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Portfolio design and optimization within the framework of the Markov chain

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Abstract

Return and risk are significant parameters in selecting an optimal portfolio, depending on the portfolio return distribution. In a stochastic process, the Markov property causes the future distribution of a random process to be measurable according to the state-transition matrix and the initial process state. According to the main idea of the present study in the optimal portfolio selection, portfolio weights are chosen in a way that the Markov property is established for the portfolio return series and the distribution of future portfolio returns is close to the distribution of investor's expected returns; hence, K-L divergence (Kullback–Leibler divergence) is utilized as a criterion of closeness. Using this idea, an optimal portfolio selection model was designed and implemented in the present study. This optimal portfolio was optimized using a Markov approach and according to historical data of 10 indices on the Tehran Stock Exchange from 2009 to 2022 in a six-member state. The optimal portfolio performance evaluation using the Sharpe ratio and value at risk criteria indicated that the research model had a higher performance than the mean-variance and weight parity models.

Keywords: Markov property, K-L divergence (Kullback–Leibler divergence) criterion, return distribution, Goodness of Fit (GoF) test
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1 Introduction

The momentary asset or portfolio price or return is a stochastic variable. A stochastic process refers to all these stochastic variables in a certain period. Therefore, the real observed value of a portfolio in that period is an objectification or path of an example of a stochastic process. The ARIMA, Wiener, and Levy processes are resulted from such view. Markov processes are important and widely-used classes of stochastic processes. In the Markov process, the future process position depends only on its current position, and thus having knowledge about previous positions of a process does not provide additional information [6]. A discrete-time Markov chain is a process in which the state space as a set of values of random variables in the process is a countable or finite set. In financial stochastic processes, an analytical framework with the help of Markov chains makes it possible to answer questions that are not possible in other analytical frameworks such as basic, technical, and time series frameworks. Answering

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questions about the probability of displacement in the state space, the necessary time for these displacements, and the measurement of stable states of a system are some of these questions.

The literature review indicates the vast use of Markov chains as a tool to predict the asset or portfolio return. Markov chains mainly contributed to Markov switching and Hidden Markov models, and are widely used in predicting a stochastic vector of return on assets. The present research uses Markov chains for the selection of an optimal portfolio. Portfolio optimization is a dynamic field in financial research, both theoretically and practically. The optimal and efficient portfolio selection is one of the main goals of investors in financial markets. Various portfolio models have been developed in the research literature each of which has its specific assumptions and limitations. The present research investigated the new view on the use of Markov chains to optimize the portfolio.

2 Theoretical bases and research background

Portfolio optimization is the process of selecting the weights of assets that constitute the portfolio based on investment environment limitations (e.g. specific budget and the impossibility of short selling) and the investor's demands [15]. Investors' demands are often modeled in two forms: return and risk. Therefore, it is expected that a portfolio can estimate the investor's expected return at a minimum level of risk, and such a portfolio is considered efficient [10]. Therefore, the research literature indicated the use of various measures to estimate the return and risk. For example, the weighted mean of returns on a single asset, or statistical and artificial intelligence models are used to estimate the expected return. Furthermore, various measures are used to measure risk, including standard deviation, half standard deviation, Value at Risk (VaR), and Expected Shortfall (ES) [13].

The present research utilizes a Markov chain for portfolio optimization; hence, it first defines the Markov chain properties to determine its capabilities for modeling the portfolio and estimating the return and risk parameters as the main research idea. A sequence of stochastic variables, like $\{X_t\}$, is called a stochastic process. A discrete-time chain is a stochastic process in which the state space, i.e. the set of values of stochastic variables in the process, is a countable or finite set and the time index is a discrete set $t \in \{0, 1, 2, \ldots\}$. It is always assumed that time is discrete. The chain $\{X_t\}$ is a Markov chain if Equation (2.1) is true:

$$P\{X_{n+1} = j \big| X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i\} = P\{X_{n+1} = j \big| X_n = i. \tag{2.1}$$

