Int. J. Nonlinear Anal. Appl. **1** (2010) No.2 , 100–124
ISSN: 2008-6822 (electronic)
http://www.ijnaa.com

INTUITIONISTIC FUZZY STABILITY OF A QUADRATIC AND QUARTIC FUNCTIONAL EQUATION

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Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

Abstract. In this paper, we prove the generalized Hyers–Ulam stability of a quadratic and quartic functional equation in intuitionistic fuzzy Banach spaces.

1. Introduction

In recent years, the fuzzy theory has emerged as the most active area of research in many branches of mathematics and engineering. This new theory was introduced by Zadeh [1], in 1965 and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers and fuzzification of many classical theories has also been made. It has also very useful application in various fields, e.g. population dynamics [2], chaos control [3], computer programming [4], nonlinear dynamical systems [5], fuzzy physics [6], fuzzy topology [7], fuzzy stability [8]-[12], nonlinear operators [13], statistical convergence [14, 15], etc.

The concept of intuitionistic fuzzy normed spaces, initially has been introduced by Saadati and Park [16]. In [17], by modifying the separation condition and strengthening some conditions in the definition of Saadati and Park, Saadati et al. have obtained a modified case of intuitionistic fuzzy normed spaces. Many authors have considered the intuitionistic fuzzy normed linear spaces, and intuitionistic fuzzy 2–normed spaces(see [18]–[21]).

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [22] in 1940 and affirmatively solved by Hyers [23]. The result of Hyers was generalized by Aoki [24] for approximate additive function and by Rassias [25] for approximate linear functions by allowing the difference Cauchy equation $\|f(x_1+x_2)-f(x_1)-f(x_2)\|$ to be controlled by $\varepsilon(\| x_1 \|^p + \| x_2 \|^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the generalized Ulam-Rassias stability or Hyers-Ulam-Rassias stability (see [26, 27, 28]). In 1994, a generalization of Rassias [29] theorem was obtained by Găvruta , who

Date: Received: January 2010; Revised: Jun 2010.

2000 *Mathematics Subject Classification.* Primary 39B82, 39B52.

Key words and phrases. Intuitionistic fuzzy normed space; Mixed functional equation; Intuitionistic fuzzy stability.

replaced $\varepsilon(\| x_1 \|^p + \| x_2 \|^p)$ by a general control function $\varphi(x_1, x_2)$. The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric bi-additive function [30, 31]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B_1 such that $f(x) = B_1(x, x)$ for all x. The bi-additive function B_1 is given by

$$B_1(x,y)=\frac{1}{4}(f(x+y)-f(x-y)).$$

In the paper [32], Czerwik proved the Hyers–Ulam–Rassias stability of the equation (1.1).

Lee et. al. [33] considered the following functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of (1.2) if and only if there exists a unique symmetric bi–quadratic function $B_2 : X \times X \longrightarrow Y$ such that $f(x) = B_2(x, x)$ for all x. The bi–quadratic function B_2 is given by

$$B_2(x,y)=\frac{1}{12}(f(x+y)+f(x-y)-2f(x)-2f(y)).$$

Obviously, the function $f(x) = cx^4$ satisfies the functional equation (1.2) which is called the quartic functional equation.

Eshaghi and Khodaei [34] have established the general solution and investigated the Hyers-Ulam-Rassias stability for a mixed type of cubic, quadratic and additive functional equation with $f(0) = 0$,

$$f(x+ky)+f(x-ky)=k^2f(x+y)+k^2f(x-y)+2(1-k^2)f(x) \tag{1.3}$$

in quasi–Banach spaces, which k is nonzero integer number with $k \neq \pm 1$. Interesting new results concerning mixed functional equations have recently been obtained by Najati et. al. [35, 36, 37] as well as for the fuzzy stability of a mixed type of additive and quadratic functional equation by Park [12].

The functional equation

$$f(nx+y)+f(nx-y)=n^2f(x+y)+n^2f(x-y)+2(f(nx)-n^2f(x))-2(n^2-1)f(y) \tag{1.4}$$

($n \in \mathbb{N}$, $n \geq 2$) is called the mixed quadratic and quartic functional equation, since the function $f(x) = ax^4 + bx^2$ is its solution. The stability problem for the mixed quadratic and quartic functional equation was proved by Eshaghi et. al. [38] for a function $f : X \to Y$, where X and Y are quasi–Banach spaces.

2. **Preliminaries**

We use the definition of intuitionistic fuzzy normed spaces given in [16, 39, 40] to investigate some stability results for the functional equation (1.4) in the intuitionistic fuzzy normed vector space setting.

Definition 2.1. ([41]). A binary operation $*: [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2. ([41]). A binary operation $\star: [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(a) $\star$ is commutative and associative;
(b) $\star$ is continuous;
(c) $a \star 0 = a$ for all $a \in [0,1]$;
(d) $a \star b \leq c \star d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [16], have introduced the concept of intuitionistic fuzzy normed space.

Definition 2.3. (Saadati and Park [16], Mursaleen and Mohiuddine [39]). The five–tuple $(X, \mu, \nu, *, \star)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\star$ is a continuous t-conorm, and μ, ν fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions: For every $x, y \in X$ and $s, t > 0$,

(IF_1) $\mu(x,t) + \nu(x,t) \leq 1$;
(IF_2) $\mu(x,t) > 0$;
(IF_3) $\mu(x,t) = 1$ if and only if $x = 0$;
(IF_4) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(IF_5) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$;
(IF_6) $\mu(x, .) : (0,\infty) \longrightarrow [0,1]$ is continuous;
(IF_7) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$;
(IF_8) $\nu(x,t) = 0$ if and only if $x = 0$;
(IF_9) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(IF_{10}) $\nu(x,t) \star \nu(y,s) \geq \nu(x+y, t+s)$;
(IF_{11}) $\nu(x, .) : (0,\infty) \longrightarrow [0,1]$ is continuous;
(IF_{12}) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

The properties of IFNS, examples of intuitionistic fuzzy norms and the concepts of convergence and Cauchy sequences in an IFNS are given in [16].

Definition 2.4. Let $(X, \mu, \nu, *, \star)$ be an IFNS. Then, a sequence $\{x_n\}$ is said to be convergent to $x \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $k \in \mathbb{N}$ such that $\mu(x_n - x, t) > 1 - \varepsilon$ and $\nu(x_n - x, t) < \varepsilon$ for all $n \geq k$. In this case we write (μ, ν)–$\lim x_n = x$.

Definition 2.5. Let $(X, \mu, \nu, *, \star)$ be an IFNS. Then, $\{x_n\}$ is said to be Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $k \in \mathbb{N}$ such that $\mu(x_n - x_m, t) > 1 - \varepsilon$ and $\nu(x_n - x_m, t) < \varepsilon$ for all $n, m \geq k$.

Definition 2.6. Let $(X, \mu, \nu, *, \star)$ be an IFNS. Then $(X, \mu, \nu, *, \star)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \star)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \star)$.

Definition 2.7. We say that a function $f : X \to Y$ between IFNS, X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [13, 39]).

In the rest of this paper, unless otherwise explicitly stated, we will assume that X is a linear space, (Z, μ', ν') is an intuitionistic fuzzy normed space and (Y, μ, ν) is an intuitionistic fuzzy Banach space. For convenience, we use the following abbreviation for a given function $f : X \to Y$,

$$\begin{aligned} Df(x,y) &= f(nx+y) + f(nx-y) - n^2 f(x+y) - n^2 f(x-y) - 2f(nx) \\ &\quad + 2n^2 f(x) + 2(n^2-1)f(y) \end{aligned}$$

for all $x, y \in X$.

3. Intuitionistic fuzzy stability of (1.4): for quadratic functions

Lemma 3.1. ([38]). *Let V_1 and V_2 be real vector spaces. If a function $f : V_1 \to V_2$ satisfies (1.4), then the function $g : V_1 \to V_2$ defined by $g(x) = f(2x) - 16f(x)$ is quadratic.*

Theorem 3.2. *Let $\ell \in \{-1, 1\}$ be fixed and let $\varphi_q : X \times X \longrightarrow Z$ be a function such that*

$$\varphi_q(2x, 2y) = \alpha \varphi_q(x, y) \tag{3.1}$$

for all $x, y \in X$ and for some positive real number α with $\alpha\ell < 4\ell$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies

$$\mu\left(Df(x,y), t\right) \geq \mu'\left(\varphi_q(x,y), t\right) \quad \& \quad \nu\left(Df(x,y), t\right) \leq \nu'\left(\varphi_q(x,y), t\right) \tag{3.2}$$

for all $x, y \in X$ and $t > 0$. Then the limit

$$Q(x) = (\mu, \nu) - \lim_{n\to\infty} \frac{1}{4^{\ell n}} \left(f(2^{\ell n+1}x) - 16 f(2^{\ell n}x) \right)$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic function such that

$$\begin{aligned} &\mu\left(f(2x) - 16f(x) - Q(x), t\right) \geq M_q\left(x, \frac{\ell(4-\alpha)}{2} t\right) \;\; \& \\ &\nu\left(f(2x) - 16f(x) - Q(x), t\right) \leq N_q\left(x, \frac{\ell(4-\alpha)}{2} t\right) \end{aligned} \tag{3.3}$$

