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Existence results for a fractional differential system with integral boundary conditions in the derivative Banach space

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Abstract

In this study, we focus on the existence of a solution for a fractional differential system with integral boundary conditions in specific fractional derivative Banach space. We establish the existence of a solution by using the Schauder fixed point theorem.

Keywords: Fractional differential system; Fractional derivative Banach space, Green's function, Fixed-point theorem

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1 Introduction

In this paper, we consider the following nonlinear system involving the Caputo's derivative

$$\begin{cases} {}^cD_0^\alpha x(t) &= f\left(t, y(t), {}^c D_0^\gamma y(t)\right) \\ {}^cD_0^\beta y(t) &= g\left(t, x(t), {}^c D_0^\gamma x(t)\right) \\ x(0)+x'(0) &= \displaystyle\int_0^1 p_1(x(s))ds \qquad x''(0)=0 \\ x(1)+x'(1) &= \displaystyle\int_0^1 p_2(x(s))ds \\ y(0)+y'(0) &= \displaystyle\int_0^1 q_1(y(s))ds \qquad y''(0)=0 \\ y(1)+y'(1) &= \displaystyle\int_0^1 q_2(y(s))ds, \end{cases} \tag{1.1}$$

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where $2 < \alpha, \beta \leq 3; 0 < \gamma \leq 1$, $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are given continous functions, $p_1, p_2, q_1, q_2 : \mathbb{R} \longrightarrow \mathbb{R}$ are given functions such that $p_1(0) = p_2(0) = q_1(0) = q_2(0) = 0$ and ${}^cD_0^\alpha$ is the standard Caputo derivative.

Recently, differential equations and systems of fractional order occupies an important place in the current researches. In fact, they model various phenomena in many fields of science and engineering as in electromagnetic, control, electrochemistry, viscoelasticity, porous media, fluid electrical, probability and statistic, etc. For details see [6, 7, 8, 11, 12, 14, 15] and we refer [10] for more properties on the fractional differential calculus.

Existence results for fractional differential system like (1.1) put into consideration large investigations, we would mention for example the work of Xinwei Su [16] and Bashir Ahmad et al. [1] and [2].

In our investigation, we have extended the result obtained by Djalal Boucenna et al. [4] to the system, all keeping the same space of study.

Our article is structured as follows: the Section 1 is devoted to basic notations and theorems. In Section 2, we will give our main result and proofs. An example is given in the last section to illustrate our result.

2 Preliminary Knowledge

In order to deal with the system (1.1), let us recall some basic definitions and theories of fractional calculus.

Definition 2.1. The Riemann-Liouville fractional integral of order α of the function $f : [0, \infty] \longrightarrow \mathbb{R}$ is defined as

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the integral exists.

Definition 2.2. The Caputo fractional derivative of order α of the function $f : [0, \infty] \longrightarrow \mathbb{R}$ is defined as

$${}^cD_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = I^{n-\alpha} f^{(n)}(t),$$

such that $n-1 < \alpha \leq n$ and $n = [\alpha] + 1$ where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3. (See [3]) For $\alpha > 0$, the general solution of the fractional differential equation ${}^cD_0^\alpha x(t) = 0$ is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \;\; c_i \in \mathbb{R} \;\text{ and }\; i = 0, 1, ..., n-1.$$

In view of Lemma 2.3, it follows that

$$I^\alpha\, {}^cD_0^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $\;i = 0, 1, ..., n-1\;$ and $\;n = [\alpha] + 1$.

Lemma 2.4. (See [5]) Let be $\mathbf{F}$ a bounded set in $L^p[0,1]$, with $0 \leq p < \infty$. Assume that

$$\lim_{|h| \to 0} \|\tau_h f - f\| = 0 \;\text{ uniformly on }\; \mathbf{F}.$$

Then, $\mathbf{F}$ is relatively compact in $L^p[0,1]$.

In what follows, we give some fundamental properties of the fractional derivative Banach space.

Definition 2.5. Let $0 < \gamma \leq 1$ and $1 < p < \infty$. The fractional derivative space $E^{\gamma,p}$ is defined by the closure of $C^\infty([0,1])$ equipped with the norm

$$\|u\|_{\gamma,p} = \left(\int_0^1 |u(t)|^p dt + \int_0^1 |{}^cD_0^\alpha u(t)|^p dt \right)^{\frac{1}{p}}.$$

Proposition 2.6. (See [9, 13]) The fractional derivative space $E^{\gamma,p}$ is a reflexive and separable Banach space.