In other words, the upcoming values of the process depend only on its current value. Suppose that P_{ij} is the probability of state transition from i to j, the P_{ij} values are usually displayed as the following matrix.

$$P = \begin{bmatrix} P_{00} & P_{01} & \ldots & P_{0n} \\ P_{10} & P_{11} & \ldots & P_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n0} & P_{n1} & \ldots & P_{nn} \end{bmatrix} \tag{2.2}$$

$P = [p_{ij}]$ matrix is called the state-transition or the transition-probability matrix. A Markov chain is obvious by specifying the transition probability matrix and the X_0 probability distribution (the chain starting state). The following equation can be obtained using the law of total probability:

$$p_{ij}^n = \sum_{k=0}^{\infty} p_{ik} p_{kj}^{(n-1)} \tag{2.3}$$

where p_{ij}^n is the probability of transition from state i to j with n steps or periods. The aforementioned formula is called the Chapman-Kolmogorov Equation. Important questions can be answered if a process is Markov. Such questions are about the possibility of state transition of the system from one to another after a certain period, the necessary mean time for such transitions, and also finding a stable system state that describes the system's behavior in the long term.

After a brief introduction to the Markov chain, we return to the main question about the Markov process capabilities in the optimal portfolio selection. As mentioned earlier, return and risk are two main parameters in portfolio determination and both are functions of the portfolio return distribution; hence, investors can achieve their investment goals by modeling the expected returns and tolerable risk in a portfolio return distribution form that is called the expected return distribution. If the portfolio return series can act as a Markov chain in a certain time horizon, then the future distribution of its return can be measured using the Chapman-Kolmogorov theorem according to Equation (2.3). It is the main idea of the present research. Portfolio weights are optimized in a portfolio selection model. The present research model decides to select the weights in a way that the portfolio return series become a Markov process in a suitable state so that its return distribution is close to the investor's expected return distribution (indicating the expected return and risk) as much as possible.

The research portfolio based on Markov analysis is a flexible model because the portfolio optimization based on the Markov chain is a non-parametric approach that does not depend on the asset return distribution or a specific market. Numerous studies on portfolio selection assume the parametric asset distribution, for example, based on normal or log-normal distribution. There is no need for these assumptions in the analysis based on the Markov chain. Finally, the portfolio risk can be controlled using a final portfolio return distribution at maturity. Therefore, desirable and undesirable risks can be simultaneously considered in an expected return distribution of the portfolio. Undesirable risk can calculate the investor's loss at a certain confidence level and it is closer to the concept of risk as the rate of loss in the investor's mind.

Several studies on the use of Markov chains in financial modeling are mentioned as follows. Mirmohammadi et al. [10] introduced the combined portfolio selection model of risk parity and factor analysis based on Markov switching. The portfolio of a sample research consisting of 8 industries (as portfolio assets) on the Tehran Stock Exchange from 2011-2020 indicated that the combined research model had a higher Sharpe ratio than the common mean-variance and weight parity models, and it was more resistant to the market crash and produces less loss than the other two models. Using two methods, Davoudi and Mirsaeidi [6] indicated that the two-week returns (fourteen working days) of the total stock market index of the Tehran Stock Exchange had a Markov property in 1997-2015 in a six-member state defined based on returns and risk. The Markov property indicates that the average time for state transition is from 4 to 13 periods (each for 14 days), and the maximum limit probabilities, indicating the long-term behavior of a process, are related to a state in which the obtained return is higher than the average. Ahmadian [1] utilized the Markov chain to analyze stock pricing. In this regard, the dividend check model is conducted using the Markov chain to obtain the intrinsic value of the company's stock. Alamatian and Vafaei Jahan [2] offered a system based on Bayesian networks and a hidden Markov model to forecast the daily trends of the Iran stock market. They used a total of 6 market indices of the Tehran Stock Exchange and 22 technical indicators. Bayesian networks were also used to find relationships between the variables, and finally, modeling was conducted using the hidden Markov model. The best accuracy percentage of this proposed system was 85.25%. Amiri and Biglari Kami [4] took advantage of the Markov chain model to predict the stock price behavior in the future and investigate the memory-less property of the stock market. To this end, three levels, positive, neutral, and negative, were considered for the percentage of stock price vitality, and percentage of traded volume, and then 9 states or situations were defined for the Markov chain model according to their interaction. The results indicated that the use of the Markov chain in stock price forecasting could be desirable, and the memory-less property could be attributed to the stock market under certain conditions.

