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Characterization of n-exact sequence of n-additive categories

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Abstract

In this paper, the concepts of n-monomorphism, n-epimorphism, n-isomorphism, n-equivalent, n-coproduct, n-product, n-injection, n-projection, n-initial object, n-terminal object, n-pushout diagram, n-inverse system, n-inverse limit, and n-homology of categories will be introduced and will be shown the relationship between them. Next some of their properties and structured characteristics will be investigated and obtained some results about them.

Keywords: Category theory, n-projective and n-injective objects, n-homomorphisms, n-epimorphisms and n-monomorphisms, n-limits, n-colimits, n-ker and n-coker
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1 Introduction

Category theory formalizes mathematical structures, and their concepts in terms of a labeled directed graph called a category, whose nodes are called objects and their edges called arrows (or morphisms). This category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. The language of category theory has been employed to formalize concepts of other high-level abstractions such as sets, rings, and groups. Several terms were utilized in category theory, including the "morphism" used differently from their usage in the rest of mathematics. In category theory, morphisms obey specific conditions of theory. Samuel Eilenberg and Saunders Mac Lane introduced the concepts of categories, functors, and natural transformations in 1942-45 in their study of algebraic topology, to understand the processes that preserve the mathematical structure. Category theory has practical applications in programming language theory, for example, the usage of monads in functional programming. It may also be used as an axiomatic foundation for mathematics, as an alternative to set theory and other proposed foundations. In mathematics, an abelian category is a category in which morphisms and objects can be added and which kernels and cokernels exist and have desirable properties. The motivating prototype example of an abelian category is the category of abelian groups, Ab. The theory originated to unify several cohomology theories by Alexander Grothendieck and independently in the slightly earlier work of David Buchsbaum. Abelian categories are very stable categories. For example, they are regular and satisfy the snake lemma. The class of Abelian categories is closed under several categorical constructions. For instance, the category of chain complexes of an Abelian category or the category of functors from a small category to an Abelian category is also Abelian. These stability properties make them inevitable in homological algebra and beyond. This theory has significant applications in algebraic geometry, cohomology, and pure category theory. The Abelian categories are named after Niels Henrik Abel. An exact sequence

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is a concept in mathematics, especially in group theory, ring, module theory, homological algebra, and differential geometry. An exact sequence is a finite or infinite sequence of objects and morphisms between them such that the image of one morphism equals the kernel of the next. Homological algebra is the branch of mathematics that studies homology in a general algebraic setting. It is a relatively young discipline whose origins can be traced to investigations in combinatorial topology (a precursor to algebraic topology) and abstract algebra (theory of modules and syzygies) at the end of the 19th century, chiefly by Henri Poincar´e and David Hilbert. The development of homological algebra has closely intertwined with the emergence of category theory. By and large, homological algebra is the study of homological functors and their intricate algebraic structures. One beneficial and ubiquitous concept in mathematics is that of chain complexes, which can be studied through their homology and cohomology. Homological algebra allows extracting information in these complexes and presenting it as homo-logical invariants of rings, modules, topological spaces, and other 'tangible' mathematical objects. Spectral sequences provide a powerful tool for doing this. From its very origins, homological algebra has played an enormous role in algebraic topology. Its sphere of influence has gradually expanded and presently includes commutative algebra, algebraic geometry, algebraic number theory, representation theory, mathematical physics, operator algebras, complex analysis, and the theory of partial differential equations. K-theory is an independent discipline that draws upon methods of homological algebra, as does the noncommutative geometry of Alain Connes.

This paper is organized as follows. In Section Two, we state and discuss the axioms and draw the main consequences. In Section Three, we define n-monomorphism, n-epimorphism, n-isomorphism, n-equivalent, and consider conditions that a chain map will be n-isomorphism. Section 4 briefly discusses categorical constructions. In this section, we introduce the concepts of n-coproduct, n-product, n-injection, n-projection, n-initial object, and n-terminal object. We prove some results about them. Finally, in Section Five, we discuss the n-homology of categories. We introduce the n-pushout diagram, n-inverse system, n-inverse limit, and n-homology of categories. We introduce n-pushout diagram, n-inverse system, n-inverse limit, and n-homology of categories.

2 Preliminaries

In this section, we recall some of the fundamental concepts and definitions, which are necessary for this paper. For details, we refer to [4,10,12,13].

Definition 2.1. Let $\mathcal{C}$ be an additive category and $d^0 : X^0 \longrightarrow X^1$ a morphism in $\mathcal{C}$. An **n-coker** of d^0 is a sequence

$$(d^1, ..., d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^n} X^{n+1}$$

such that, , for all $Y \in \mathcal{C}$ the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{C}(X^{n+1}, Y) \xrightarrow{d^n.?} \mathcal{C}(X^n, Y) \xrightarrow{d^{n-1}.?} ... \xrightarrow{d^1.?} \mathcal{C}(X^1, Y) \xrightarrow{d^0.?} \mathcal{C}(X^0, Y)$$

is exact. Equivalently, the sequence $(d^1, ..., d^n)$ is an n-coker of d^0 if, , for all $1 \leq k \leq n-1$ the morphism d^k is a weak cokernel of d^{k-1}, and d^n is moreover a cokernel of d^{n-1}. The concept of n-ker of morphism is defined dually.

Definition 2.2. Let $\mathcal{C}$ be an additive category. An **n-exact sequence** in $\mathcal{C}$ is a complex

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

in $Ch^n(\mathcal{C})$ such that $(d^0, ..., d^{n-1})$ is an n-ker of d^n, and $(d^1, ..., d^n)$ is an n-coker of d^0.