Lemma 2.7. (See [9, 13]) For all $u \in E^{\gamma,p}$, we have

$$\|u\|_p \leq \frac{1}{\Gamma(\gamma+1)} \|^c D_0^\gamma u\|_p.$$

So, we can introduce the equivalent norm in $E^{\gamma,p}$ by

$$\|u\|_{\gamma,p} = \|^c D_0^\gamma u\|_p = \left(\int_0^1 |^c D_0^\alpha u(t)|^p dt \right)^{\frac{1}{p}}, \quad u \in E^{\gamma,p}.$$

In the sequel, X denotes the product space $E^{\gamma,p} \times E^{\gamma,q}$ endowed with the norm

$$\|(x,y)\|_X = \max\left(\|x\|_{E^{\gamma,p}}, \|y\|_{E^{\gamma,q}}\right).$$

3 Main results and proofs

Let us start by presenting the green's function for the system (1.1).

Lemma 3.1. For any given functions ϕ, p_1 and $p_2 \in C[0,1]$, the solution of the boundary value problem

$$\begin{cases} {}^c D_0^\alpha x(t) &= \phi(t), \\ x(0) + x'(0) &= \displaystyle\int_0^1 p_1(s) ds, \\ x(1) + x'(1) &= \displaystyle\int_0^1 p_2(s) ds, \\ x''(0) = 0, \end{cases} \tag{3.1}$$

is given by

$$x(t) = \int_0^1 G_1(t,s)\phi(s) + (2-t) \int_0^1 p_1(s) ds + (t-1) \int_0^1 p_2(s) ds,$$

where $G_1(t,s)$ is the Green's function given by the expression

$$G_1(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1} + (1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & \text{if } 0 \leq s \leq t \leq 1, \\ \dfrac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

Moreover, we set

$$G_2(t,s) = \begin{cases} \dfrac{(t-s)^{\beta-1} + (1-t)(1-s)^{\beta-1}}{\Gamma(\beta)} + \dfrac{(1-t)(1-s)^{\beta-2}}{\Gamma(\beta-1)} & \text{if } 0 \leq s \leq t \leq 1, \\ \dfrac{(1-t)(1-s)^{\beta-1}}{\Gamma(\beta)} + \dfrac{(1-t)(1-s)^{\beta-2}}{\Gamma(\beta-1)} & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

then, $G_1(t,s), G_2(t,s)$ are called Green's functions of the boundary value problem (1.1).

Proof . In view of Lemma 2.3, the first equation in (3.1) is reduced to an equivalent integral equation

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi(s) + c_0 + c_1 t + c_2 t^2,$$

for some $c_0, c_1, c_2 \in \mathbb{R}$. Since $x''(0) = 0$,we get $c_2 = 0$. Applying the boundary conditions, we find that

$$c_0 = 2\int_0^1 p_1(s)ds - \int_0^1 p_2(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi(s) + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi(s)$$

and

$$c_1 = \int_0^1 p_2(s)ds - \int_0^1 p_1(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi(s) - \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\phi(s).$$

Hence the solution of (3.1) is given by

$$\begin{aligned} x(t) &= \int_0^t \left[\frac{(t-s)^{\alpha-1} + (1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\phi(s)ds \\ &\int_t^1 \left[\frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\phi(s)ds + (2-t)\int_0^1 p_1(s)ds + (t-1)\int_0^1 p_2(s)ds. \end{aligned} \tag{3.2}$$

So,

$$x(t) = \int_0^1 G_1(t,s)\phi(s)ds + (2-t)\int_0^1 p_1(s)ds + (t-1)\int_0^1 p_2(s)ds. \tag{3.3}$$

In the same way, if

$$\begin{cases} {}^cD_0^\alpha y(t) = \psi(t), \\ y(0) + y'(0) = \int_0^1 q_1(s)ds, \\ y(1) + x'(1) = \int_0^1 q_2(s)ds, \\ y''(0) = 0, \end{cases} \tag{3.4}$$

then, we obtain

$$y(t) = \int_0^1 G_2(t,s)\psi(s)ds + (2-t)\int_0^1 q_1(s)ds + (t-1)\int_0^1 q_2(s)ds. \tag{3.5}$$

□

Thereafter, we use the notations $G_1^\star = \max_{[0,1]\times[0,1]} G_1(t,s)$; $G_2^\star = \max_{[0,1]\times[0,1]} G_2(t,s)$.