Kostadinova et al. [8] utilized the Markov chain to predict stock price trends. Markov chain models were obtained for 3 different stock prices based on the state-transition matrix and initial state vector for 2019. The analyzed stocks were combined in an optimal mean-risk portfolio, and the analysis of the risk aversion coefficient and its effect on the portfolio selection was performed completely. Pasricha et al. [11] developed an approach based on the semi-Markov process for an optimal portfolio selection consisting of bonds. The credit portfolio optimization criteria were based on the infinite norm, and the proposed optimization model was transformed into a linear programming problem, assuming that the bond credit rating followed a semi-Markov process. Finally, the portfolio was optimized on 10 bonds at a maturity of 9 years. Rahmani and Dehghani [12] proposed a two-step method for enhanced portfolio indexing of the Tehran Stock Exchange price index. First, a discrete Markov chain model was designed to filter stocks based on their high probability of profit compared to the benchmark index, and then the optimal weights were assigned to the filtered assets. The sample included weekly data from March 2013 to March 2020. The data were classified into 26 time-frames, including 52 in-sample and 12 out-of-sample data. The results indicated that not only the portfolio return was positively correlated with the index return and could be traced completely, but it can also achieve a higher return. Burkett et al. [5] introduced an innovative new approach to design portfolios that applied a discrete state-based method to define market states and asset allocation decisions according to current and future state membership. The transition dynamics of derived states were modeled as a Markov process. Asset weighting and portfolio allocation decisions were made using an approach based on heuristic optimization. The research portfolio was optimized on a 7-share portfolio from 2004 to 2019. Ryou et al. [14] presented an investment strategy that utilized a hidden Markov model for stock selection in bullish mode. Identifying the stock state, the stock in the bullish mode was first purchased and rebalanced after the holding period. The study also indicated that investment strategy was useful in the Korean stock exchange using a hidden Markov network. Ruiz-Cruz et al. [13] proposed a trading portfolio strategy that was mainly based on the K-Means clustering algorithm to determine and learn internal hidden patterns in time series of stock market prices, the predictive algorithm based on a simple Markov chain, and a Fuzzy inference system to make decisions in transactions. The trading algorithm performance was confirmed through simulation using real prices of the Mexican stock exchange. Zandieh [15] conducted portfolio optimization in the Belgrade Stock Exchange and thus utilized the Markov chain as a simple and non-parametric method. They considered a portfolio with 10 assets and 252 daily data in 2015. All returns were combined and classified to find the Markov process states. Considering the three states, stocks with a maximum possibility of being in the range of expected returns were in the portfolio. Bebarta et al. [3] used a unified system to forecast stock prices. In this system, they utilized neural networks to adjust the input size for the hidden Markov model, and then the basic parameters of the system were optimized using the genetic algorithm. The model test result on four shares in 2014 indicated that the performance of this unified system was better than the performance of single neural networks. Gupta and Dhingra [7] used the hidden Markov model to predict stock prices. Their model used a Gaussian Mixture as the system output. The output was three-dimensional and defined based on the opening price, closing, highest, and lowest prices. Finally, the network accuracy was investigated on four stocks and the researchers reported satisfactory results.