Definition 2.3. Let $\mathcal{C}$ be an additive category, X a complex in $Ch^{n-1}(\mathcal{C})$, and $f^0 : X^0 \longrightarrow Y^0$ a morphism in $\mathcal{C}$. An **n-pushout diagram** of X along f^0 is a morphism of complexes

$$\begin{tikzcd}
X \arrow[d, "f"] & X^0 \arrow[r] \arrow[d, "f^0"] & X^1 \arrow[r] \arrow[d] & X^2 \arrow[r] \arrow[d] & ... \arrow[r] & X^{n-1} \arrow[r] \arrow[d] & X^n \arrow[d, "f^n"] \\
Y & Y^0 \arrow[r] & Y^1 \arrow[r] & Y^2 \arrow[r] & ... \arrow[r] & Y^{n-1} \arrow[r] & Y^n
\end{tikzcd}$$

such that in the mapping cone $C = C(f)$

$$X^0 \xrightarrow{d_c^{-1}} X^1 \bigoplus Y^0 \xrightarrow{d_c^0} \dots \xrightarrow{d_c^{n-2}} X^n \bigoplus Y^{n-1} \xrightarrow{d_c^{n-1}} Y^n.$$

the sequence $(d_c^0, ..., d_c^{n-1})$ is an n-coker of d_c^{-1}, where we define

$$d_c^k := \begin{bmatrix} -d_X^{k+1} & 0 \\ f^{k+1} & d_Y^k \end{bmatrix} : X^{k+1} \bigoplus Y^k \longrightarrow X^{k+2} \bigoplus Y^{k+1}$$

for each $k \in \{-1, 0, 1, ..., n-1\}$. In particular

$$d_c^{-1} = \begin{bmatrix} -d_X^0 \\ f^0 \end{bmatrix} \text{ and } d_c^{n-1} = \begin{bmatrix} f^n & d_Y^{n-1} \end{bmatrix}$$

Note that the fact that $C(f)$ is a complex encodes precisely that X and Y are complexes and that f is a morphism of complexes.The concept of n-pullback diagram is defined dually.

Definition 2.4. Let n be a positive integer. An n**-abelian category** is an additive category $\mathcal{C}$ which satisfies the following axioms;

$(A0)$ The category $\mathcal{C}$ is idempotent complete.

$(A1)$ Every morphism in $\mathcal{C}$ has n-ker and n-coker.

$(A2)$ for every monomorphism $f^0 : X^0 \longrightarrow X^1$ in $\mathcal{C}$ and, for every n-coker $(f^0, f^1, ..., f^{n-1})$ of f^0, the following sequence n-exact:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \dots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

$(A2^{op})$ for every epimorphism $g^n : X^n \longrightarrow X^{n+1}$ in $\mathcal{C}$ and, for every n-ker $(g^0, g^1, ..., g^{n-1})$ of g^n, the following sequence n-exact:

$$X^0 \xrightarrow{g^0} X^1 \xrightarrow{g^1} \dots \xrightarrow{g^{n-1}} X^n \xrightarrow{g^n} X^{n+1}.$$

3 Special n-chain maps

Definition 3.1. Let $\mathcal{C}$ be an additive category and $A, B, C \in Ch^n(\mathcal{C})$, a morphism $u : B \longrightarrow C$ in a category $\mathcal{C}$ is an **n-monomorphism** (or is **n-monic**) if u can be canceled from the left; that is, , for all objects $\mathcal{C}$ and all morphism $f, g : A \longrightarrow B$, we have that $uf = ug$ implies f=g

$$\begin{array}{ccccccccccccc}
A & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \dots & \longrightarrow & A^n & \longrightarrow & A^{n+1} \\
{\scriptstyle g}\downarrow\downarrow{\scriptstyle f} & & {\scriptstyle g^0}\downarrow\downarrow{\scriptstyle f^0} & & \downarrow\downarrow & & \downarrow\downarrow & & & & \downarrow\downarrow & & {\scriptstyle g^{n+1}}\downarrow\downarrow{\scriptstyle f^{n+1}} \\
B & & B^0 & \longrightarrow & B^1 & \longrightarrow & B^2 & \longrightarrow & \dots & \longrightarrow & B^n & \longrightarrow & B^{n+1} \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u^0} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle u^{n+1}} \\
C & & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \dots & \longrightarrow & C^n & \longrightarrow & C^{n+1}
\end{array}$$

and, for all $0 \leq i \leq n$, $u^i : B^i \longrightarrow C^i$ are monomorphisms. It is clear that $u : B \longrightarrow C$ is an n-monomorphism if and only if, for all A, the induced map $u_* : Hom(A, B) \longrightarrow Hom(A, C)$ is an injection. $Hom(A, B)$ is an abelian groups, because of $\mathcal{C}$ is additive category, and so u is an n-monic if and only if $ug = 0$ implies $g = 0$

Definition 3.2. Let $\mathcal{C}$ be an additive category and $B, C, D \in Ch^n(\mathcal{C})$, a morphism $v : B \longrightarrow C$ in a category $\mathcal{C}$ be an **n-epimorphism** (or is **n-epic**) if v can be canceled from the right; that is, , for all objects D and all morphism $h, k : C \longrightarrow D$, we have that $hv = kv$ implies $h = k$.

$$
\begin{tikzcd}
B \arrow[d, "v"] & B^0 \arrow[r] \arrow[d, "v^0"] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & \dots \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, "v^{n+1}"] \\
C \arrow[d, shift right, "k"'] \arrow[d, shift left, "h"] & C^0 \arrow[r] \arrow[d, shift right, "k^0"'] \arrow[d, shift left, "h^0"] & C^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & C^2 \arrow[r] \arrow[d] & \dots \arrow[r] & C^n \arrow[r] \arrow[d] & C^{n+1} \arrow[d, shift right, "k^{n+1}"'] \arrow[d, shift left, "h^{n+1}"] \\
D & D^0 \arrow[r] & D^1 \arrow[r] & D^2 \arrow[r] & \dots \arrow[r] & D^n \arrow[r] & C^{n+1}
\end{tikzcd}
$$

and, for all $0 \le i \le n$, $v^i : B^i \longrightarrow C^i$ are epimorphisms. It is clear that, $v : B \longrightarrow C$ is an n-epimorphism if and only if, for all D, the induced map $v^* : Hom(C, D) \longrightarrow Hom(B, D)$ is an injection. $Hom(A, B)$ is an abelian groups, because of $\mathcal{C}$ is additive category, and so v is n-monic if and only if $gv = 0$ implies $g = 0$.