Lemma 3.2. (See [4]) The functions $G_1, G_2, \frac{\partial^\gamma G_1}{\partial t}, \frac{\partial^\gamma G_2}{\partial t}$ are continuous and satisfy, for all $t, s \in [0,1]$,

1. $|G_1(t,s)| \leq \frac{3}{\Gamma(\alpha-1)}$.
2. $|G_2(t,s)| \leq \frac{3}{\Gamma(\alpha-1)}$.
3. $\left|\frac{\partial^\gamma G_1}{\partial t}\right| \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} + \frac{2}{\Gamma(2-\gamma)\Gamma(\alpha-1)}$.
4. $\left|\frac{\partial^\gamma G_2}{\partial t}\right| \leq \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)} + \frac{2}{\Gamma(2-\gamma)\Gamma(\beta-1)}$
5. ${}^cD_0^\gamma(t-1) = -\frac{1}{\Gamma(2-\gamma)}t^{1-\gamma}$, ${}^cD_0^\gamma(2-t) = -\frac{1}{\Gamma(2-\gamma)}t^{1-\gamma}$.

Lemma 3.3. Suppose that $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. Then, $(x, y) \in X$ is solution of the system (1.1) if and only if (x, y) is a fixe point of the operator defined as

$$\begin{array}{rcl} T : X & \to & X \\ (x(t), y(t)) & \mapsto & \left(T_1\left(x(t), y(t)\right), T_2\left(x(t), y(t)\right)\right) \end{array}$$

where

$$T_1\left(x(t), y(t)\right) = \int_0^1 G_1(t,s) f\left(s, y(s), {}^c D_0^\gamma y(s)\right)(s) ds + (2-t) \int_0^1 p_1(x(s)) ds + (t-1) \int_0^1 p_2(x(s)) ds$$

and

$$T_2\left(x(t), y(t)\right) = \int_0^1 G_2(t,s) g\left(s, x(s), {}^c D_0^\gamma x(s)\right)(s) ds + (2-t) \int_0^1 q_1(y(s)) ds + (t-1) \int_0^1 q_2(y(s)) ds.$$

Proof . Let $(x, y) \in X$ be a solution of the system (1.1). In view of lemma 3.1, we obtain that (x, y) is fixed point of the operator T. The other direction is obvious. $\square$

Theorem 3.4. Assume that the following hypotheses hold:

$(\mathcal{H}_1)$ $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions.

$(\mathcal{H}_2)$ $|f(t,x,y)| \le w_1(t) + c_1\left(|x| + |y|\right), \quad |g(t,x,y)| \le w_2(t) + c_2\left(|x| + |y|\right).$

$(\mathcal{H}_3)$ $|p_1(x) - p_1(y)| \le k_p |x-y|, \quad |p_2(x) - p_2(y)| \le k'_p |x-y|, \quad |q_1(x) - q_1(y)| \le k_q |x-y|, |q_2(x) - q_2(y)| \le k'_q |x-y|,$

where $w_1, w_2 \in L^1[0,1]$ and $c_1, c_2, k_p, k'_p, k_q, k'_q$ are approximately small constants. Then, the problem (1.1) has a solution.

Proof . We prove the existence solution for the system (1.1) by using Schauder fixed point theorem. First of all, let us define B_R as follows

$$B_R = \{(x,y) \in X; \|(x,y)\| \le R\},$$

where

$$R \ge \frac{l_1 \|w_1\|_1 + l_2 \|w_2\|_1}{1-K}$$

and K is constant will be fixed later. So, it is obvious that B_R is convex, closed and bounded subset of X. Now, we show that $T(B_R) \subset B_R$. By lemma 3.2, for each $(x,y) \in B_R$, one has

$$|{}^c D_0^\gamma T_1(x(t), y(t))| = \left| \int_0^1 \frac{\partial^\gamma G_1}{\partial t}(t,s) f\left(s, y(s), {}^c D_0^\gamma y(s)\right)(s) ds + {}^c D_0^\gamma (2-t) \int_0^1 p_1(x(s)) ds + {}^c D_0^\gamma (t-1) \int_0^1 p_2(x(s)) ds \right|.$$

Hence,

$$|{}^c D_0^\gamma T_1(x(t), y(t))| \le l_1 \left[\int_0^1 |w_1| + c_1 \left[\int_0^1 |y(t)| + \int_0^1 |{}^c D_0^\gamma y(t)| \right] \right] + \frac{1}{\Gamma(2-\gamma)} t^{-\gamma+1} \int_0^1 |p_1(x(s)) ds|$$

$$+ \frac{1}{\Gamma(2-\gamma)} t^{-\gamma+1} \int_0^1 |p_2(x(s)) ds|$$

Consequently, employing the Hölder inequality and the hypotheses $(\mathcal{H}_1) - (\mathcal{H}_3)$ the above estimate becomes

$$\begin{aligned} \|T_1(x,y)\| \le & l_1 \|w_1\|_1 + c_1 l_1 \left[\|y(t)\|_q + \|{}^c D_0^\gamma y(t)\|_q + \frac{1}{\Gamma(2-\gamma)} k_p \|x\|_p + k'_p \frac{1}{\Gamma(2-\gamma)} \|x\|_p \right] \\ \le & l_1 \|w_1\|_1 + 2 c_1 l_1 \|y\|_{\gamma,q} + \frac{1}{\Gamma(2-\gamma)} \left(k_p + k'_p\right) \|x\|_{\gamma,p} \\ \le & l_1 \|w_1\|_1 + k_1 \|(x,y)\|. \end{aligned}$$