for all $x \in X$ *and* $t > 0$, *where*

$$\begin{aligned}
M_q(x,t) = {} & \mu^{'}(\varphi_q(x,(n+2)x), \frac{n^2(n^2-1)t}{17}) \\
& * \mu^{'}(\varphi_q(x,(n-2)x), \frac{n^2(n^2-1)t}{17}) * \mu^{'}(\varphi_q(x,(n+1)x), \frac{n^2(n^2-1)t}{68}) \\
& * \mu^{'}(\varphi_q(x,(n-1)x), \frac{n^2(n^2-1)t}{68}) * \mu^{'}(\varphi_q(x,nx), \frac{n^2(n^2-1)t}{170}) \\
& * \mu^{'}(\varphi_q(2x,2x), \frac{n^2(n^2-1)t}{17}) * \mu^{'}(\varphi_q(2x,x), \frac{n^2(n^2-1)t}{68}) \\
& * \mu^{'}(\varphi_q(x,3x), \frac{(n^2-1)t}{17}) * \mu^{'}(\varphi_q(x,2x), \frac{n^2(n^2-1)t}{28(3n^2-1)}) \\
& * \mu^{'}(\varphi_q(x,x), \frac{n^2(n^2-1)t}{17(17n^2-8)}) * \mu^{'}(\varphi_q(0,(n+1)x), \frac{(n^2-1)^2t}{17}) \\
& * \mu^{'}(\varphi_q(0,(n-3)x), \frac{(n^2-1)^2t}{17}) * \mu^{'}(\varphi_q(0,(n-1)x), \frac{(n^2-1)^2t}{170}) \\
& * \mu^{'}(\varphi_q(0,nx), \frac{(n^2-1)^2t}{68}) * \mu^{'}(\varphi_q(0,(n-2)x), \frac{(n^2-1)^2t}{68}) \\
& * \mu^{'}(\varphi_q(0,2x), \frac{n^2(n^2-1)^2t}{17(n^4+1)}) * \mu^{'}(\varphi_q(0,x), \frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}) \quad \& \\
N_q(x,t) = {} & \nu^{'}(\varphi_q(x,(n+2)x), \frac{n^2(n^2-1)t}{17}) \\
& \star \nu^{'}(\varphi_q(x,(n-2)x), \frac{n^2(n^2-1)t}{17}) \star \nu^{'}(\varphi_q(x,(n+1)x), \frac{n^2(n^2-1)t}{68}) \\
& \star \nu^{'}(\varphi_q(x,(n-1)x), \frac{n^2(n^2-1)t}{68}) \star \nu^{'}(\varphi_q(x,nx), \frac{n^2(n^2-1)t}{170}) \\
& \star \nu^{'}(\varphi_q(2x,2x), \frac{n^2(n^2-1)t}{17}) \star \nu^{'}(\varphi_q(2x,x), \frac{n^2(n^2-1)t}{68}) \\
& \star \nu^{'}(\varphi_q(x,3x), \frac{(n^2-1)t}{17}) \star \nu^{'}(\varphi_q(x,2x), \frac{n^2(n^2-1)t}{28(3n^2-1)}) \\
& \star \nu^{'}(\varphi_q(x,x), \frac{n^2(n^2-1)t}{17(17n^2-8)}) \star \nu^{'}(\varphi_q(0,(n+1)x), \frac{(n^2-1)^2t}{17}) \\
& \star \nu^{'}(\varphi_q(0,(n-3)x), \frac{(n^2-1)^2t}{17}) \star \nu^{'}(\varphi_q(0,(n-1)x), \frac{(n^2-1)^2t}{170}) \\
& \star \nu^{'}(\varphi_q(0,nx), \frac{(n^2-1)^2t}{68}) \star \nu^{'}(\varphi_q(0,(n-2)x), \frac{(n^2-1)^2t}{68}) \\
& \star \nu^{'}(\varphi_q(0,2x), \frac{n^2(n^2-1)^2t}{17(n^4+1)}) \star \nu^{'}(\varphi_q(0,x), \frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}).
\end{aligned}$$

Proof. Case (1): $\ell = 1$. Set $x = 0$ in (3.2) and then interchange x with y to get

$$\begin{aligned}
&\mu((n^2-1)f(x) - (n^2-1)f(-x), t) \geq \mu^{'}(\varphi_q(0,x), t) \quad \& \\
&\nu((n^2-1)f(x) - (n^2-1)f(-x), t) \leq \nu^{'}(\varphi_q(0,x), t)
\end{aligned} \tag{3.4}$$

for all $x \in X$ and $t > 0$. Replacing y by x, $2x$, nx, $(n+1)x$, $(n-1)x$, $(n+2)x$, $(n-2)x$ and $3x$ in (3.2), respectively, we get

$$\begin{aligned} &\mu(f((n+1)x) + f((n-1)x) - n^2 f(2x) - 2f(nx) + (4n^2-2)f(x), t) \\ &\qquad\qquad \geq \mu'(\varphi_q(x,x),t) \quad \& \\ &\nu(f((n+1)x) + f((n-1)x) - n^2 f(2x) - 2f(nx) + (4n^2-2)f(x), t) \\ &\qquad\qquad \leq \nu'(\varphi_q(x,x),t), \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\mu(f((n+2)x) + f((n-2)x) - n^2 f(3x) - n^2 f(-x) - 2f(nx) + 2n^2 f(x) \\ &\qquad + 2(n^2-1)f(2x), t) \geq \mu'(\varphi_q(x,2x),t) \quad \& \\ &\nu(f((n+2)x) + f((n-2)x) - n^2 f(3x) - n^2 f(-x) - 2f(nx) + 2n^2 f(x) \\ &\qquad + 2(n^2-1)f(2x), t) \leq \nu'(\varphi_q(x,2x),t), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\mu(f(2nx) - n^2 f((n+1)x) - n^2 f((1-n)x) + 2(n^2-2)f(nx) + 2n^2 f(x), t) \\ &\qquad\qquad \geq \mu'(\varphi_q(x,nx),t) \quad \& \\ &\nu(f(2nx) - n^2 f((n+1)x) - n^2 f((1-n)x) + 2(n^2-2)f(nx) + 2n^2 f(x), t) \\ &\qquad\qquad \leq \nu'(\varphi_q(x,nx),t), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\mu(f((2n+1)x) + f(-x) - n^2 f((n+2)x) - n^2 f(-nx) - 2f(nx) + 2n^2 f(x) \\ &\qquad + 2(n^2-1)f((n+1)x), t) \geq \mu'(\varphi_q(x,(n+1)x),t) \quad \& \\ &\nu(f((2n+1)x) + f(-x) - n^2 f((n+2)x) - n^2 f(-nx) - 2f(nx) + 2n^2 f(x) \\ &\qquad + 2(n^2-1)f((n+1)x), t) \leq \nu'(\varphi_q(x,(n+1)x),t), \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\mu(f((2n-1)x) + f(x) - n^2 f((2-n)x) - (n^2+2)f(nx) + 2n^2 f(x) \\ &\qquad + 2(n^2-1)f((n-1)x), t) \geq \mu'(\varphi_q(x,(n-1)x),t) \quad \& \\ &\nu(f((2n-1)x) + f(x) - n^2 f((2-n)x) - (n^2+2)f(nx) + 2n^2 f(x) \\ &\qquad + 2(n^2-1)f((n-1)x), t) \leq \nu'(\varphi_q(x,(n-1)x),t), \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\mu(f(2(n+1)x) + f(-2x) - n^2 f((n+3)x) - n^2 f(-(n+1)x) - 2f(nx) \\ &\qquad + 2n^2 f(x) + 2(n^2-1)f((n+2)x), t) \geq \mu'(\varphi_q(x,(n+2)x),t) \quad \& \\ &\nu(f(2(n+1)x) + f(-2x) - n^2 f((n+3)x) - n^2 f(-(n+1)x) - 2f(nx) \\ &\qquad + 2n^2 f(x) + 2(n^2-1)f((n+2)x), t) \leq \nu'(\varphi_q(x,(n+2)x),t), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\mu(f((2(n-1)x) + f(2x) - n^2 f((n-1)x) - n^2 f(-(n-3)x) - 2f(nx) \\ &\qquad + 2n^2 f(x) + 2(n^2-1)f((n-2)x), t) \geq \mu'(\varphi_q(x,(n-2)x),t) \quad \& \\ &\nu(f((2(n-1)x) + f(2x) - n^2 f((n-1)x) - n^2 f(-(n-3)x) - 2f(nx) \\ &\qquad + 2n^2 f(x) + 2(n^2-1)f((n-2)x), t) \leq \nu'(\varphi_q(x,(n-2)x),t) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\mu(f((n+3)x)+f((n-3)x)-n^2f(4x)-n^2f(-2x)-2f(nx)+2n^2f(x)\\
&\qquad\qquad+2(n^2-1)f(3x),t)\geq\mu^{'}(\varphi_q(x,3x),t)\quad\&\\
&\nu(f((n+3)x)+f((n-3)x)-n^2f(4x)-n^2f(-2x)-2f(nx)+2n^2f(x)\\
&\qquad\qquad+2(n^2-1)f(3x),t)\leq\nu^{'}(\varphi_q(x,3x),t)
\end{aligned}\tag{3.12}$$

for all $x\in X$ and $t>0$. Combining (3.4) with (3.5)–(3.12), respectively, yields the following inequalities:

$$\begin{aligned}
&\mu(f((n+2)x)+f((n-2)x)-n^2f(3x)-n^2f(x)-2f(nx)+2n^2f(x)\\
&\qquad+2(n^2-1)f(2x),t)\geq\mu^{'}(\varphi_q(x,2x),\frac{t}{2})*\mu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,x),\frac{t}{2})\quad\&\\
&\nu(f((n+2)x)+f((n-2)x)-n^2f(3x)-n^2f(x)-2f(nx)+2n^2f(x)\\
&\qquad+2(n^2-1)f(2x),t)\leq\nu^{'}(\varphi_q(x,2x),\frac{t}{2})\star\nu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,x),\frac{t}{2}),
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
&\mu(f(2nx)-n^2f((n+1)x)-n^2f((n-1)x)+2(n^2-2)f(nx)+2n^2f(x),t)\\
&\qquad\geq\mu^{'}(\varphi_q(x,nx),\frac{t}{2})*\mu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,(n-1)x),\frac{t}{2})\quad\&\\
&\nu(f(2nx)-n^2f((n+1)x)-n^2f((n-1)x)+2(n^2-2)f(nx)+2n^2f(x),t)\\
&\qquad\leq\nu^{'}(\varphi_q(x,nx),\frac{t}{2})\star\nu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,(n-1)x),\frac{t}{2}),
\end{aligned}\tag{3.14}$$

$$\begin{aligned}
&\mu(f((2n+1)x)+f(x)-n^2f((n+2)x)-n^2f(nx)-2f(nx)+2n^2f(x)\\
&\qquad+2(n^2-1)f((n+1)x),t)\geq\mu^{'}(\varphi_q(x,(n+1)x),\frac{t}{3})\\
&\qquad*\mu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,nx),\frac{t}{3})*\mu^{'}(\frac{1}{n^2-1}\varphi_q(0,x),\frac{t}{3})\quad\&\\
&\nu(f((2n+1)x)+f(x)-n^2f((n+2)x)-n^2f(nx)-2f(nx)+2n^2f(x)\\
&\qquad+2(n^2-1)f((n+1)x),t)\leq\nu^{'}(\varphi_q(x,(n+1)x),\frac{t}{3})\\
&\qquad\star\nu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,nx),\frac{t}{3})\star\nu^{'}(\frac{1}{n^2-1}\varphi_q(0,x),\frac{t}{3}),
\end{aligned}\tag{3.15}$$