Definition 3.3. Let $\mathcal{C}$ be an additive category and $A, B \in Ch^n(\mathcal{C})$, $f : A \longrightarrow B$ is an **n-isomorphism** if there are the chain maps $h : B \longrightarrow A$ and $k : B \longrightarrow A$ such that $B \xrightarrow{h} A \xrightarrow{f} B$ and $A \xrightarrow{f} B \xrightarrow{k} A$ are identity chain maps and, for all $0 \le i \le n$, $f^i : A^i \longrightarrow B^i$ are isomorphisms.

It is easy to see that, the composition of n-isomorphisms is an n-isomorphism.

Proposition 3.4.

(a) Let $\mathcal{C}$ be an additive category and $A, B, C \in Ch^n(\mathcal{C})$, $A \longrightarrow B \longrightarrow C$ be an n-monomorphism, then so is $A \longrightarrow B$. If both $A \longrightarrow B$ and $B \longrightarrow C$ are n-monomorphism, then so is $A \longrightarrow B \longrightarrow C$.

(b) Let $\mathcal{C}$ be an additive category and $A, B, C \in Ch^n(\mathcal{C})$, $A \longrightarrow B \longrightarrow C$ be an n-epimorphism, then so is $B \longrightarrow C$. If both $A \longrightarrow B$ and $B \longrightarrow C$ are n-epimorphism,, then so is $A \longrightarrow B \longrightarrow C$.

Proof .

(a) By proposition 1.41 [12] this holds for k-monomorphisms, where $0 \le k \le n$. So, The the proof holds for n-monomorphism.
(b) By proposition 1.42 [12] this holds for k-epimorphisms, where $0 \le k \le n$. So, The the proof holds for n-epimorphism.

□

Proposition 3.5. Let $\mathcal{C}$ be an abelian additive category. A chain maps is an n-isomorphism if and only if is both an n-monomorphism and an n-epimorphism.

Proof . If

$$
\begin{tikzcd}
A^0 \arrow[r] \arrow[d, "f^0"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & \dots \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "f^{n+1}"] \\
B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \dots \arrow[r] & B^n \arrow[r] & B^{n+1}
\end{tikzcd}
$$

is an n-isomorphism, then there are n-chain maps such that

$$
\begin{tikzcd}
B^0 \arrow[r] \arrow[d, shift right, "g^0"'] \arrow[d, shift left, "K^0"] & B^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & B^2 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & \dots \arrow[r] & B^n \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & B^{n+1} \arrow[d, shift right, "g^{n+1}"'] \arrow[d, shift left, "K^{n+1}"] \\
A^0 \arrow[r] \arrow[d, "f^0"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & \dots \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "f^{n+1}"] \\
B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \dots \arrow[r] & B^n \arrow[r] & B^{n+1}
\end{tikzcd}
$$

is an n-monomorphism and

$$\begin{tikzcd}
A^0 \arrow[r] \arrow[d, "f^0"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & ... \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "f^{n+1}"] \\
B^0 \arrow[r] \arrow[d, shift right, "k^0"'] \arrow[d, shift left, "g^0"] & B^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & B^2 \arrow[r] \arrow[d] & ... \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, shift right, "k^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] \\
A^0 \arrow[r] & A^1 \arrow[r] & A^2 \arrow[r] & ... \arrow[r] & A^n \arrow[r] & A^{n+1}
\end{tikzcd}$$

is an n-epimorphism.

Conversely, let $A, B \in Ch^n(\mathcal{C})$, and

$$\begin{tikzcd}
A \arrow[d, "f"] & A^0 \arrow[r] \arrow[d, "f^0"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & ... \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "f^{n+1}"] \\
B & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & ... \arrow[r] & B^n \arrow[r] & B^{n+1}
\end{tikzcd}$$

clearly be an n-monomorphism and n-epimorphism. $B^i \longrightarrow 0$ is the cokernel of $f^i : A^i \longrightarrow B^i$, , for all $i \in \{0, 1, ..., n+1\}$. $1_{B^i} : B^i \longrightarrow B^i$ is clearly a kernel of $B^i \longrightarrow 0$. By the theorem 2.11 [4] so is $f^i : A^i \longrightarrow B^i$, , for all $i \in \{0, 1, ..., n+1\}$.