Similarly, we can show that

$$\|T_2(x,y)\| \leq l_2 \|w_2\|_1 + k_2 \|(x,y)\|.$$

That is, we get

$$\|T_1(x,y)\| \leq l_1 \|w_1\|_1 + l_2 \|w_2\|_1 + \max(k_1,k_2) R \leq l_1 \|w_1\|_1 + l_2 \|w_2\|_1 + KR \leq R$$

and

$$\|T_2(x,y)\| \leq l_1 \|w_1\|_1 + l_2 \|w_2\|_1 + \max(k_1,k_2) R \leq l_1 \|w_1\|_1 + l_2 \|w_2\|_1 + KR \leq R$$

where

$$l_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} + \frac{2}{\Gamma(2-\gamma)\Gamma(\alpha-1)}.$$

$$l_2 = \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)} + \frac{2}{\Gamma(2-\gamma)\Gamma(\beta-1)}.$$

Therefore, we conclude that $\|T(x,y)\| \leq R$. which means that the operator T transforms B_R into B_R. Finally, let us prove that $T_1(B_R)$ and $T_2(B_R)$ are relatively compact in $E^{\gamma,p}, E^{\gamma,q}$ respectively. Indeed, let $t \in [0,1]$ and $h > 0$, where $t+h < 1$ and $(x,y) \in B_R$, then

$$\begin{aligned}
|{}^cD_0^\gamma T_1(x(t+h), y(t+h)) -{}^c D_0^\gamma T_1(x(t),y(t))| &= \Big| \int_0^1 \left[{}^cD_0^\gamma G_1(t+h,s)) -{}^c D_0^\gamma G_1(t,s))\right] f\left(s, y(s), {}^c D_0^\gamma y(s)\right)(s) ds \\
&\quad + {}^cD_0^\gamma(h) \int_0^1 p_1(x(s))ds + {}^c D_0^\gamma(h) \int_0^1 p_2(x(s))ds \Big| \\
&\leq \sup_{s,t\in[0,1]} |G_\gamma(t+h,s)) - G_\gamma(t,s))| \left[\|w_1\|_1 + R\right].
\end{aligned}$$

Thereby, by Lemma 2.4, we deduce that $T_1(B_R)$ is relatively compact. In the same manner, we also obtain $T_2(B_R)$ is relatively compact too. Thus, arising from this, T is Compact. So, the Schauder fixed point theorem implies that the system (1.1) has a solution on $[0,1]$. This completes the proof. □

4 Example

In order to illustrate our study, we consider the following fractional differential system

$$\begin{cases} {}^cD_0^{2.5}x(t) = \dfrac{1}{\sqrt{t^2+748}} \sin\left(y(t) +{}^c D_0^{0.5} y(t)\right), \\ {}^cD_0^{2.7}y(t) = \sin(t) + \dfrac{e^{-t}}{10^5+e^t}\left[x(t) +{}^c D_0^{0.5}x(t)\right], \end{cases} \tag{4.1}$$

with $p_1(x) = \frac{1}{10^3}x$, $p_2(x) = \frac{1}{2\times10^4}\sin x$, $q_1(x) = \frac{1}{10^4}\frac{|x|}{x^2+5}$ and $q_2 = \frac{1}{664}\sin x$.

Here $\alpha = 2.5$, $\beta = 2.7$, $\gamma = 0.5$, $c_1 = 10^{-5}$, $c_2 = \frac{1}{\sqrt{748}}$, $k_p = 10^{-3}$, $k_p' = 2\times10^{-4}$, $k_q = 10^{-4}$, $k_q' = \frac{1}{664}$, $l_1 = 3.91$, $l_2 = 3.93$, $w_1 = 0$ and $w_2 = \sin t$.

If we choose $R = 2$ then, we have

$$R \geq \frac{l_2\|w_2\|}{1-K} = \frac{0.0033}{1-0.28} = 0.0045.$$

In view of Theorem 3.4, we conclude that the operator $T : B_2 \to B_2$ is compact, therefore the fractional boundary value problem (4.1) admits a solution.

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