$$\begin{aligned}
&\mu(f((2n-1)x)+f(x)-n^2f((n-2)x)-(n^2+2)f(nx)+2n^2f(x)\\
&\qquad +2(n^2-1)f((n-1)x),t)\geq \mu^{'}(\varphi_q(x,(n-1)x),\frac{t}{2})\\
&\qquad * \mu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,(n-2)x),\frac{t}{2}) \quad \& \\
&\nu(f((2n-1)x)+f(x)-n^2f((n-2)x)-(n^2+2)f(nx)+2n^2f(x)\\
&\qquad +2(n^2-1)f((n-1)x),t)\leq \nu^{'}(\varphi_q(x,(n-1)x),\frac{t}{2})\\
&\qquad \star \nu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,(n-2)x),\frac{t}{2}),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\mu(f(2(n+1)x)+f(2x)-n^2f((n+3)x)-n^2f((n+1)x)-2f(nx)+2n^2f(x)\\
&\qquad +2(n^2-1)f((n+2)x),t)\geq \mu^{'}(\varphi_q(x,(n+2)x),\frac{t}{3})\\
&\qquad * \mu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,(n+1)x),\frac{t}{3}) * \mu^{'}(\varphi_q(0,2x),\frac{t}{3}) \quad \& \\
&\nu(f(2(n+1)x)+f(2x)-n^2f((n+3)x)-n^2f((n+1)x)-2f(nx)+2n^2f(x)\\
&\qquad +2(n^2-1)f((n+2)x),t)\leq \nu^{'}(\varphi_q(x,(n+2)x),\frac{t}{3})\\
&\qquad \star \nu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,(n+1)x),\frac{t}{3}) \star \nu^{'}(\varphi_q(0,2x),\frac{t}{3}),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\mu(f(2(n-1)x)+f(2x)-n^2f((n-1)x)-n^2f((n-3)x)-2f(nx)+2n^2f(x)\\
&\qquad +2(n^2-1)f((n-2)x),t)\geq \mu^{'}(\varphi_q(x,(n-2)x),\frac{t}{2})\\
&\qquad * \mu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,(n-3)x),\frac{t}{2}) \quad \& \\
&\nu(f(2(n-1)x)+f(2x)-n^2f((n-1)x)-n^2f((n-3)x)-2f(nx)+2n^2f(x)\\
&\qquad +2(n^2-1)f((n-2)x),t)\leq \nu^{'}(\varphi_q(x,(n-2)x),\frac{t}{2})\\
&\qquad \star \nu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,(n-3)x),\frac{t}{2}),
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&\mu(f((n+3)x)+f((n-3)x)-n^2f(4x)-n^2f(2x)-2f(nx)+2n^2f(x)\\
&\qquad +2(n^2-1)f(3x),t)\geq \mu^{'}(\varphi_q(x,3x),\frac{t}{2}) * \mu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,2x),\frac{t}{2}) \quad \& \\
&\nu(f((n+3)x)+f((n-3)x)-n^2f(4x)-n^2f(2x)-2f(nx)+2n^2f(x)\\
&\qquad +2(n^2-1)f(3x),t)\leq \nu^{'}(\varphi_q(x,3x),\frac{t}{2}) \star \nu^{'}(\frac{n^2}{n^2-1}\varphi_q(0,2x),\frac{t}{2})
\end{aligned} \tag{3.19}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2x$ and x in (3.2), respectively, we obtain

$$\begin{aligned}&\mu(f((2n+1)x)+f((2n-1)x)-n^2f(3x)-2f(2nx)+2n^2f(2x)\\&\qquad+(n^2-2)f(x),t)\geq\mu^{'}(\varphi_q(2x,x),t)\quad\&\\&\nu(f((2n+1)x)+f((2n-1)x)-n^2f(3x)-2f(2nx)+2n^2f(2x)\\&\qquad+(n^2-2)f(x),t)\leq\nu^{'}(\varphi_q(2x,x),t)\end{aligned}\tag{3.20}$$

for all $x \in X$ and $t > 0$. Putting $2x$ instead of x and y in (3.2), we obtain

$$\begin{aligned}&\mu(f(2(n+1)x)+f(2(n-1)x)-n^2f(4x)-2f(2nx)+2(2n^2-1)f(2x),t)\\&\qquad\geq\mu^{'}(\varphi_q(2x,2x),t)\quad\&\\&\nu(f(2(n+1)x)+f(2(n-1)x)-n^2f(4x)-2f(2nx)+2(2n^2-1)f(2x),t)\\&\qquad\leq\nu^{'}(\varphi_q(2x,2x),t)\end{aligned}\tag{3.21}$$

for all $x \in X$ and $t > 0$. It follows from (3.5), (3.13), (3.14), (3.15), (3.16) and (3.20) that

$$\begin{aligned}&\mu(f(3x)-6f(2x)+15f(x),t)\geq\mu^{'}(\varphi_q(x,(n+1)x),\frac{n^2(n^2-1)t}{30})\\&\quad*\mu^{'}(\varphi_q(2x,x),\frac{n^2(n^2-1)t}{10})*\mu^{'}(\varphi_q(x,nx),\frac{n^2(n^2-1)t}{40})\\&\quad*\mu^{'}(\varphi_q(x,2x),\frac{(n^2-1)t}{20})*\mu^{'}(\varphi_q(x,x),\frac{n^2(n^2-1)t}{20(2n^2-1)})\\&\quad*\mu^{'}(\varphi_q(0,(n-1)x),\frac{(n^2-1)^2t}{40})*\mu^{'}(\varphi_q(0,nx),\frac{n^2(n^2-1)t}{30})\\&\quad*\mu^{'}(\varphi_q(0,(n-2)x)),\frac{n^2(n^2-1)t}{20})*\mu^{'}(\varphi_q(0,x),\frac{n^2(n^2-1)^2t}{30(n^4+1)})\\&\quad*\mu^{'}(\varphi_q(x,(n-1)x),\frac{n^2(n^2-1)t}{20})\quad\&\\&\nu(f(3x)-6f(2x)+15f(x),t)\leq\nu^{'}(\varphi_q(x,(n+1)x),\frac{n^2(n^2-1)t}{30})\\&\quad\star\nu^{'}(\varphi_q(2x,x),\frac{n^2(n^2-1)t}{10})\star\nu^{'}(\varphi_q(x,nx),\frac{n^2(n^2-1)t}{40})\\&\quad\star\nu^{'}(\varphi_q(x,2x),\frac{(n^2-1)t}{20})\star\nu^{'}(\varphi_q(x,x),\frac{n^2(n^2-1)t}{20(2n^2-1)})\\&\quad\star\nu^{'}(\varphi_q(0,(n-1)x),\frac{(n^2-1)^2t}{40})\star\nu^{'}(\varphi_q(0,nx),\frac{n^2(n^2-1)t}{30})\\&\quad\star\nu^{'}(\varphi_q(0,(n-2)x)),\frac{n^2(n^2-1)t}{20})\star\nu^{'}(\varphi_q(0,x),\frac{n^2(n^2-1)^2t}{30(n^4+1)})\\&\quad\star\nu^{'}(\varphi_q(x,(n-1)x),\frac{n^2(n^2-1)t}{20})\end{aligned}\tag{3.22}$$

for all $x \in X$ and $t > 0$. Also, from (3.5), (3.13), (3.14), (3.17), (3.18), (3.19) and (3.21), we conclude

$$
\begin{aligned}
&\mu(f(4x) - 4f(3x) + 4f(2x) + 4f(x), t) \\
&\qquad \geq \mu^{'}(\varphi_q(2x, 2x), \frac{n^2(n^2-1)t}{12}) * \mu^{'}(\varphi_q(0, x), \frac{(n^2-1)t}{48}) \\
&\qquad * \mu^{'}(\varphi_q(x, 3x), \frac{(n^2-1)t}{24}) * \mu^{'}(\varphi_q(x, x), \frac{n^2(n^2-1)t}{12}) \\
&\qquad * \mu^{'}(\varphi_q(x, nx), \frac{n^2(n^2-1)t}{48}) * \mu^{'}(\varphi_q(x, 2x), \frac{n^2 t}{48}) \\
&\qquad * \mu^{'}(\varphi_q(0, (n+1)x), \frac{(n^2-1)^2 t}{36}) \\
&\qquad * \mu^{'}(\varphi_q(x, (n-2)x), \frac{n^2(n^2-1)t}{24}) \\
&\qquad * \mu^{'}(\varphi_q(x, (n+2)x), \frac{n^2(n^2-1)t}{36}) \\
&\qquad * \mu^{'}(\varphi_q(0, (n-1)x), \frac{(n^2-1)^2 t}{48}) \\
&\qquad * \mu^{'}(\varphi_q(0, (n-3)x), \frac{(n^2-1)^2 t}{48}) \\
&\qquad * \mu^{'}(\varphi_q(0, 2x), \frac{n^2(n^2-1)^2 t}{36(n^4+1)}) \quad \& \\
&\nu(f(4x) - 4f(3x) + 4f(2x) + 4f(x), t) \\
&\qquad \leq \nu^{'}(\varphi_q(2x, 2x), \frac{n^2(n^2-1)t}{12}) \star \nu^{'}(\varphi_q(0, x), \frac{(n^2-1)t}{48}) \\
&\qquad \star \nu^{'}(\varphi_q(x, 3x), \frac{(n^2-1)t}{24}) \star \nu^{'}(\varphi_q(x, x), \frac{n^2(n^2-1)t}{12}) \\
&\qquad \star \nu^{'}(\varphi_q(x, nx), \frac{n^2(n^2-1)t}{48}) \star \nu^{'}(\varphi_q(x, 2x), \frac{n^2 t}{48}) \\
&\qquad \star \nu^{'}(\varphi_q(0, (n+1)x), \frac{(n^2-1)^2 t}{36}) \\
&\qquad \star \nu^{'}(\varphi_q(x, (n-2)x), \frac{n^2(n^2-1)t}{24}) \\
&\qquad \star \nu^{'}(\varphi_q(x, (n+2)x), \frac{n^2(n^2-1)t}{36}) \\
&\qquad \star \nu^{'}(\varphi_q(0, (n-1)x), \frac{(n^2-1)^2 t}{48}) \\
&\qquad \star \nu^{'}(\varphi_q(0, (n-3)x), \frac{(n^2-1)^2 t}{48}) \\
&\qquad \star \nu^{'}(\varphi_q(0, 2x), \frac{n^2(n^2-1)^2 t}{36(n^4+1)})
\end{aligned}
\tag{3.23}
$$