Already, we have the same chain maps. The theorem 2.11 [4] asserts that the chain maps $f : A \longrightarrow B$ is an n-isomorphism. Hence there is a chain maps $g_1^i : B^i \longrightarrow A^i$, , for all $i \in \{0, 1, ..., n+1\}$ such that

$$\begin{tikzcd}
B \arrow[d, "g_1"] & B^0 \arrow[r] \arrow[d, "g_1^0"] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & ... \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, "g_1^{n+1}"] \\
A \arrow[d, "f"] & A^0 \arrow[r] \arrow[d, "f^0"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & ... \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "f^{n+1}"] \\
B \arrow[d, equal] & B^0 \arrow[r] \arrow[d, equal] & B^1 \arrow[r] \arrow[d, equal] & B^2 \arrow[r] \arrow[d, equal] & ... \arrow[r] & B^n \arrow[r] \arrow[d, equal] & B^{n+1} \arrow[d, equal] \\
B \arrow[d, "1_B"] & B^0 \arrow[r] \arrow[d, "1_{B^0}"] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & ... \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, "1_{B^{n+1}}"] \\
B & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & ... \arrow[r] & B^n \arrow[r] & B^{n+1}
\end{tikzcd}$$

Dually we note that $0 \longrightarrow A^i$ is a kernel of $f^i : A^i \longrightarrow B^i$, and that both $f^i : A^i \longrightarrow B^i$ and $1_{A^i} : A^i \longrightarrow A^i$ are cokernel of $0 \longrightarrow A^i$, for all $i \in \{0, 1, ..., n+1\}$. Hence there is a chain maps $g_1^i : B^i \longrightarrow A^i$, for all $i \in \{0, 1, ..., n+1\}$ such that

$$\begin{tikzcd}
A \arrow[d, "f_1"] & A^0 \arrow[r] \arrow[d, "f_1^0"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & ... \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "f_1^{n+1}"] \\
B \arrow[d, "g_2"] & B^0 \arrow[r] \arrow[d, "g_2^0"] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & ... \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, "g_2^{n+1}"] \\
A \arrow[d, equal] & A^0 \arrow[r] \arrow[d, equal] & A^1 \arrow[r] \arrow[d, equal] & A^2 \arrow[r] \arrow[d, equal] & ... \arrow[r] & A^n \arrow[r] \arrow[d, equal] & A^{n+1} \arrow[d, equal] \\
A \arrow[d, "1_A"] & A^0 \arrow[r] \arrow[d, "1_{A^0}"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & ... \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "1_{A^{n+1}}"] \\
A & A^0 \arrow[r] & A^1 \arrow[r] & A^2 \arrow[r] & ... \arrow[r] & A^n \arrow[r] & A^{n+1}
\end{tikzcd}$$

By the definition of n-isomorphism, $f : A \longrightarrow B$ is. $\square$

Proposition 3.6. Let $\mathcal{C}$ be an additive category and $A, B \in Ch^n(\mathcal{C})$. If $f : A \longrightarrow B$ is an n-isomorphism, then there is a unique n-chain map $g : B \longrightarrow A$ such that $A \xrightarrow{f} B \xrightarrow{g} A$ and $B \xrightarrow{g} A \xrightarrow{f} B$ are identity chain maps and $g : B \longrightarrow A$ is an n-isomorphism.

Proof . Let h and k be as in the definition of n-isomorphism. Then we have the following diagram:

$$\begin{tikzcd}
B^0 \arrow[r] \arrow[d, "h^0"'] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & ... \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, "h^{n+1}"] \\
A^0 \arrow[r] \arrow[d, equal] & A^1 \arrow[r] \arrow[d, equal] & A^2 \arrow[r] \arrow[d, equal] & ... \arrow[r] & A^n \arrow[r] \arrow[d, equal] & A^{n+1} \arrow[d, equal] \\
B^0 \arrow[r] \arrow[d, "k^0"'] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & ... \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, "k^{n+1}"] \\
A^0 \arrow[r] \arrow[d, "1"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & ... \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "1"] \\
A^0 \arrow[r] & A^1 \arrow[r] & A^2 \arrow[r] & ... \arrow[r] & A^n \arrow[r] & A^{n+1}.
\end{tikzcd}$$

According to the above diagram, the proof is complete. $\square$

Proposition 3.7. Let $\mathcal{C}$ be an additive category, $A, B \in Ch^n(\mathcal{C})$, and let $U : A \longrightarrow B$ be an morphism in chain maps. Then the following hold,

(i) if $\ker u$ exists, then u is an n-monic if and only if $\ker u = 0$.

(ii) if $cokeru$ exists, then u is an n-epic if and only if $cokeru = 0$.

Proof . We refer to the diagrams in the definition of kernel and cokernel. Let $\ker u^i$ be $p^i : K^i \longrightarrow A^i$, and assume that , for all $0 \leq i \leq n+1$, $p^i = 0$. If $g^i : X^i \longrightarrow A^i$ satisfies $u^i g^i = 0$, for all $0 \leq i \leq n+1$, $p^i = 0$, then the universal property of kernel provides a morphism of chain maps $\theta^i : X^i \longrightarrow K^i$ with $g^i = p^i\theta^i = 0$ (because $p^i = 0$), for all $0 \leq i \leq n+1$. Hence, u^i is a monic and then u is an n-monic. Conversely, if u is an n-monic, consider

$$\begin{tikzcd}
K^0 \arrow[r] \arrow[d, shift right, "0"'] \arrow[d, shift left, "p^0"] & K^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & K^2 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & ... \arrow[r] & K^n \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & K^{n+1} \arrow[d, shift right, "0"'] \arrow[d, shift left, "p^{n+1}"] \\
A^0 \arrow[r] \arrow[d, "u^0"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & ... \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "u^{n+1}"] \\
B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & ... \arrow[r] & B^n \arrow[r] & B^{n+1}.
\end{tikzcd}$$

Since u^i is a monic, $u^i p^i = 0 = u^i 0 = 0$, for all $0 \leq i \leq n+1$, we have $p^i = 0$. The proof for n-epimorphisms and cokernels are similar. $\square$