The literature review indicated the innovation of the present research as the current issue had not been investigated in any study. In this study, the portfolio was considered as a synthetic asset that could change in its return distribution by changing the vector weights of the assets. The Markov property can be examined using statistical tests and the distribution of the future portfolio returns can be measured for future time steps according to relations in the Markov chain theory for a certain portfolio (certain weights) and a certain state space. Markov chain-based optimization model considers the final distribution of portfolio returns and thus allows investors to determine a balance between their return and risk by specifying the final portfolio return distribution function at maturity. Therefore, the value at risk (VaR) is considered the maximum portfolio loss at a certain confidence level in the final portfolio wealth distribution according to the possibility of negative returns.

3 Research model

This section introduces the optimal portfolio selection model based on the Markov chain. Suppose that the portfolio consists of N assets $i \in \{1, 2, \dots, N\}$, and $\{r_{it}\}_{t=1}^{T}$ represents the time series of historical returns on asset i for $t \in \{1, 2, \dots, T\}$. The historical portfolio return is as follows for an arbitrary selection of portfolio weights, $w_1, w_2, w_3, \dots, w_N$, as $\forall\ i w_i \geq 0$, $\sum_{i=1}^{N} w_i = 1$

$$r_t = w_1 r_{1t} + w_2 r_{2t} + \dots + w_N r_{Nt}, \quad t \in \{1, 2, \dots, T\}. \tag{3.1}$$

The resulting series of portfolio returns should be examined for the Markov property. To this end, six states were considered a set of states for the Markov chain according to the table below.

Table 1: Markov chain state

State	Area
0	$r_i < \mu + \sigma$
1	$\mu + 0.5\sigma \leq r_i < \mu + \sigma$
2	$\mu \leq r_i < \mu + 0.5\sigma$
3	$\mu - 0.5\sigma \leq r_i < \mu$
4	$\mu - \sigma \leq r_i < \mu - 0.5\sigma$
5	$r_i < \mu - \sigma$

In Table 1, μ is the mean historical portfolio returns and σ refers to its standard deviation. The state-transition matrix is first calculated to examine the Markov property according to the state space of Table 1. Therefore, the frequency of each state in the data series is first counted, and then the frequency of state transition is measured. For example, consider two states i, j and suppose that the frequency of state i is equal to $n(i)$ in the data series, and the frequency of transition of state $i \to j$ is equal to n_{ij},then, the maximum likelihood estimation (MLE) for the probability of transition of state $i \to j$, which is represented by p_{ij}, is equal to $p_{ij} = \frac{n_{ij}}{n(i)}$. Accordingly, the transition probability matrix is calculated according to Equation (3.2).

$$P = \begin{bmatrix} P_{00} & P_{01} & \dots & P_{05} \\ P_{10} & P_{11} & \dots & P_{15} \\ \vdots & \vdots & \ddots & \vdots \\ P_{50} & P_{51} & \dots & P_{55} \end{bmatrix} \tag{3.2}$$

The stationarity property was examined to check the Markov property. The time interval, in which the data were sampled, was divided into several sub-intervals to check the stationarity. Four sub-intervals were selected in the present research. If the process had stationarity, the distribution of state-transition probabilities of the chain in the sub-intervals should not be significantly different from the state-transition distribution for the whole data. If $P_{ij,r}$ is the state transition in the sub-interval r, the goodness-of-fit test is conducted to check the stationarity of the process according to Equation (3.3).

$$\begin{cases} H_0 : \forall\ ij : P_{ij,r} = P_{ij} \\ H_1 : \exists\ i,j : P_{ij,r} \neq P_{ij}. \end{cases} \tag{3.3}$$