for all $x \in X$ and $t > 0$. Finally, combining (3.20) and (3.21) yields

$$\begin{aligned}
\mu(f(4x)-24f(2x)+64f(x),t) &\geq \mu^{'}(\varphi_q(x,(n+2)x),\frac{n^2(n^2-1)t}{17}) \\
&* \mu^{'}(\varphi_q(x,(n-1)x),\frac{n^2(n^2-1)t}{68}) * \mu^{'}(\varphi_q(x,nx),\frac{n^2(n^2-1)t}{170}) \\
&* \mu^{'}(\varphi_q(2x,2x),\frac{n^2(n^2-1)t}{17}) * \mu^{'}(\varphi_q(2x,x),\frac{n^2(n^2-1)t}{68}) \\
&* \mu^{'}(\varphi_q(x,3x),\frac{(n^2-1)t}{17}) * \mu^{'}(\varphi_q(x,2x),\frac{n^2(n^2-1)t}{28(3n^2-1)}) \\
&* \mu^{'}(\varphi_q(x,x),\frac{n^2(n^2-1)t}{17(17n^2-8)}) * \mu^{'}(\varphi_q(0,(n+1)x),\frac{(n^2-1)^2t}{17}) \\
&* \mu^{'}(\varphi_q(0,(n-3)x),\frac{(n^2-1)^2t}{17}) * \mu^{'}(\varphi_q(0,(n-1)x),\frac{(n^2-1)^2t}{170}) \\
&* \mu^{'}(\varphi_q(0,nx),\frac{(n^2-1)^2t}{68}) * \mu^{'}(\varphi_q(0,(n-2)x),\frac{(n^2-1)^2t}{68}) \\
&* \mu^{'}(\varphi_q(0,2x),\frac{n^2(n^2-1)^2t}{17(n^4+1)}) * \mu^{'}(\varphi_q(0,x),\frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}) \\
&* \mu^{'}(\varphi_q(x,(n-2)x),\frac{n^2(n^2-1)t}{17}) \\
&* \mu^{'}(\varphi_q(x,(n+1)x),\frac{n^2(n^2-1)t}{68}) \quad \& \\
\nu(f(4x)-24f(2x)+64f(x),t) &\leq \nu^{'}(\varphi_q(x,(n+2)x),\frac{n^2(n^2-1)t}{17}) \\
&\star \nu^{'}(\varphi_q(x,(n-1)x),\frac{n^2(n^2-1)t}{68}) \star \nu^{'}(\varphi_q(x,nx),\frac{n^2(n^2-1)t}{170}) \\
&\star \nu^{'}(\varphi_q(2x,2x),\frac{n^2(n^2-1)t}{17}) \star \nu^{'}(\varphi_q(2x,x),\frac{n^2(n^2-1)t}{68}) \\
&\star \nu^{'}(\varphi_q(x,3x),\frac{(n^2-1)t}{17}) \star \nu^{'}(\varphi_q(x,2x),\frac{n^2(n^2-1)t}{28(3n^2-1)}) \\
&\star \nu^{'}(\varphi_q(x,x),\frac{n^2(n^2-1)t}{17(17n^2-8)}) \star \nu^{'}(\varphi_q(0,(n+1)x),\frac{(n^2-1)^2t}{17}) \\
&\star \nu^{'}(\varphi_q(0,(n-3)x),\frac{(n^2-1)^2t}{17}) \star \nu^{'}(\varphi_q(0,(n-1)x),\frac{(n^2-1)^2t}{170}) \\
&\star \nu^{'}(\varphi_q(0,nx),\frac{(n^2-1)^2t}{68}) \star \nu^{'}(\varphi_q(0,(n-2)x),\frac{(n^2-1)^2t}{68}) \\
&\star \nu^{'}(\varphi_q(0,2x),\frac{n^2(n^2-1)^2t}{17(n^4+1)}) \star \nu^{'}(\varphi_q(0,x),\frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}) \\
&\star \nu^{'}(\varphi_q(x,(n-2)x),\frac{n^2(n^2-1)t}{17}) \\
&\star \nu^{'}(\varphi_q(x,(n+1)x),\frac{n^2(n^2-1)t}{68})
\end{aligned} \tag{3.24}$$

for all $x \in X$ and $t > 0$. Let

$$\begin{aligned}
M_q(x,t) = &\mu^{'}(\varphi_q(x,(n+2)x), \frac{n^2(n^2-1)t}{17}) \\
&* \mu^{'}(\varphi_q(x,(n-2)x), \frac{n^2(n^2-1)t}{17}) * \mu^{'}(\varphi_q(x,(n+1)x), \frac{n^2(n^2-1)t}{68}) \\
&* \mu^{'}(\varphi_q(x,(n-1)x), \frac{n^2(n^2-1)t}{68}) * \mu^{'}(\varphi_q(x,nx), \frac{n^2(n^2-1)t}{170}) \\
&* \mu^{'}(\varphi_q(2x,2x), \frac{n^2(n^2-1)t}{17}) * \mu^{'}(\varphi_q(2x,x), \frac{n^2(n^2-1)t}{68}) \\
&* \mu^{'}(\varphi_q(x,3x), \frac{(n^2-1)t}{17}) * \mu^{'}(\varphi_q(x,2x), \frac{n^2(n^2-1)t}{28(3n^2-1)}) \\
&* \mu^{'}(\varphi_q(x,x), \frac{n^2(n^2-1)t}{17(17n^2-8)}) * \mu^{'}(\varphi_q(0,(n+1)x), \frac{(n^2-1)^2t}{17}) \\
&* \mu^{'}(\varphi_q(0,(n-3)x), \frac{(n^2-1)^2t}{17}) * \mu^{'}(\varphi_q(0,(n-1)x), \frac{(n^2-1)^2t}{170}) \\
&* \mu^{'}(\varphi_q(0,nx), \frac{(n^2-1)^2t}{68}) * \mu^{'}(\varphi_q(0,(n-2)x), \frac{(n^2-1)^2t}{68}) \\
&* \mu^{'}(\varphi_q(0,2x), \frac{n^2(n^2-1)^2t}{17(n^4+1)}) * \mu^{'}(\varphi_q(0,x), \frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}) \quad \&
\end{aligned}$$

$$\begin{aligned}
N_q(x,t) = &\nu^{'}(\varphi_q(x,(n+2)x), \frac{n^2(n^2-1)t}{17}) \\
&\star \nu^{'}(\varphi_q(x,(n-2)x), \frac{n^2(n^2-1)t}{17}) \star \nu^{'}(\varphi_q(x,(n+1)x), \frac{n^2(n^2-1)t}{68}) \\
&\star \nu^{'}(\varphi_q(x,(n-1)x), \frac{n^2(n^2-1)t}{68}) \star \nu^{'}(\varphi_q(x,nx), \frac{n^2(n^2-1)t}{170}) \\
&\star \nu^{'}(\varphi_q(2x,2x), \frac{n^2(n^2-1)t}{17}) \star \nu^{'}(\varphi_q(2x,x), \frac{n^2(n^2-1)t}{68}) \\
&\star \nu^{'}(\varphi_q(x,3x), \frac{(n^2-1)t}{17}) \star \nu^{'}(\varphi_q(x,2x), \frac{n^2(n^2-1)t}{28(3n^2-1)}) \\
&\star \nu^{'}(\varphi_q(x,x), \frac{n^2(n^2-1)t}{17(17n^2-8)}) \star \nu^{'}(\varphi_q(0,(n+1)x), \frac{(n^2-1)^2t}{17}) \\
&\star \nu^{'}(\varphi_q(0,(n-3)x), \frac{(n^2-1)^2t}{17}) \star \nu^{'}(\varphi_q(0,(n-1)x), \frac{(n^2-1)^2t}{170}) \\
&\star \nu^{'}(\varphi_q(0,nx), \frac{(n^2-1)^2t}{68}) \star \nu^{'}(\varphi_q(0,(n-2)x), \frac{(n^2-1)^2t}{68}) \\
&\star \nu^{'}(\varphi_q(0,2x), \frac{n^2(n^2-1)^2t}{17(n^4+1)}) \star \nu^{'}(\varphi_q(0,x), \frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}).
\end{aligned}$$

Then in (3.24) we will have

$$\begin{aligned}
&\mu(f(4x)-20f(2x)+64f(x),t) \geq M_q(x,t) \quad \& \\
&\nu(f(4x)-20f(2x)+64f(x),t) \leq N_q(x,t)
\end{aligned} \tag{3.25}$$

for all $x \in X$ and $t > 0$. Let $g : X \to Y$ be a function defined by $g(x) := f(2x) - 16f(x)$ for all $x \in X$. So, from (3.25), we conclude that

$$\mu(g(2x) - 4g(x), t) \geq M_q(x,t) \;\;\&\;\; \nu(g(2x) - 4g(x), t) \leq N_q(x,t) \tag{3.26}$$

for all $x \in X$ and $t > 0$. So

$$\mu(\frac{g(2x)}{4} - g(x), \frac{t}{4}) \geq M_q(x,t) \;\;\&\;\; \nu(\frac{g(2x)}{4} - g(x), \frac{t}{4}) \leq N_q(x,t) \tag{3.27}$$

for all $x \in X$ and $t > 0$. Then by our assumption

$$M_q(2x,t) = M_q(x, \frac{t}{\alpha}) \;\;\&\;\; N_q(2x,t) = N_q(x, \frac{t}{\alpha}) \tag{3.28}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^k x$ in (3.27) and using (3.28), we obtain

$$\begin{aligned} &\mu(\frac{g(2^{k+1}x)}{4^{k+1}} - \frac{g(2^k x)}{4^k}, \frac{t}{4(4^k)}) \geq M_q(2^k x, t) = M_q(x, \frac{t}{\alpha^k}) \;\;\& \\ &\nu(\frac{g(2^{k+1}x)}{4^{k+1}} - \frac{g(2^k x)}{4^k}, \frac{t}{4(4^k)}) \leq N_q(2^k x, t) = N_q(x, \frac{t}{\alpha^k}) \end{aligned} \tag{3.29}$$

