frac{1}{2}\big(a + \frac{1}{a}\big) \leqslant -1 \quad \text{when} \quad \arg\{p(z_0)\} = -\frac{\pi\beta}{2}, \tag{2.13}$$

where

$$\big\{p(z_0)\big\}^{\frac{1}{\beta}} = \pm ia, \qquad (a > 0).$$

Note that from Lemma 2.1,

$$|\arg\{f'(z)\}| < \frac{\pi}{4\sqrt{\alpha}}, \qquad (z \in \mathbb{U}). \tag{2.14}$$

Now, suppose $\arg\{p(z_0)\} = \frac{\pi\beta}{2}$. Since

$$f'(z_0) = p(z_0) + z_0 p'(z_0) = p(z_0)\Big[1 + \frac{z_0 p'(z_0)}{p(z_0)}\Big],$$

from (2.10) and (2.11) we deduce that

$$\begin{aligned} \arg\{f'(z_0)\} &= \arg\{p(z_0)\} + \arg\{1 + i\ell\beta\} = \frac{\pi\beta}{2} + \arctan\{\ell\beta\} \\ &\geqslant \frac{\pi}{4\sqrt{\alpha}} + \arctan\big\{\frac{1}{2\sqrt{\alpha}}\big\} > \frac{\pi}{4\sqrt{\alpha}}, \end{aligned}$$

which is contradictory with the relation (2.14). If $\arg\{p(z_0)\} = -\frac{\pi\beta}{2}$, then from (2.10) and (2.13) we deduce that

$$\begin{aligned} \arg\{f'(z_0)\} &= \arg\{p(z_0)\} + \arg\{1 + i\ell\beta\} = -\frac{\pi\beta}{2} + \arctan\{\ell\beta\} \\ &\leqslant -\frac{\pi}{4\sqrt{\alpha}} + \arctan\big\{-\frac{1}{2\sqrt{\alpha}}\big\} < -\frac{\pi}{4\sqrt{\alpha}}, \end{aligned}$$

which leads to contradiction with the relation (2.14). Therefore $\frac{f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\frac{1}{2\sqrt{\alpha}}}$.

c) Let $p(z)=\sqrt{\frac{f(z)}{z}}$. With considering the branch of square root we have $p\in\mathcal{H}[1,1]$ and

$$p^2(z)+2zp'(z)p(z)=f'(z),\qquad (z\in\mathbb{U}). \tag{2.15}$$

Since $f\in\mathcal{KS}(\alpha)$, by the relation (2.3) for $\alpha\in[\frac{1}{4},1)$ we have $\operatorname{Re}\{f'(z)\}>0$. Put $h(z)=\frac{1-z}{1+z}$. Thus from the relation (2.15) we can conclude that

$$p^2(z)+2zp'(z)p(z)\prec\frac{1-z}{1+z},$$

then by Lemma 1.9,

$$\sqrt{\frac{f(z)}{z}}\prec q(z)=\sqrt{Q(z)},$$

where

$$\begin{aligned}Q(z)&=\frac{1}{z}\int_0^z h(t)dt=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt\\&=\frac{2}{z}\ln(1+z)-1,\end{aligned}$$

and the function q is the best $(1,1)$–dominant. Moreover, we have

$$Q(z)+zQ'(z)=h(z)\prec h(z)$$

and $Q\in\mathcal{H}[1,1]$. Therefore from Lemma 1.7, we follow that $Q\prec h$ and then $\operatorname{Re}\{Q(z)\}>0$. Since h is convex, it is clear that Q is convex (see [6], Theorem 2.6h, part(ii)). On the Other hand, since the function Q has real coefficients, we deduce that

$$\min_{|z|\leqslant 1}\operatorname{Re}\{q(z)\}=\sqrt{Q(1)}=\sqrt{2\ln 2-1}, \tag{2.16}$$

then $\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z}}\right\}>\sqrt{2\ln 2-1}$.

□

Theorem 2.4. Let $f\in\mathcal{BS}(\alpha)$. If $\alpha\in(0,1)$, then

$$\frac{f(z)}{z}\prec\left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{\frac{1}{2\sqrt{\alpha}}}=q(z), \tag{2.17}$$

and q is the best $(1,1)$–dominant. Also if $\alpha=0$, then

$$\frac{f(z)}{z}\prec\exp z, \tag{2.18}$$

and $\exp(z)$ is the best $(1,1)$–dominant.

Proof . Let $p(z)=\frac{f(z)}{z}$ and $\alpha\in(0,1)$. Since $f\in\mathcal{BS}(\alpha)$ we obtain

$$\frac{zp'(z)}{p(z)}=\frac{zf'(z)}{f(z)}-1\prec F_\alpha(z).$$

We know that $F_\alpha(z)$ is starlike and $F_\alpha(0)=0$. Thus by Lemma 1.8,

$$\frac{f(z)}{z}\prec q(z)=\exp\left[\int_0^z\frac{F_\alpha(t)}{t}dt\right]=\left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{\frac{1}{2\sqrt{\alpha}}},$$

and the function q is the best $(1,1)$–dominant. For proving the relation (2.18), we perform the former process with $F_\alpha(z)=z$. □

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