In the goodness-of-fit test, Pearson's chi-square statistic is used as $\chi^2 = \sum_{i=0}^{5}\sum_{j=0}^{5}(e_{ij,r} - f_{ij,r})^2/e_{ij,r}$ in which $f_{ij,r}$ is the frequency of transition from the state $i \to j$ in the sub-interval r, and $e_{ij,r}$ is the expected frequency equal to $e_{ij,r} = N_{ij,r} \times P_{ij}$ if the null hypothesis is accepted. The Pearson chi-square statistic has a chi-square distribution with $(6-1)(6-1) = 25$ degrees of freedom. The null hypothesis is confirmed at a significant level of $1-\alpha$ if $\chi^2 < \chi^2_{1-\alpha}(25) = 37.652$. Furthermore, if the null hypothesis is accepted in all sub-intervals, then we can claim that the process is stationary at the desired significance level [6]. Examining the memory-less property is the second step of investigating the Markov property. To this end, Equation (3.4) must be confirmed from a statistical point of view.

$$P(X_{n+1} = k \big| X_n = i\ \&\ X_{n-1} = j) = P(X_{n+1} = j \big| X_n = i). \tag{3.4}$$

Therefore, $P_{ijk} = P(X_{n+1} = k \big| X_n = i\ \&\ X_{n-1} = j)$ is first estimated based on $P_{ijk} = \frac{n_{ijk}}{n(i,j)}$ in which n_{ijk} is the number of transition of state $i \to j$ and then to k. Furthermore, $n(i,j)$ is the frequency when the process is in state i at that time and in state j at a later time. If the process has a Markov property, the distribution of probabilities for two-step transitions should be independent of the starting point, or it should be a memoryless process. The goodness of fit test is arranged as follows to examine this issue for each state k.

$$H_0 : \forall\ ij : P_{ijk} = P_{jk} H_1 : \exists\ i,j : P_{ikj} \neq P_{jk}. \tag{3.5}$$

We use Pearson's chi-square statistic for the goodness-of-fit test as follows:

$$\chi^2 = \sum_{i=0}^{5}\sum_{j=0}^{5}(e_{ijk} - f_{ijk})^2/e_{ijk}$$

where f_{ijk} is the frequency of transitions from i to j, and then k, and e_{ijk} is the expected frequency of transition from $i \to j$, and then to k. The following equation is obtained assuming the null hypothesis. The introduced statistic has a chi-square distribution with $(6-1)(6-1) = 25$ degrees of freedom, and the null hypothesis is confirmed if $\chi^2 < \chi^2_{1-\alpha}(25) = 37.652$. If the null hypothesis is accepted for all states of k, then it can be claimed that the process has Markov properties [4].

The research portfolio model presents the return and risk by expressing the distribution function of the portfolio return probability on the state space in Table 1. Assume that P_{des} is the investor's optimal probability size and Q is the portfolio return distribution due to its Markov property. The research model seeks to make the two distributions as close as possible, and it thus uses the criterion of the proximity of two probability sizes, namely K-L divergence. K-L divergence, which is also called relative entropy and L-divergence, shown as $D_{\text{KL}}(P\Box Q)$, represents a type of statistical distance as a criterion for the difference between a probability distribution P with a second probability distribution of reference Q. A simple interpretation of the divergence $D_{\text{KL}}(P\Box Q)$ indicates the expected surprise in the use of size Q as a model when the true distribution of the model has a size P. In a simple state, K-L divergence with zero value indicates that the two distributions have the same amount of information. K-L divergence from size Q to P shown by $D_{\text{KL}}(P\Box Q)$ is defined as follows [9].

$$D_{\text{KL}}(P\Box Q) = \sum_{x\in X} P(x)\log\Big(\frac{P(x)}{Q(x)}\Big). \tag{3.6}$$

The research portfolio selection model is according to Equation (3.7) based on the aforementioned assumptions.