for all $x \in X$, $t > 0$ and $k \geq 0$. Replacing t by $\alpha^k t$ in (3.29), we see that

$$\begin{aligned} &\mu(\frac{g(2^{k+1}x)}{4^{k+1}} - \frac{g(2^k x)}{4^k}, \frac{\alpha^k t}{4(4^k)}) \geq M_q(x,t) \;\;\& \\ &\nu(\frac{g(2^{k+1}x)}{4^{k+1}} - \frac{g(2^k x)}{4^k}, \frac{\alpha^k t}{4(4^k)}) \leq N_q(x,t) \end{aligned} \tag{3.30}$$

for all $x \in X$, $t > 0$ and $k > 0$. It follows from $\frac{g(2^k x)}{4^k} - g(x) = \sum_{j=0}^{k-1}(\frac{g(2^{j+1}x)}{4^{j+1}} - \frac{g(2^j x)}{4^j})$ and (3.30) that

$$\begin{aligned} &\mu(\frac{g(2^k x)}{4^k} - g(x), \sum_{j=0}^{k-1} \frac{\alpha^j t}{4(4)^j}) \geq \prod_{j=0}^{k-1} \mu(\frac{g(2^{j+1}x)}{4^{j+1}} - \frac{g(2^j x)}{4^j}, \frac{\alpha^j t}{4(4)^j}) \geq M_q(x,t) \;\;\& \\ &\nu(\frac{g(2^k x)}{4^k} - g(x), \sum_{j=0}^{k-1} \frac{\alpha^j t}{4(4)^j}) \leq \coprod_{j=0}^{k-1} \nu(\frac{g(2^{j+1}x)}{4^{j+1}} - \frac{g(2^j x)}{4^j}, \frac{\alpha^j t}{4(4)^j}) \leq N_q(x,t) \end{aligned} \tag{3.31}$$

for all $x \in X$, $t > 0$ and $k > 0$, where $\prod_{k=1}^{n} a_k = a_1 * a_2 * ... * a_n$ and $\coprod_{k=1}^{n} a_k = a_1 \star a_2 \star ... \star a_n$. Replacing x by $2^m x$ in (3.31), we observe that

$$\begin{aligned} &\mu(\frac{g(2^{k+m}x)}{4^{k+m}} - \frac{g(2^m x)}{4^m}, \sum_{j=0}^{k-1} \frac{\alpha^j t}{4(4)^{j+m}}) \geq M_q(2^m x, t) = M_q(x, \frac{t}{\alpha^m}) \;\;\& \\ &\nu(\frac{g(2^{k+m}x)}{4^{k+m}} - \frac{g(2^m x)}{4^m}, \sum_{j=0}^{k-1} \frac{\alpha^j t}{4(4)^{j+m}}) \leq N_q(2^m x, t) = N_q(x, \frac{t}{\alpha^m}) \end{aligned}$$

for all $x \in X$, $t > 0$ and all $m \geq 0,\ k > 0$. Hence

$$\mu(\frac{g(2^{k+m}x)}{4^{k+m}} - \frac{g(2^m x)}{4^m}, \sum_{j=m}^{k+m-1} \frac{\alpha^j t}{4(4)^j}) \geq M_q(x,t) \quad \&$$
$$\nu(\frac{g(2^{k+m}x)}{4^{k+m}} - \frac{g(2^m x)}{4^m}, \sum_{j=m}^{k+m-1} \frac{\alpha^j t}{4(4)^j}) \leq N_q(x,t)$$

for all $x \in X$, $t > 0$ and all $m \geq 0,\ k > 0$. By last inequality, we obtain

$$\begin{aligned} \mu(\frac{g(2^{k+m}x)}{4^{k+m}} - \frac{g(2^m x)}{4^m}, t) \geq M_q(x, \frac{t}{\sum_{j=m}^{k+m-1} \frac{\alpha^j}{4(4)^j}}) \quad \& \\ \nu(\frac{g(2^{k+m}x)}{4^{k+m}} - \frac{g(2^m x)}{4^m}, t) \leq N_q(x, \frac{t}{\sum_{j=m}^{k+m-1} \frac{\alpha^j}{4(4)^j}}) \end{aligned} \tag{3.32}$$

for all $x \in X$, $t > 0$ and all $m \geq 0,\ k > 0$. Since $0 < \alpha < 4$ and $\sum_{j=0}^{\infty} (\frac{\alpha}{4})^j < \infty$, the Cauchy criterion for convergence in IFNS shows that $\{\frac{g(2^k x)}{4^k}\}$ is a Cauchy sequence in Y. Since Y is complete, this sequence converges to some point $Q(x) \in Y$. So one can define the function $Q : X \to Y$ by

$$Q(x) = (\mu,\nu) - \lim_{k\to\infty} \frac{1}{4^k} g(2^k x) = (\mu,\nu) - \lim_{k\to\infty} \frac{1}{4^k}(f(2^{k+1}x) - 16 f(2^k x)) \tag{3.33}$$

for all $x \in X$. Fix $x \in X$ and put m=0 in (3.33) to obtain

$$\mu(\frac{g(2^k x)}{4^k} - g(x), t) \geq M_q(x, \frac{t}{\sum_{j=0}^{k-1} \frac{\alpha^j}{4(4)^j}}) \ \ \& \ \ \nu(\frac{g(2^k x)}{4^k} - g(x), t) \leq N_q(x, \frac{t}{\sum_{j=0}^{k-1} \frac{\alpha^j}{4(4)^j}})$$

for all $x \in X$, $t > 0$ and all $k > 0$. From which we obtain

$$\begin{aligned} \mu(Q(x) - g(x), t) &\geq \mu(Q(x) - \frac{g(2^k x)}{4^k}, \frac{t}{2}) * \mu(\frac{g(2^k x)}{4^k} - g(x), \frac{t}{2}) \\ &\geq M_q(x, \frac{t}{\sum_{j=0}^{k-1} \frac{\alpha^j}{2(4)^j}}) \quad \& \\ \nu(Q(x) - g(x), t) &\leq \nu(Q(x) - \frac{g(2^k x)}{4^k}, \frac{t}{2}) \star \nu(\frac{g(2^k x)}{4^k} - g(x), \frac{t}{2}) \\ &\leq N_q(x, \frac{t}{\sum_{j=0}^{k-1} \frac{\alpha^j}{2(4)^j}}) \end{aligned} \tag{3.34}$$

for k large enough. Taking the limit as $k \to \infty$ in (3.35) and using the definition of IFNS, we obtain

$$\mu(Q(x) - g(x), t) \geq M_q(x, \frac{(4-\alpha)t}{2}) \ \ \& \ \ \nu(Q(x) - g(x), t) \leq N_q(x, \frac{(4-\alpha)t}{2}) \tag{3.35}$$

for all $x \in X$ and $t > 0$. On the other hand, we have

$$\begin{aligned}\mu(\frac{Q(2x)}{4} - Q(x), t) &\geq \mu(\frac{Q(2x)}{4} - \frac{g(2^{k+1}x)}{4^{k+1}}, \frac{t}{3}) * \mu(\frac{g(2^k x)}{4^k} - Q(x), \frac{t}{3}) \\ &\quad * \mu(\frac{g(2^{k+1}x)}{4^{k+1}} - \frac{g(2^k x)}{4^k}, \frac{t}{3}) \ \& \\ \nu(\frac{Q(2x)}{4} - Q(x), t) &\leq \nu(\frac{Q(2x)}{4} - \frac{g(2^{k+1}x)}{4^{k+1}}, \frac{t}{3}) \star \nu(\frac{g(2^k x)}{4^k} - Q(x), \frac{t}{3}) \\ &\quad \star \nu(\frac{g(2^{k+1}x)}{4^{k+1}} - \frac{g(2^k x)}{4^k}, \frac{t}{3})\end{aligned}$$

for all $x \in X$ and $t > 0$. So it follows from (3.29) and (3.34) that

$$Q(2x) = 4Q(x) \tag{3.36}$$

for all $x \in X$. By (3.1) and (3.2), we obtain

$$\begin{aligned}\mu(\frac{1}{4^k} Dg(2^k x, 2^k y), t) &= \mu(\frac{1}{4^k} Df(2^{k+1}x, 2^{k+1}y) - \frac{16}{4^k} Df(2^k x, 2^k y), t) \\ &\geq \mu(Df(2^{k+1}x, 2^{k+1}y), \frac{4^k t}{2}) * \mu(Df(2^k x, 2^k y), \frac{4^k t}{32}) \\ &\geq \mu'(\varphi_q(2^{k+1}x, 2^{k+1}y), \frac{4^k t}{2}) * \mu'(\varphi_q(2^k x, 2^k y), \frac{4^k t}{32}) \\ &= \mu'(\varphi_q(x, y), \frac{4^k t}{2\alpha^{k+1}}) * \mu'(\varphi_q(x, y), \frac{4^k t}{32\alpha^k})\end{aligned} \tag{3.37}$$

for all $x, y \in X$ and $t > 0$. Letting $k \to \infty$ in (3.37), we obtain

$$\mu(DQ(x, y), t) = 1$$

for all $x, y \in X$ and $t > 0$. Similarly, we obtain

$$\nu(DQ(x, y), t) = 0.$$

This means that Q satisfies (1.4). Thus by Lemma 3.1, the function $x \rightsquigarrow Q(2x) - 16Q(x)$ is quadratic. Therefore (3.36) implies that the function Q is quadratic.