Definition 3.8. Let $\mathcal{C}$ be an additive category, let $A, B \in Ch^{n-1}(\mathcal{C})$. If B is an object in an additive category $\mathcal{C}$, consider all ordered pairs (A, f), where $f : A \longrightarrow B$ is an n-monomorphism. Call two such pairs (A, f) and (A', f') **n-equivalent** if there exists an n-isomorphism $g : A' \longrightarrow A$ whit $f' = fg$.

$$\begin{tikzcd}
A'^0 \arrow[r] \arrow[d, "g^0"'] \arrow[ddr, "f'^0"] & A'^1 \arrow[r] \arrow[d] \arrow[ddr] & A'^2 \arrow[r] \arrow[d] \arrow[ddr] & ... \arrow[r] & A'^{n-1} \arrow[r] \arrow[d] \arrow[ddr] & A'^n \arrow[d, "g^n"'] \arrow[ddr, "f'^n"] & \\
A^0 \arrow[r] \arrow[dr, "f^0"'] & A^1 \arrow[r] \arrow[dr] & A^2 \arrow[r] \arrow[dr] & ... \arrow[r] & A^{n-1} \arrow[r] \arrow[dr] & A^n \arrow[dr, "f^n"'] & \\
 & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & ... \arrow[r] & B^{n-1} \arrow[r] & B^n
\end{tikzcd}$$

A **n-subgadget of** B is an n-equivalence class $[(A, f)]$ such that A^i **subgadget of** is an equivalence class $[(A^i, f^i)]$ for all $0 \leq i \leq n$, and we call A a **n-subobject of** B. Note that if (A'^i, f'^i) is equivalent to (A^i, f^i), then $A'^i \cong A^i$, for all $0 \leq i \leq n$ then (A', f') is an n-equivalent to (A, f) and $A' \cong A$.

Definition 3.9. Let $\mathcal{C}$ be an additive category, let $C, B \in Ch^{n-1}(\mathcal{C})$. If B is an object in an additive category $\mathcal{C}$, consider all ordered pairs (f, C), where $f : B \longrightarrow C$ is an n-epimorphism. Call two such pairs (f, C) and (f', C') **n-equivalent** if there exists an n-isomorphism $g : C \longrightarrow C'$ whit $f' = gf$.

$$\begin{tikzcd}
B^0 \arrow[rr] \arrow[d, "f^0"'] \arrow[ddr, "f'^0"] & & B^1 \arrow[rr] \arrow[d] \arrow[ddr] & & B^2 \arrow[rr] \arrow[d] \arrow[ddr] & & \cdots \arrow[rr] & & B^{n-1} \arrow[rr] \arrow[d] \arrow[ddr] & & B^n \arrow[d, "f^n"'] \arrow[ddr, "f'^n"] & \\
C^0 \arrow[rr] \arrow[dr, "g^0"'] & & C^1 \arrow[rr] \arrow[dr] & & C^2 \arrow[rr] \arrow[dr] & & \cdots \arrow[rr] & & C^{n-1} \arrow[rr] \arrow[dr] & & C^n \arrow[dr, "g^n"] & \\
& C'^0 \arrow[rr] & & C'^1 \arrow[rr] & & C'^2 \arrow[rr] & & \cdots \arrow[rr] & & C'^{n-1} \arrow[rr] & & C'^n
\end{tikzcd}$$

A **n-quotient of** B is an n-equivalence class $[(f, C)]$ such that A^i **quotient of** is an equivalence class $[(f^i, C^i)]$ for all $0 \leq i \leq n$, and we call C a **n-quotient of** B. Note that if (f'^i, C'^i) is equivalent to (f^i, C^i), then $C^i \cong C'^i$ for all $0 \leq i \leq n$ then (f', C') is an n-equivalent to (f, C) and $C' \cong C$

By the above definition, we define difference n-ker as follows:

Definition 3.10. Let $\mathcal{C}$ be an additive category and $d^n : X^n \longrightarrow X^{n+1}$, $d'^n : X^n \longrightarrow X^{n+1}$ two morphisms in $\mathcal{C}$. We say that $X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow ... \longrightarrow X^n$ is a **difference n-ker** of d^n and d'^n if

- **(i)** we have

$$\begin{tikzcd}
K^0 \arrow[r] \arrow[d, "h^0"] & K^1 \arrow[r] \arrow[d] & K^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & K^n \arrow[r] \arrow[d] & K^{n+1} \arrow[d, "h^{n+1}"] \\
A^0 \arrow[r] \arrow[d, shift right, "f^0"'] \arrow[d, shift left, "g^0"] & A^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & A^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] \\
B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \cdots \arrow[r] & B^n \arrow[r] & B^{n+1}
\end{tikzcd}$$

- **D n-K2** , for all $k : X \longrightarrow A$

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d, "k^0"] & X^1 \arrow[r] \arrow[d] & X^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[d, "k^{n+1}"] \\
A^0 \arrow[r] \arrow[d, shift right, "f^0"'] \arrow[d, shift left, "g^0"] & A^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & A^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] \\
B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \cdots \arrow[r] & B^n \arrow[r] & A^{n+1}
\end{tikzcd}$$

There is a unique $X \longrightarrow K$ such that

$$\begin{tikzcd}
X^0 \arrow[rr] \arrow[d] \arrow[ddr] & & X^1 \arrow[rr] \arrow[d] \arrow[ddr] & & X^2 \arrow[rr] \arrow[d] \arrow[ddr] & & \cdots \arrow[rr] & & X^n \arrow[rr] \arrow[d] \arrow[ddr] & & X^{n+1} \arrow[d] \arrow[ddr] & \\
C^0 \arrow[rr] \arrow[dr] & & C^1 \arrow[rr] \arrow[dr] & & C^2 \arrow[rr] \arrow[dr] & & \cdots \arrow[rr] & & C^n \arrow[rr] \arrow[dr] & & C^{n+1} \arrow[dr] & \\
& A^0 \arrow[rr] & & A^1 \arrow[rr] & & A^2 \arrow[rr] & & \cdots \arrow[rr] & & A^n \arrow[rr] & & A^{n+1}
\end{tikzcd}$$

is commute.