$$\begin{array}{ll} \min & \text{KL}(P_{r_{T+1}} \big| S_T = \varphi(r_T) \| P_{\text{des}}) \\ \text{s.t:} & \\ & \{r_t\}_{t=1}^{T} \text{ is a Markov chain w.r.t state set } S\ (1) \\ & \varphi(r_T) = \text{state of } r_T\ (2) \\ & \forall\ i : x_i \geq 0, \quad \sum_{i=1}^{n} x_i = 1\ (3) \\ & \text{Horizon} \in \{1,2,3,\ldots,14\}\ (4) \end{array} \tag{3.7}$$

This model assumes that T is the portfolio closing moment. P_{des} is the investor's desirable portfolio return distribution in the next time step (the return due to the movement from T to $T+1$), and $P_{r_{T+1}} \big| S_T = \varphi(r_T)$ is the portfolio return distribution as a Markov chain in a later time step, provided that the portfolio is in the state $\varphi(r_T)$ at time closing T. In fact, φ function depicts the return of each time on its state according to Table 1. Therefore, the research portfolio depends on the type of return state at the closing time. The first limitation indicates that the output series must be a Markov process according to the state space. To this end, the stationary and memory-less hypothesis tests are used. The time horizon of the portfolio may vary from one to 14 days. In fact, its model chooses that time horizon when the historical returns of the portfolio in that time horizon become a Markov chain according to the state space of Table 1. The particle swarm optimization (PSO) algorithm was used to optimize the model (3.7). This algorithm consists of a mass of particles. Each particle is settled in a region of the search space. The objective function value for each particle shows the degree of fitness of that particle's location. Particles in the search area move at a certain speed. The speed (direction and speed value) of each particle is affected by two factors, first, the best experience that particle has ever had (the best fitness level), and second, the best experience that neighboring particles have ever had. Finally, the movement of the particles will converge toward an optimal point.

4 Results

The present section provides details of the implementation of a portfolio selection model in the studies on the Tehran Stock Exchange. The portfolio of the present research consists of 10 assets according to Table 2 and each asset is an index of the Tehran Stock Exchange. Choosing an index as an asset means a diverse portfolio from its subset stock. For example, the index of metal minerals is a diverse portfolio from a subset stock of that index so that the research portfolio was diverse.

Table 2: List of research portfolio assets

Number of assets	Index name	Number of assets	Index name
1	Metallic minerals	6	Chemical
2	Non-metallic minerals	7	Medicine
3	Oil products	8	Machinery
4	Car	9	Food
5	Cement	10	Technical

The historical data of 10 assets, consisting of 3442 daily data, were extracted from the beginning of 2009 to the end of 2022. After data extraction, the model (3.7) was optimized using particle swarm optimization (PSO). A total of 3000 data were used as training sets to optimize the model, and 342 data were as test data. According to Model (3.7), the time horizon of the research portfolio can change from 1-14 days. The optimal time horizon, in which the return series of the portfolio indicates Markov behavior (stationary and memory-less), is a 10-day horizon based on the set of states in Table 1. Given the 10-day time horizon, the training data set consists of 300 10-day training data and 44 10-day test data. The average portfolio return in a 10-day horizon, represented by μ and the standard deviation (risk) represented by σ were calculated as follows:$\mu = 0.0172$, $\sigma = 0.0531$. On this basis, the process state space in the optimal portfolio includes six states according to Table 3. The last column of Table 3 indicates the investor's expected return distribution for the next step.

Table 3: State space in the optimal portfolio

State	Area	Amount	Expected desirable return distribution
0	$r > \mu + \sigma$	$r > 0.0704$	0.20
1	$\mu + 0.5\sigma \leq r < \mu + \sigma$	$0.04385 \leq r < 0.0704$	0.40
2	$\mu \leq r < \mu + 0.5\sigma$	$0.0173 \leq r < 0.04385$	0.15
3	$\mu - 0.5\sigma \leq r < \mu$	$-0.00925 \leq r < 0.0173$	0.10
4	$\mu - \sigma \leq r < \mu - 0.5\sigma$	$-0.0358 \leq r < -0.00925$	0.10
5	$r < \mu - \sigma$	$r < -0.0358$	0.05

Figure 1 shows the state transition of a step (each step equal to 10 days) in the optimal solution for training data.