Now, to prove the uniqueness property of Q, let $Q' : X \to Y$ be another quadratic function satisfying (3.3). It follows from (3.3), (3.28) and (3.36) that

$$\begin{aligned}\mu(Q(x) - Q'(x), t) &= \mu(\frac{Q(2^k x)}{4^k} - \frac{Q'(2^k x)}{4^k}, t) \\ &\geq \mu(\frac{Q(2^k x)}{4^k} - \frac{g(2^k x)}{4^k}, \frac{t}{2}) * \mu(\frac{g(2^k x)}{4^k} - \frac{Q'(2^k x)}{4^k}, \frac{t}{2}) \\ &\geq M_q(2^k x, \frac{4^k(4-\alpha)t}{4}) * M_q(2^k x, \frac{4^k(4-\alpha)t}{4}) \\ &= M_q(x, \frac{4^k(4-\alpha)t}{4\alpha^k}) * M_q(x, \frac{4^k(4-\alpha)t}{4\alpha^k}) \ \& \\ \nu(Q(x) - Q'(x), t) &\leq N_q(2^k x, \frac{4^k(4-\alpha)t}{4}) \star N_q(2^k x, \frac{4^k(4-\alpha)t}{4}) \\ &= N_q(x, \frac{4^k(4-\alpha)t}{4\alpha^k}) \star N_q(x, \frac{4^k(4-\alpha)t}{4\alpha^k})\end{aligned}$$

for all $x \in X$ and $t > 0$. Since $\alpha < 4$, gives $\lim_{k\to\infty} M_q(x, \frac{4^k(4-\alpha)t}{4\alpha^k}) = 1$ and $\lim_{k\to\infty} N_q(x, \frac{4^k(4-\alpha)t}{4\alpha^k}) = 0$. Thus, $\mu(Q(x)-Q^{'}(x), t) = 1$ and $\nu(Q(x)-Q^{'}(x), t) = 0$, therefore $Q(x) = Q^{'}(x)$.

Case (2): $\ell = -1$. We can state the proof in the same pattern as we did in the first case.
Replacing x by $\frac{x}{2}$ in (3.26), we obtain

$$\mu(g(x) - 4g(\frac{x}{2}), t) \geq M_q(\frac{x}{2}, t) \ \ \& \ \ \nu(g(x) - 4g(\frac{x}{2}), t) \leq N_q(\frac{x}{2}, t) \tag{3.38}$$

for all $x \in X$ and $t > 0$. Replacing x and t by $\frac{x}{2^k}$ and $\frac{t}{4^k}$ in (3.38), respectively, we obtain

$$\begin{aligned} &\mu(4^k g(\frac{x}{2^k}) - 4^{k+1}g(\frac{x}{2^{k+1}}), t) \geq M_q(\frac{x}{2^{k+1}}, \frac{t}{4^k}) = M_q(x, (\frac{\alpha}{4})^k \alpha t) \ \ \& \\ &\nu(4^k g(\frac{x}{2^k}) - 4^{k+1}g(\frac{x}{2^{k+1}}), t) \leq N_q(\frac{x}{2^{k+1}}, \frac{t}{4^k}) = N_q(x, (\frac{\alpha}{4})^k \alpha t) \end{aligned}$$

for all $x \in X$, $t > 0$ and all $k > 0$. One can deduce

$$\begin{aligned} &\mu(4^{k+m}g(\frac{x}{2^{k+m}}) - 4^m g(\frac{x}{2^m}), t) \geq M_q(x, \frac{t}{\sum_{j=m+1}^{k+m} \frac{4^j}{4\alpha^j})}) \ \& \\ &\nu(4^{k+m}g(\frac{x}{2^{k+m}}) - 4^m g(\frac{x}{2^m}), t) \leq N_q(x, \frac{t}{\sum_{j=m+1}^{k+m} \frac{4^j}{4\alpha^j})}) \end{aligned} \tag{3.39}$$

for all $x \in X$, $t > 0$ and all $m, k \geq 0$. From which we conclude that $\{4^k g(\frac{x}{2^k})\}$ is a Cauchy sequence in the intuitionistic fuzzy Banach space (Y, μ, ν). Therefore, there is a function $Q : X \longrightarrow Y$ defined by $Q(x) := (\mu, \nu) - \lim_{k\to\infty} 4^k g(\frac{x}{2^k})$. Employing (3.39) with $m = 0$, we obtain

$$\mu(Q(x) - g(x), t) \geq M_q(x, \frac{(\alpha-4)t}{2}) \ \ \& \ \ \nu(Q(x) - g(x), t) \leq N_q(x, \frac{(\alpha-4)t}{2})$$

for all $x \in X$ and $t > 0$. The proof for uniqueness of Q for this case proceeds similarly to that in the previous case, hence it is omitted. □

4. Intuitionistic fuzzy stability of (1.4): for quartic functions

In this section, we prove the generalized Hyers–Ulam stability of the functional equation (1.4) in intuitionistic fuzzy Banach space for quartic functions.

Lemma 4.1. ([38]). *Let V_1 and V_2 be real vector spaces. If a function $f : V_1 \to V_2$ satisfies (1.4), then the function $h : V_1 \to V_2$ defined by $h(x) = f(2x) - 4f(x)$ is quartic.*

Theorem 4.2. *Let $\ell \in \{-1, 1\}$ be fixed and let $\varphi_v : X \times X \longrightarrow Z$ be a function such that*

$$\varphi_v(2x, 2y) = \alpha\varphi_v(x, y) \tag{4.1}$$

for all $x, y \in X$ and for some positive real number α with $\alpha\ell < 16\ell$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies

$$\mu\left(Df(x, y), t\right) \geq \mu^{'}\left(\varphi_v(x, y), t\right) \ \ \& \ \ \nu\left(Df(x, y), t\right) \leq \nu^{'}\left(\varphi_v(x, y), t\right) \tag{4.2}$$

for all $x, y \in X$ and $t > 0$. Then the limit

$$V(x) = (\mu, \nu) - \lim_{k\to\infty} \frac{1}{16^{\ell k}} \left(f(2^{\ell k+1}x) - 4f(2^{\ell k}x)\right)$$

exists for all $x \in X$ and $V : X \to Y$ is a unique quartic function such that

$$\begin{aligned} &\mu\left(f(2x) - 4f(x) - V(x), t\right) \geq M_v\left(x, \frac{\ell(16-\alpha)}{2}t\right) \ \ \& \\ &\nu\left(f(2x) - 4f(x) - V(x), t\right) \leq N_v\left(x, \frac{\ell(16-\alpha)}{2}t\right) \end{aligned} \tag{4.3}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned} M_v(x,t) = {} & \mu^{'}(\varphi_v(x,(n+2)x), \frac{n^2(n^2-1)t}{17}) \\ & * \mu^{'}(\varphi_v(x,(n-2)x), \frac{n^2(n^2-1)t}{17}) * \mu^{'}(\varphi_v(x,(n+1)x), \frac{n^2(n^2-1)t}{68}) \\ & * \mu^{'}(\varphi_v(x,(n-1)x), \frac{n^2(n^2-1)t}{68}) * \mu^{'}(\varphi_v(x,nx), \frac{n^2(n^2-1)t}{170}) \\ & * \mu^{'}(\varphi_v(2x,2x), \frac{n^2(n^2-1)t}{17}) * \mu^{'}(\varphi_v(2x,x), \frac{n^2(n^2-1)t}{68}) \\ & * \mu^{'}(\varphi_v(x,3x), \frac{(n^2-1)t}{17}) * \mu^{'}(\varphi_v(x,2x), \frac{n^2(n^2-1)t}{28(3n^2-1)}) \\ & * \mu^{'}(\varphi_v(x,x), \frac{n^2(n^2-1)t}{17(17n^2-8)}) * \mu^{'}(\varphi_v(0,(n+1)x), \frac{(n^2-1)^2t}{17}) \\ & * \mu^{'}(\varphi_v(0,(n-3)x), \frac{(n^2-1)^2t}{17}) * \mu^{'}(\varphi_v(0,(n-1)x), \frac{(n^2-1)^2t}{170}) \\ & * \mu^{'}(\varphi_v(0,nx), \frac{(n^2-1)^2t}{68}) * \mu^{'}(\varphi_v(0,(n-2)x), \frac{(n^2-1)^2t}{68}) \\ & * \mu^{'}(\varphi_v(0,2x), \frac{n^2(n^2-1)^2t}{17(n^4+1)}) * \mu^{'}(\varphi_v(0,x), \frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}) \ \ \& \end{aligned}$$

$$\begin{aligned}
N_v(x,t) = {} & \nu'(\varphi_v(x,(n+2)x), \frac{n^2(n^2-1)t}{17}) \\
& \star \nu'(\varphi_v(x,(n-2)x), \frac{n^2(n^2-1)t}{17}) \star \nu'(\varphi_v(x,(n+1)x), \frac{n^2(n^2-1)t}{68}) \\
& \star \nu'(\varphi_v(x,(n-1)x), \frac{n^2(n^2-1)t}{68}) \star \nu'(\varphi_v(x,nx), \frac{n^2(n^2-1)t}{170}) \\
& \star \nu'(\varphi_v(2x,2x), \frac{n^2(n^2-1)t}{17}) \star \nu'(\varphi_v(2x,x), \frac{n^2(n^2-1)t}{68}) \\
& \star \nu'(\varphi_v(x,3x), \frac{(n^2-1)t}{17}) \star \nu'(\varphi_v(x,2x), \frac{n^2(n^2-1)t}{28(3n^2-1)}) \\
& \star \nu'(\varphi_v(x,x), \frac{n^2(n^2-1)t}{17(17n^2-8)}) \star \nu'(\varphi_v(0,(n+1)x), \frac{(n^2-1)^2t}{17}) \\
& \star \nu'(\varphi_v(0,(n-3)x), \frac{(n^2-1)^2t}{17}) \star \nu'(\varphi_v(0,(n-1)x), \frac{(n^2-1)^2t}{170}) \\
& \star \nu'(\varphi_v(0,nx), \frac{(n^2-1)^2t}{68}) \star \nu'(\varphi_v(0,(n-2)x), \frac{(n^2-1)^2t}{68}) \\
& \star \nu'(\varphi_v(0,2x), \frac{n^2(n^2-1)^2t}{17(n^4+1)}) \star \nu'(\varphi_v(0,x), \frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}).
\end{aligned}$$