Proposition 3.11. Let $\mathcal{C}$ be an additive category and $K, A, B \in Ch^n(\mathcal{C})$. If $f : K \longrightarrow A$ is a difference n-ker of $x : A \longrightarrow B$ and $y : A \longrightarrow B$. Then it is an n-monomorphism and represents the largest n-subobjects S of A, $S \xrightarrow{u} A \underset{y}{\overset{x}{\rightrightarrows}} B$.

Proof . Let for $X \in Ch^n(\mathcal{C})$, then we have the following;

$$\begin{tikzcd}
C^0 \arrow[r] \arrow[d, shift right, "b^0"'] \arrow[d, shift left, "a^0"] & C^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & C^2 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & \dots \arrow[r] & C^n \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & C^{n+1} \arrow[d, shift right, "b^{n+1}"'] \arrow[d, shift left, "a^{n+1}"] \\
K^0 \arrow[r] \arrow[d, "f^0"] & K^1 \arrow[r] \arrow[d] & K^2 \arrow[r] \arrow[d] & \dots \arrow[r] & K^n \arrow[r] \arrow[d] & K^{n+1} \arrow[d, "f^{n+1}"] \\
A^0 \arrow[r] \arrow[d, equal] & A^1 \arrow[r] \arrow[d, equal] & A^2 \arrow[r] \arrow[d, equal] & \dots \arrow[r] & A^n \arrow[r] \arrow[d, equal] & A^{n+1} \arrow[d, equal] \\
C^0 \arrow[r] \arrow[d, "c^0"'] & C^1 \arrow[r] \arrow[d] & C^2 \arrow[r] \arrow[d] & \dots \arrow[r] & C^n \arrow[r] \arrow[d] & C^{n+1} \arrow[d, "c^{n+1}"'] \\
A^0 \arrow[r] & A^1 \arrow[r] & A^2 \arrow[r] & \dots \arrow[r] & A^n \arrow[r] & A^{n+1}
\end{tikzcd}$$

thus, leads to the following diagram:

$$\begin{tikzcd}
C^0 \arrow[r] \arrow[d, "c^0"] & C^1 \arrow[r] \arrow[d] & C^2 \arrow[r] \arrow[d] & \dots \arrow[r] & C^n \arrow[r] \arrow[d] & C^{n+1} \arrow[d, "c^{n+1}"] \\
A^0 \arrow[r] \arrow[d, shift right, "y^0"'] \arrow[d, shift left, "x^0"] & A^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & A^2 \arrow[r] \arrow[d] & \dots \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, shift right, "y^{n+1}"'] \arrow[d, shift left, "x^{n+1}"] \\
B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \dots \arrow[r] & B^n \arrow[r] & B^{n+1},
\end{tikzcd}$$

by D n-K1. But by D n-K2,

$$\begin{tikzcd}
C^0 \arrow[r] \arrow[d, "a^0"'] \arrow[ddr, "c^0"] & C^1 \arrow[r] \arrow[d] \arrow[ddr] & C^2 \arrow[r] \arrow[d] \arrow[ddr] & \dots \arrow[r] & C^n \arrow[r] \arrow[d] \arrow[ddr] & C^{n+1} \arrow[d, "a^{n+1}"'] \arrow[ddr, "c^{n+1}"] & \\
K^0 \arrow[r] \arrow[dr, "f^0"'] & K^1 \arrow[r] \arrow[dr] & K^2 \arrow[r] \arrow[dr] & \dots \arrow[r] & K^n \arrow[r] \arrow[dr] & K^{n+1} \arrow[dr, "f^{n+1}"'] & \\
& A^0 \arrow[r] & A^1 \arrow[r] & A^2 \arrow[r] & \dots \arrow[r] & A^n \arrow[r] & A^{n+1}
\end{tikzcd}$$

and

$$\begin{tikzcd}
C^0 \arrow[r] \arrow[d, "b^0"'] \arrow[ddr, "c^0"] & C^1 \arrow[r] \arrow[d] \arrow[ddr] & C^2 \arrow[r] \arrow[d] \arrow[ddr] & \dots \arrow[r] & C^n \arrow[r] \arrow[d] \arrow[ddr] & C^{n+1} \arrow[d, "b^{n+1}"'] \arrow[ddr, "c^{n+1}"] & \\
K^0 \arrow[r] \arrow[dr, "f^0"'] & K^1 \arrow[r] \arrow[dr] & K^2 \arrow[r] \arrow[dr] & \dots \arrow[r] & K^n \arrow[r] \arrow[dr] & K^{n+1} \arrow[dr, "f^{n+1}"'] & \\
& A^0 \arrow[r] & A^1 \arrow[r] & A^2 \arrow[r] & \dots \arrow[r] & A^n \arrow[r] & A^{n+1}
\end{tikzcd}$$

is commute, then $fa = fb$ implies $a = b$ and $f : K \longrightarrow A$ is an n-monomorphism. $\square$

All difference n-ker of $x : A \longrightarrow B$ and $y : A \longrightarrow B$ represent the same n-subobject, and conversely, if $K \longrightarrow A$ is a difference n-ker of $x : A \longrightarrow B$ and $y : A \longrightarrow B$, if $K' \longrightarrow A$ represent the same object, Then $K' \longrightarrow A$ is difference n-ker of $x : A \longrightarrow B$ and $y : A \longrightarrow B$.