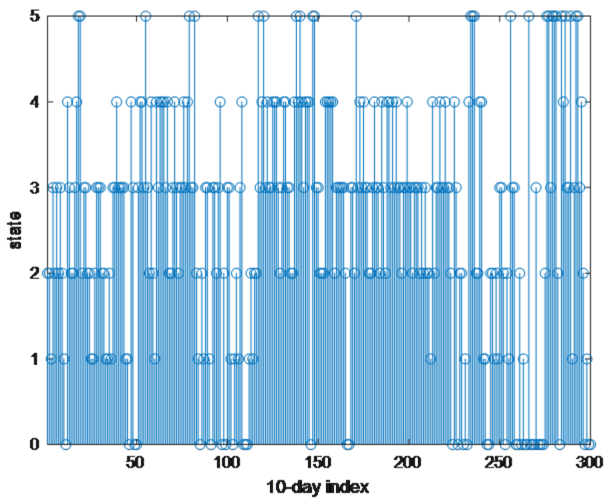

Figure 1: One-step state transition

The one-step transition probability matrix P in the optimal solution was measured as follows:

$$P = \begin{bmatrix} 0.342 & 0.132 & 0.184 & 0.132 & 0.105 & 0.079 \\ 0.321 & 0.143 & 0.214 & 0.214 & 0.071 & 0.036 \\ 0.131 & 0.230 & 0.230 & 0.262 & 0.098 & 0.049 \\ 0.043 & 0.032 & 0.258 & 0.355 & 0.258 & 0.054 \\ 0.057 & 0.019 & 0.094 & 0.434 & 0.264 & 0.132 \\ 0.037 & 0.037 & 0.148 & 0.370 & 0.111 & 0.296 \end{bmatrix}$$

The data were classified into 4 groups of 75 for the optimal solution to examine the validity. Table 4 presents the result of the goodness of fit test to determine the stationary of the process.

Table 4: The result of the goodness of fit test to examine the stationary

Null hypothesis	df	Pearson Chi-Square statistic	Critical value of the statistic at a 95% significance level	Test result
The distribution of the state-transition probability matrix in the first subsample follows matrix p	25	31.54	37.652	Null hypothesis accepted
The distribution of the state-transition probability matrix in the second subsample follows matrix p	25	24.013	37.652	Null hypothesis accepted
The distribution of the state-transition probability matrix in the third subsample follows matrix p	25	18.360	37.652	Null hypothesis accepted
The distribution of the state-transition probability matrix in the fourth subsample follows matrix p	25	35.1201	37.652	Null hypothesis accepted

According to the acceptance of the null hypothesis in all sub-intervals, the process is stationary. Table 5 presents the results of the memorylessness test in the optimal solution with the help of the goodness of fit test.

Acceptance of the null hypothesis in all states indicated that the process was memoryless. As stated in the research model section, the optimal portfolio depended on the portfolio state at the closing time; hence, six optimal portfolios were measured according to Table 6, depending that which six states presented the process state when the portfolio was closed.

Figure 2 presents the 10-day returns obtained by the optimal portfolio using the Markov approach on 44 test data.

Table 5: The result of the goodness of fit test to examine memorylessness

Null hypothesis	df	Chi-Square statistic	Critical value at a 95% significance level	Test result
$\forall\ i,j : p_{ij0} = p_j0$	25	21.7321	37.652	Null hypothesis accepted
$\forall\ i,j : p_{ij1} = p_j1$	25	25.1012	37.652	Null hypothesis
$\forall\ i,j : p_{ij2} = p_j2$	25	18.5109	37.652	Null hypothesis
$\forall\ i,j : p_{ij3} = p_j3$	25	23.7213	37.652	Null hypothesis
$\forall\ i,j : p_{ij4} = p_j4$	25	30.3012	37.652	Null hypothesis
$\forall\ i,j : p_{ij5} = p_j5$	25	18.5012	37.652	Null hypothesis