Proof. Case (1): $\ell = 1$. Similar to the proof Theorem 3.2, we have

$$\begin{aligned}
&\mu(f(4x) - 20f(2x) + 64f(x), t) \geq M_v(x,t) \ \ \& \\
&\nu(f(4x) - 20f(2x) + 64f(x), t) \leq N_v(x,t)
\end{aligned} \tag{4.4}$$

for all $x \in X$ and $t > 0$. Let $h : X \to Y$ be a function defined by $h(x) := f(2x) - 4f(x)$ for all $x \in X$. From (4.4), we conclude that

$$\mu(h(2x) - 16h(x), t) \geq M_v(x,t) \ \ \& \ \ \nu(h(2x) - 16h(x), t) \leq N_v(x,t) \tag{4.5}$$

for all $x \in X$ and $t > 0$. So

$$\mu(\frac{h(2x)}{16} - h(x), \frac{t}{16}) \geq M_v(x,t) \ \ \& \ \ \nu(\frac{h(2x)}{16} - h(x), \frac{t}{16}) \leq N_v(x,t) \tag{4.6}$$

for all $x \in X$ and $t > 0$. Then by our assumption

$$M_v(2x,t) = M_v(x, \frac{t}{\alpha}) \ \ \& \ \ N_v(2x,t) = N_v(x, \frac{t}{\alpha}) \tag{4.7}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^k x$ in (4.6) and using (4.7), we obtain

$$\begin{aligned}
&\mu(\frac{h(2^{k+1}x)}{16^{k+1}} - \frac{h(2^k x)}{16^k}, \frac{t}{16(16^k)}) \geq M_v(2^k x, t) = M_v(x, \frac{t}{\alpha^k}) \ \& \\
&\nu(\frac{h(2^{k+1}x)}{16^{k+1}} - \frac{h(2^k x)}{16^k}, \frac{t}{16(16^k)}) \leq N_v(2^k x, t) = N_v(x, \frac{t}{\alpha^k})
\end{aligned} \tag{4.8}$$

for all $x \in X$, $t > 0$ and $k \geq 0$. Replacing t by $\alpha^k t$ in (4.8), we see that

$$\begin{aligned}
&\mu(\frac{h(2^{k+1}x)}{16^{k+1}} - \frac{h(2^k x)}{16^k}, \frac{\alpha^k t}{16(16^k)}) \geq M_v(x,t) \ \ \& \\
&\nu(\frac{h(2^{k+1}x)}{16^{k+1}} - \frac{h(2^k x)}{16^k}, \frac{\alpha^k t}{16(16^k)}) \leq N_v(x,t)
\end{aligned} \tag{4.9}$$

for all $x \in X$, $t > 0$ and $k > 0$. It follows from $\frac{h(2^kx)}{16^k} - h(x) = \sum_{j=0}^{k-1}(\frac{h(2^{j+1}x)}{16^{j+1}} - \frac{h(2^jx)}{16^j})$ and (4.9) that

$$\begin{aligned} \mu(\frac{h(2^kx)}{16^k} - h(x), \sum_{j=0}^{k-1}\frac{\alpha^j t}{16(16)^j}) &\geq \prod_{j=0}^{k-1}\mu(\frac{h(2^{j+1}x)}{16^{j+1}} - \frac{h(2^jx)}{16^j}, \frac{\alpha^j t}{16(16)^j}) \\ &\geq M_v(x,t) \quad \& \\ \nu(\frac{h(2^kx)}{16^k} - h(x), \sum_{j=0}^{k-1}\frac{\alpha^j t}{16(16)^j}) &\leq \coprod_{j=0}^{k-1}\nu(\frac{h(2^{j+1}x)}{16^{j+1}} - \frac{h(2^jx)}{16^j}, \frac{\alpha^j t}{16(16)^j}) \\ &\leq N_v(x,t) \end{aligned} \tag{4.10}$$

for all $x \in X$, $t > 0$ and $k > 0$. Replacing x by $2^m x$ in (4.10), we observe that

$$\mu(\frac{h(2^{k+m}x)}{16^{k+m}} - \frac{h(2^mx)}{16^m}, \sum_{j=0}^{k-1}\frac{\alpha^j t}{16(16)^{j+m}}) \geq M_v(2^mx,t) = M_v(x, \frac{t}{\alpha^m}) \ \&$$

$$\nu(\frac{h(2^{k+m}x)}{16^{k+m}} - \frac{h(2^mx)}{16^m}, \sum_{j=0}^{k-1}\frac{\alpha^j t}{16(16)^{j+m}}) \leq N_v(2^mx,t) = N_v(x, \frac{t}{\alpha^m})$$

for all $x \in X$, $t > 0$ and all $m \geq 0,\ k > 0$. Hence

$$\mu(\frac{h(2^{k+m}x)}{16^{k+m}} - \frac{h(2^mx)}{16^m}, \sum_{j=m}^{k+m-1}\frac{\alpha^j t}{16(16)^j}) \geq M_v(x,t) \quad \&$$

$$\nu(\frac{h(2^{k+m}x)}{16^{k+m}} - \frac{h(2^mx)}{16^m}, \sum_{j=m}^{k+m-1}\frac{\alpha^j t}{16(16)^j}) \leq N_v(x,t)$$

for all $x \in X$, $t > 0$ and all $m \geq 0,\ k > 0$. By last inequality, we obtain

$$\begin{aligned} \mu(\frac{h(2^{k+m}x)}{16^{k+m}} - \frac{h(2^mx)}{16^m}, t) &\geq M_v(x, \frac{t}{\sum_{j=m}^{k+m-1}\frac{\alpha^j}{16(16)^j})}) \ \& \\ \nu(\frac{h(2^{k+m}x)}{16^{k+m}} - \frac{h(2^mx)}{16^m}, t) &\leq N_v(x, \frac{t}{\sum_{j=m}^{k+m-1}\frac{\alpha^j}{16(16)^j})}) \end{aligned} \tag{4.11}$$

for all $x \in X$, $t > 0$ and all $m \geq 0,\ k > 0$. Since $0 < \alpha < 16$ and $\sum_{j=0}^{\infty}(\frac{\alpha}{16})^j < \infty$, the Cauchy criterion for convergence in IFNS shows that $\{\frac{h(2^kx)}{16^k}\}$ is a Cauchy sequence in Y. Since Y is complete, this sequence converges to some point $V(x) \in Y$. So one can define the function $V : X \to Y$ by

$$V(x) = (\mu,\nu) - \lim_{k\to\infty}\frac{1}{16^k}h(2^kx) = (\mu,\nu) - \lim_{k\to\infty}\frac{1}{16^k}(f(2^{k+1}x) - 4f(2^kx)) \tag{4.12}$$

for all $x \in X$. Fix $x \in X$ and put m=0 in (4.11) to obtain

$$\mu(\frac{h(2^kx)}{16^k} - h(x), t) \geq M_v(x, \frac{t}{\sum_{j=0}^{k-1}\frac{\alpha^j}{16(16)^j})}) \quad \&$$

$$\nu(\frac{h(2^kx)}{16^k} - h(x), t) \leq N_v(x, \frac{t}{\sum_{j=0}^{k-1}\frac{\alpha^j}{16(16)^j})})$$

for all $x \in X$, $t > 0$ and all $k > 0$. From which we obtain

$$\begin{aligned} \mu(V(x) - h(x), t) &\geq \mu(V(x) - \frac{h(2^k x)}{16^k}, \frac{t}{2}) * \mu(\frac{h(2^k x)}{16^k} - h(x), \frac{t}{2}) \\ &\geq M_v(x, \frac{t}{\sum_{j=0}^{k-1} \frac{\alpha^j}{8(16)^j}}) \quad \& \\ \nu(V(x) - h(x), t) &\leq \nu(V(x) - \frac{h(2^k x)}{16^k}, \frac{t}{2}) \star \nu(\frac{h(2^k x)}{16^k} - h(x), \frac{t}{2}) \\ &\leq N_v(x, \frac{t}{\sum_{j=0}^{k-1} \frac{\alpha^j}{8(16)^j}}) \end{aligned} \tag{4.13}$$

for k large enough. Taking the limit as $k \to \infty$ in (4.13) and using the definition of IFNS, we obtain

$$\begin{aligned} \mu(V(x) - h(x), t) &\geq M_v(x, \frac{(16-\alpha)t}{2}) \quad \& \\ \nu(V(x) - h(x), t) &\leq N_v(x, \frac{(16-\alpha)t}{2}) \end{aligned} \tag{4.14}$$

for all $x \in X$ and $t > 0$. On the other hand, we have

$$\begin{aligned} \mu(\frac{V(2x)}{16} - V(x), t) &\geq \mu(\frac{V(2x)}{16} - \frac{h(2^{k+1}x)}{16^{k+1}}, \frac{t}{3}) * \mu(\frac{h(2^k x)}{16^k} - V(x), \frac{t}{3}) \\ &\quad * \mu(\frac{h(2^{k+1}x)}{16^{k+1}} - \frac{h(2^k x)}{16^k}, \frac{t}{3}) \quad \& \\ \nu(\frac{V(2x)}{16} - V(x), t) &\leq \nu(\frac{V(2x)}{16} - \frac{h(2^{k+1}x)}{16^{k+1}}, \frac{t}{3}) \star \nu(\frac{h(2^k x)}{16^k} - V(x), \frac{t}{3}) \\ &\quad \star \nu(\frac{h(2^{k+1}x)}{16^{k+1}} - \frac{h(2^k x)}{16^k}, \frac{t}{3}) \end{aligned}$$

for all $x \in X$ and $t > 0$. So it follows from (4.8) and (4.12) that

$$V(2x) = 16V(x) \tag{4.15}$$

for all $x \in X$. By (4.1) and (4.2), we obtain

$$\begin{aligned} \mu(\frac{1}{16^k} Dh(2^k x, 2^k y), t) &= \mu(\frac{1}{16^k} Df(2^{k+1}x, 2^{k+1}y) - \frac{4}{16^k} Df(2^k x, 2^k y), t) \\ &\geq \mu(Df(2^{k+1}x, 2^{k+1}y), \frac{16^k t}{2}) * \mu(Df(2^k x, 2^k y), \frac{16^k t}{8}) \\ &\geq \mu'(\varphi_v(2^{k+1}x, 2^{k+1}y), \frac{16^k t}{2}) * \mu'(\varphi_v(2^k x, 2^k y), \frac{16^k t}{8}) \\ &= \mu'(\varphi_v(x, y), \frac{16^k t}{2\alpha^{k+1}}) * \mu'(\varphi_v(x, y), \frac{16^k t}{8\alpha^k}) \end{aligned} \tag{4.16}$$

for all $x, y \in X$ and $t > 0$. Letting $k \to \infty$ in (4.16), we obtain

$$\mu(DV(x, y), t) = 1$$

for all $x, y \in X$ and $t > 0$. Similarly, we obtain

$$\nu(DV(x, y), t) = 0.$$

This means that V satisfies (1.4). Thus by Lemma 4.1, the function $x \rightsquigarrow V(2x) - 4V(x)$ is quartic. Therefore (4.15) implies that the function V is quartic.