Definition 3.12. Let $\mathcal{C}$ be an additive category and $A, B, C, X \in \mathcal{C}$. Given two chain maps $f : A \longrightarrow B$ and $g : A \longrightarrow B$ we say that $h : B \longrightarrow C$ is a **difference n-coker** of f and g if

- **D n-C1** we have

$$\begin{tikzcd}
A^0 \arrow[r] \arrow[d, shift right, "g^0"'] \arrow[d, shift left, "f^0"] & A^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & A^2 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & ... \arrow[r] & A^n \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & A^{n+1} \arrow[d, shift right, "g^{n+1}"'] \arrow[d, shift left, "f^{n+1}"] \\
B^0 \arrow[r] \arrow[d, "h^0"] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & ... \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, "h^{n+1}"] \\
C^0 \arrow[r] & C^1 \arrow[r] & C^2 \arrow[r] & ... \arrow[r] & C^n \arrow[r] & C^{n+1}
\end{tikzcd}$$

- **D n-C2** , for all $k : B \longrightarrow X$ such that

$$\begin{tikzcd}
A^0 \arrow[r] \arrow[d, shift right, "g^0"'] \arrow[d, shift left, "f^0"] & A^1 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & A^2 \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & ... \arrow[r] & A^n \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & A^{n+1} \arrow[d, shift right, "g^{n+1}"'] \arrow[d, shift left, "f^{n+1}"] \\
B^0 \arrow[r] \arrow[d, "h^0"] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & ... \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, "h^{n+1}"] \\
X^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & ... \arrow[r] & X^n \arrow[r] & X^{n+1}
\end{tikzcd}$$

There is a unique $C \longrightarrow X$ such that

$$\begin{tikzcd}
B^0 \arrow[r] \arrow[d] \arrow[ddr] & B^1 \arrow[r] \arrow[d] \arrow[ddr] & B^2 \arrow[r] \arrow[d] \arrow[ddr] & ... \arrow[r] & B^n \arrow[r] \arrow[d] \arrow[ddr] & B^{n+1} \arrow[d] \arrow[ddr] & \\
C^0 \arrow[r] \arrow[dr] & C^1 \arrow[r] \arrow[dr] & C^2 \arrow[r] \arrow[dr] & ... \arrow[r] & C^n \arrow[r] \arrow[dr] & C^{n+1} \arrow[dr] & \\
& X^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & ... \arrow[r] & X^n \arrow[r] & X^{n+1}
\end{tikzcd}$$

is commute.

Definition 3.13. Let $\mathcal{C}$ be an additive category and $d^{n-1} : X^{n-1} \longrightarrow X^n$ a morphism in $\mathcal{C}$ its cokerd^{n-1} is $d^n : X^n \longrightarrow X^{n+1}$. An **n-image** of d^{n-1} is a sequence

$$(d^0, ..., d^{n-1}) : X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} X^n$$

such that

$$n - im(d^{n-1}) = n - \ker(\mathrm{coker} d^{n-1}).$$

Definition 3.14. Let $\mathcal{C}$ be an additive category. An we say the complex

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

in $Ch^n(\mathcal{C})$ from the left, it is strongly n-exact sequence $(d^0, ..., d^{n-1})$ is an n-ker of d^n, and d^0 is an injection.

Definition 3.15. Let $\mathcal{C}$ be an additive category. An we say the complex

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

in $Ch^n(\mathcal{C})$ from the right, it is strongly n-exact sequence $(d^1, ..., d^n)$ is an n-coker of d^0, and d^n is a projection.

Proposition 3.16. Let $\mathcal{C}$ be an abelian additive category. The composition of kernel and n-*coker*, $(n-1)$-ker and *cokernel*, *cokernel* and n-ker, $(n-1)$-*coker* and kernel are identity.

Proof . The prove is by induction on $n \geq 1$. The base step $n = 1$, by theorem 2.11 [4] is clear. If $n = 2$, , for all $X \in Ch^2(\mathcal{C})$, then (d^0, d^1) is a 2-coker of d^2. Let $X^0 \longrightarrow X^1$ be the kernel of $X^1 \longrightarrow X^2 \longrightarrow X^3$ and, , for all $K \in obj(\mathcal{C})$, $K \longrightarrow X^1$, also. We have to prove that $X^0 \longrightarrow X^1$ and $K \longrightarrow X^1$ are equal. We shall apply the definition of kernel and n-coker a number of times. If $X^0 \longrightarrow X^1$ is a monomorphism, by definition of abelian category [4] it is a kernel of some map $X^1 \longrightarrow B \longrightarrow 0$. By definition of n-coker, we have $X^0 \longrightarrow X^1 \longrightarrow B \longrightarrow 0$ that $X^2 \longrightarrow B$ and $X^3 \longrightarrow 0$ yielding a commutative diagram:

$$\begin{tikzcd}
\ker(X^1 \longrightarrow B \longrightarrow 0) \arrow[r, equal] & X^0 \arrow[dr] & & X^2 \arrow[r] \arrow[dd] & X^3 \arrow[r, equal] \arrow[dd] & 2-\text{coker}(X^0 \longrightarrow X^1) \\
& & X^1 \arrow[ur] \arrow[dr] & & & \\
\ker(X^1 \longrightarrow X^2 \longrightarrow X^3) \arrow[r, equal] & K \arrow[ur] & & B \arrow[r] & 0 &
\end{tikzcd} \tag{3.1}$$

.