Table 6: Optimal portfolios depending on the six states

Number of assets \ State	0	1	2	3	4	5
1	0.276	0.124	0.102	0.397	0.077	0.297
2	0.000	0.051	0.000	0.240	0.000	0.140
3	0.134	0.151	0.250	0.011	0.253	0.011
4	0.111	0.001	0.200	0.069	0.070	0.089
5	0.106	0.106	0.164	0.107	0.276	0.117
6	0.000	0.110	0.100	0.000	0.000	0.000
7	0.077	0.193	0.000	0.000	0.114	0.203
8	0.000	0.101	0.118	0.118	0.101	0.048
9	0.154	0.070	0.001	0.023	0.106	0.043
10	0.140	0.100	0.070	0.036	0.010	0.060

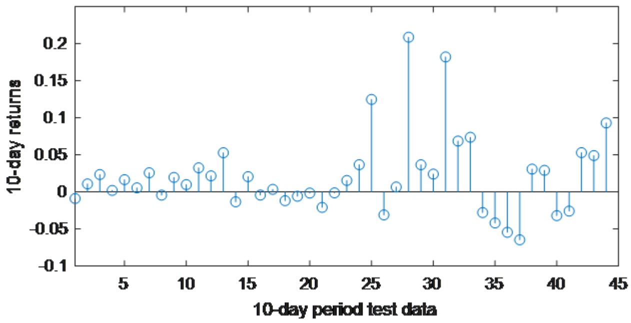

Figure 2: The returns obtained by the optimal research portfolio on the test data

Table 7 presents the performance of the research model on the test data using the mean-variance portfolio (Markowitz) and the weight parity portfolio (in which all assets have equal weights) in the criteria, namely mean return, standard deviation, and value at risk (as criteria of risk measurement), and Sharpe ratio (as a criterion of risk-adjusted return).

Table 7: Performance comparison of the research model with the mean-variance and weight parity models

Criterion	Research portfolio	Markowitz portfolio	Weight parity portfolio
Mean return (10-day)	0.02113	0.00951	0.00513
sd	0.05334	0.03799	0.04974
Sharpe ratio	0.39619	0.25023	0.10322
VaR (0.95)	0.039	0.0455	0.05981

Based on Table 7, the research portfolio with a Markov chain approach had a better performance in the mean return, Sharpe ratio, and value at risk than the mean-variance and risk parity portfolios.

Conclusion

The optimal portfolio selection is an important sensitive financial field. Therefore, various models of optimal portfolio selection have been developed. Return and risk are two important parameters in the optimal portfolio selection, depending on portfolio return distribution. In a random process, the Markov property allows the future distribution of a random process to be calculated according to the state-transition matrix and the initial state of the process. The main idea of the present research in selecting the optimal portfolio was to choose the portfolio weights in a way that the Markov property was established for portfolio returns series, and distribution of the future portfolio returns, which could be measured owing to the Markov property and the Chapman-Kolmogorov equation, corresponded to the investor's expected future return distribution; hence K-L divergence was utilized as a measure of closeness of two distributions. A suitable state space was defined to convert the portfolio return series into a Markov chain so that the price series movement between those state spaces in a suitable time horizon was according to the Markov property (stationary and memorylessness). The stationary and memorylessness properties were also examined with the help of the goodness of fit test. The present research presented an optimal portfolio selection model based on the Markov chain and its performance was evaluated in a diverse sample portfolio with 10 indices on the Tehran Stock Exchange. The results of the profitability evaluation indicated that the research model had a better performance than the mean-variance and weight parity models in the mean return, Sharpe ratio, and value at risk criteria. Therefore, the use of this model is recommended, especially for risk-averse investors.

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