The rest of the proof is similar to the proof of Theorem 3.2 and we omit the details. □

5. Intuitionistic fuzzy stability of (1.4)

In this section, we prove the main results concerning the generalized Hyers–Ulam stability of a mixed quadratic and quartic functional equation (1.4) in IFNS.

Lemma 5.1. ([38]). *A function $f : V_1 \to V_1$ satisfies (1.4) for all $x, y \in V_1$ if and only if there exist a unique symmetric bi–additive function $B_1 : V_1 \times V_1 \longrightarrow V_2$ and a unique symmetric bi–quadratic function $B_2 : V_1 \times V_1 \longrightarrow V_2$ such that $f(x) = B_1(x, x) + B_2(x, x)$ for all $x \in V_1$.*

Theorem 5.2. *Let $\varphi : X \times X \longrightarrow Z$ be a function such that*

$$\varphi(2x, y) = \alpha\varphi(x, y) \tag{5.1}$$

for all $x, y \in X$ and for some positive real number α. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\mu\left(Df(x, y), t\right) \geq \mu^{'}\left(\varphi(x, y), t\right) \quad \& \quad \nu\left(Df(x, y), t\right) \leq \nu^{'}\left(\varphi(x, y), t\right) \tag{5.2}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic function $Q : X \to Y$ and a unique quartic function $V : X \to Y$ such that satisfying

$$\mu(f(x) - Q(x) - V(x), t) \geq \begin{cases} M(x, 3t(4 - \alpha)) * M(x, 3t(16 - \alpha)), & 0 < \alpha < 4 \\ M(x, 3t(\alpha - 4)) * M(x, 3t(16 - \alpha)), & 4 < \alpha < 16 \\ M(x, 3t(\alpha - 4)) * M(x, 3t(\alpha - 16)), & \alpha > 16 \end{cases} \quad \&$$
$$\nu(f(x) - Q(x) - V(x), t) \leq \begin{cases} N(x, 3t(4 - \alpha)) \star N(x, 3t(16 - \alpha)), & 0 < \alpha < 4 \\ N(x, 3t(\alpha - 4)) \star N(x, 3t(16 - \alpha)), & 4 < \alpha < 16 \\ N(x, 3t(\alpha - 4)) \star N(x, 3t(\alpha - 16)), & \alpha > 16 \end{cases} \tag{5.3}$$

for all $x \in X$ *and* $t > 0$, *where*

$$\begin{aligned}
M(x,t) = {} & \mu^{'}(\varphi(x,(n+2)x), \frac{n^2(n^2-1)t}{17}) \\
& * \mu^{'}(\varphi(x,(n-2)x), \frac{n^2(n^2-1)t}{17}) * \mu^{'}(\varphi(x,(n+1)x), \frac{n^2(n^2-1)t}{68}) \\
& * \mu^{'}(\varphi(x,(n-1)x), \frac{n^2(n^2-1)t}{68}) * \mu^{'}(\varphi(x,nx), \frac{n^2(n^2-1)t}{170}) \\
& * \mu^{'}(\varphi(2x,2x), \frac{n^2(n^2-1)t}{17}) * \mu^{'}(\varphi(2x,x), \frac{n^2(n^2-1)t}{68}) \\
& * \mu^{'}(\varphi(x,3x), \frac{(n^2-1)t}{17}) * \mu^{'}(\varphi(x,2x), \frac{n^2(n^2-1)t}{28(3n^2-1)}) \\
& * \mu^{'}(\varphi(x,x), \frac{n^2(n^2-1)t}{17(17n^2-8)}) * \mu^{'}(\varphi(0,(n+1)x), \frac{(n^2-1)^2t}{17}) \\
& * \mu^{'}(\varphi(0,(n-3)x), \frac{(n^2-1)^2t}{17}) * \mu^{'}(\varphi(0,(n-1)x), \frac{(n^2-1)^2t}{170}) \\
& * \mu^{'}(\varphi(0,nx), \frac{(n^2-1)^2t}{68}) * \mu^{'}(\varphi(0,(n-2)x), \frac{(n^2-1)^2t}{68}) \\
& * \mu^{'}(\varphi(0,2x), \frac{n^2(n^2-1)^2t}{17(n^4+1)}) * \mu^{'}(\varphi(0,x), \frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}) \quad \&
\end{aligned}$$

$$\begin{aligned}
N(x,t) = {} & \nu^{'}(\varphi(x,(n+2)x), \frac{n^2(n^2-1)t}{17}) \\
& \star \nu^{'}(\varphi(x,(n-2)x), \frac{n^2(n^2-1)t}{17}) \star \nu^{'}(\varphi(x,(n+1)x), \frac{n^2(n^2-1)t}{68}) \\
& \star \nu^{'}(\varphi(x,(n-1)x), \frac{n^2(n^2-1)t}{68}) \star \nu^{'}(\varphi(x,nx), \frac{n^2(n^2-1)t}{170}) \\
& \star \nu^{'}(\varphi(2x,2x), \frac{n^2(n^2-1)t}{17}) \star \nu^{'}(\varphi(2x,x), \frac{n^2(n^2-1)t}{68}) \\
& \star \nu^{'}(\varphi(x,3x), \frac{(n^2-1)t}{17}) \star \nu^{'}(\varphi(x,2x), \frac{n^2(n^2-1)t}{28(3n^2-1)}) \\
& \star \nu^{'}(\varphi(x,x), \frac{n^2(n^2-1)t}{17(17n^2-8)}) \star \nu^{'}(\varphi(0,(n+1)x), \frac{(n^2-1)^2t}{17}) \\
& \star \nu^{'}(\varphi(0,(n-3)x), \frac{(n^2-1)^2t}{17}) \star \nu^{'}(\varphi(0,(n-1)x), \frac{(n^2-1)^2t}{170}) \\
& \star \nu^{'}(\varphi(0,nx), \frac{(n^2-1)^2t}{68}) \star \nu^{'}(\varphi(0,(n-2)x), \frac{(n^2-1)^2t}{68}) \\
& \star \nu^{'}(\varphi(0,2x), \frac{n^2(n^2-1)^2t}{17(n^4+1)}) \star \nu^{'}(\varphi(0,x), \frac{n^2(n^2-1)^2t}{28(3n^4-n^2+2)}).
\end{aligned}$$

Proof. Case (1): $0 < \alpha < 4$. By Theorems 3.2 and 4.2, there exists a quadratic function $Q_0 : X \to Y$ and a quartic function $V_0 : X \to Y$ such that

$$\begin{aligned} &\mu(f(2x) - 16f(x) - Q_0(x), t) \geq M(x, \frac{t(4-\alpha)}{2}) \quad \& \\ &\nu(f(2x) - 16f(x) - Q_0(x), t) \leq N(x, \frac{t(4-\alpha)}{2}) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\mu(f(2x) - 4f(x) - V_0(x), t) \geq M(x, \frac{t(16-\alpha)}{2}) \quad \& \\ &\nu(f(2x) - 4f(x) - V_0(x), t) \leq N(x, \frac{t(16-\alpha)}{2}) \end{aligned} \tag{5.5}$$

for all $x \in X$ and $t > 0$. It follows from (5.4) and (5.5) that

$$\begin{aligned} &\mu(f(x) + \frac{1}{12}Q_0(x) - \frac{1}{12}V_0(x), t) \geq M(x, 3t(4-\alpha)) * M(x, 3t(16-\alpha)) \quad \& \\ &\nu(f(x) + \frac{1}{12}Q_0(x) - \frac{1}{12}V_0(x), t) \leq N(x, 3t(4-\alpha)) \star N(x, 3t(16-\alpha)) \end{aligned} \tag{5.6}$$

for all $x \in X$ and $t > 0$. Letting $Q(x) = -\frac{1}{12}Q_0(x)$ and $V(x) = \frac{1}{12}V_0(x)$ in (5.6), we obtain

$$\begin{aligned} &\mu(f(x) - Q(x) - V(x), t) \geq M(x, 3t(4-\alpha)) * M(x, 3t(16-\alpha)) \quad \& \\ &\nu(f(x) - Q(x) - V(x), t) \leq N(x, 3t(4-\alpha)) \star N(x, 3t(16-\alpha)) \end{aligned} \tag{5.7}$$

for all $x \in X$ and $t > 0$.

To prove the uniqueness of Q and V, let $Q', V' : X \to Y$ be another quadratic and quartic functions satisfying (5.7). Let $\overline{Q} = Q - Q'$ and $\overline{V} = V - V'$. So

$$\begin{aligned} \mu(\overline{Q}(x) + \overline{V}(x), t) &\geq \mu(f(x) - Q(x) - V(x), \frac{t}{2}) * \mu(f(x) - Q'(x) - V'(x), \frac{t}{2})) \\ &\geq M(x, \frac{3t(4-\alpha)}{2}) * M(x, \frac{3t(16-\alpha)}{2}) * M(x, \frac{3t(4-\alpha)}{2}) \\ &* M(x, \frac{3t(16-\alpha)}{2}) \end{aligned}$$

for all $x \in X$ and $t > 0$. Therefore it follows from the last inequalities that

$$\begin{aligned} &\mu(\overline{Q}(2^k x) + \overline{V}(2^k x), 16^k t) \\ &\geq M(2^k x, \frac{3(4-\alpha)16^k t}{2}) * M(2^k x, \frac{3(16-\alpha)16^k t}{2}) \\ &* M(2^k x, \frac{3(4-\alpha)16^k t}{2}) * M(2^k x, \frac{3(16-\alpha)16^k t}{2}) \\ &= M(x, \frac{3(4-\alpha)16^k t}{\alpha^k}) * M(x, \frac{3(16-\alpha)16^k t}{\alpha^k}) \\ &* M(x, \frac{3(4-\alpha)16^k t}{\alpha^k}) * M(x, \frac{3(16-\alpha)16^k t}{\alpha^k}) \end{aligned}$$

for all $x \in X$ and $t > 0$. So

$$\lim_{k\to\infty} \mu(\frac{1}{16^k}(\overline{Q}(2^k x) + \overline{V}(2^k x)), t) = 1$$

for all $x \in X$ and $t > 0$. Similarly, we obtain

$$\lim_{k\to\infty} \nu(\frac{1}{16^k}(\overline{Q}(2^k x) + \overline{V}(2^k x)), t) = 0.$$

Hence $\overline{V} = 0$ and then $\overline{Q} = 0$.

The proof for Case (2): $4 < \alpha < 16$ and Case (3): $\alpha > 16$ proceeds similarly to that in the Case (1). □

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