The above diagram implies there are the chain maps $X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow X^3 \longrightarrow 0$, and $X^0 \longrightarrow K$ such that

$$\begin{tikzcd}
X^0 \arrow[dd] \arrow[dr] & \\
& X^1 \\
K \arrow[ur] &
\end{tikzcd}$$

commutes. Again by (3.1), there are the chain maps $K \longrightarrow X^1 \longrightarrow B \longrightarrow 0$ and $K \longrightarrow X^1$ such that

$$\begin{tikzcd}
X^0 \arrow[dr] & \\
& X^1 \\
K \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

commutes. Thus the subobjects represented by $X^0 \longrightarrow X^1$ and $K \longrightarrow X^1$ are contained in each other and hence equal. Since $X^0 \longrightarrow X^1$ is a kernel of $X^1 \longrightarrow X^2 \longrightarrow X^3$, ker(2-coker)=Id. We now turn to the inductive step. Assume inductively, that $n = k+1$, where $k \geq 2$, and the result has been proved in the case where $n = k$. In the case in which $n = k+1$, then, , for all $X \in Ch^{k+1}(\mathcal{C})$, then $(d^0, d^1, d^2, ..., d^k)$ be a $(k+1)$-coker of d^{k+1}. Let $X^0 \longrightarrow X^1$ be the kernel of $X^1 \longrightarrow X^2 \longrightarrow X^3 \longrightarrow ... \longrightarrow X^k \longrightarrow X^{k+1}$ and, for all $K \in obj(\mathcal{C})$, $K' \longrightarrow X^1$ also. We have to prove that $X^0 \longrightarrow X^1$ and $K' \longrightarrow X^1$ is equal. We shall apply the definition of n-ker and n-coker a number of times. If $X^0 \longrightarrow X^1$ a monomorphism. By definition of abelian category [4] it is a kernel of some map $X^1 \longrightarrow B \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ... \longrightarrow 0$. By definition of n-coker, we have $X^0 \longrightarrow X^1 \longrightarrow B \longrightarrow 0 \longrightarrow 0 \longrightarrow ... \longrightarrow 0$ that $X^2 \longrightarrow B$, $X^3 \longrightarrow 0$, $X^4 \longrightarrow 0$,..., $X^k \longrightarrow 0$, $X^{k+1} \longrightarrow 0$ yielding a commutative diagram:

$$\begin{tikzcd}
\ker(X^1 \to B \to 0 \to ... \to 0) \arrow[r, equal] & X^0 \arrow[rr] \arrow[dr] & & X^2 \arrow[r] \arrow[dd] & X^3 \arrow[r] \arrow[dd] & ... \arrow[r] & X^{k+1} \arrow[r, equal] \arrow[dd] & (k+1)-\text{coker}(X^0 \to X^1) \\
& & X^1 \arrow[ur] \arrow[dr] & & & & & \\
\ker(X^1 \to X^2 ... \to X^{k+1}) \arrow[r, equal] & K' \arrow[ur] \arrow[rr] & & B \arrow[r] & 0 \arrow[r] & ... \arrow[r] & 0 &
\end{tikzcd}$$

By the chain maps, there is the chain map and $X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow X^3 \longrightarrow ... \longrightarrow X^k \longrightarrow X^{k+1} \longrightarrow 0$, there is a map $X^0 \longrightarrow K'$ such that

$$\begin{tikzcd}
X^0 \arrow[dd] \arrow[dr] & \\
& X^1 \\
K' \arrow[ur] &
\end{tikzcd}$$

commutes. $K' \longrightarrow X^1 \longrightarrow B \longrightarrow 0 \longrightarrow 0 \longrightarrow ... \longrightarrow 0$; there is a chain map $K' \longrightarrow X^1$ such that

$$\begin{tikzcd}
X^0 \arrow[dr] & \\
& X^1 \\
K' \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Thus the subobjects represented by $X^0 \longrightarrow X^1$ and $K' \longrightarrow X^1$ are contained in each other and hence equal. $X^0 \longrightarrow X^1$ is a kernel of $X^1 \longrightarrow X^2 \longrightarrow X^3 \longrightarrow ... \longrightarrow X^k \longrightarrow X^{k+1}$. Thus ker ((k+1)-coker)=Identity, and dually ($(n-1)$-ker) and cokernel, cokernel and (n-ker), ((n-1)-coker) and kernel are identities. □

4 Categorical constructions

Definition 4.1. Let $\mathcal{C}$ be an additive category, if A and B are complexes in Ch^{n-2}, then their **n-coproduct** is a triple $(A \sqcup B, \alpha, \beta)$, where $A \sqcup B$ is an complex in Ch^{n-2} and $\alpha : A \longrightarrow A \sqcup B$, $\beta : B \longrightarrow A \sqcup B$ are morphisms in chain maps, called **n-injections**, such that,

(i) for every complex X in Ch^{n-2} and every pair of morphism in chain maps $f : A \longrightarrow X$ and $g : B \longrightarrow X$, there exists a unique morphism in chain maps $\theta : A \sqcup B \longrightarrow X$ making the diagram commute: $\theta\alpha = f$ and $\theta\beta = g$.

$$\begin{tikzcd}
& A^0 \arrow[rr] \arrow[dl, "\alpha^0"'] \arrow[dd, "f^0"] & & A^1 \arrow[rr] \arrow[dl] \arrow[dd] & & A^2 \arrow[r] \arrow[dl] \arrow[dd] & \cdots \arrow[rrr] & & & A^{n-2} \arrow[rr] \arrow[dl] \arrow[dd] & & A^{n-1} \arrow[dl, "\alpha^{n-1}"'] \arrow[dd, "f^{n-1}"] \\
A^0 \sqcup B^0 \arrow[rr, dotted] \arrow[dr, dotted, "\theta^0"] & & A^1 \sqcup B^1 \arrow[rr, dotted] \arrow[dr, dotted] & & A^2 \sqcup B^2 \arrow[rr, dotted] \arrow[dr, dotted] & & \cdots \arrow[rr, dotted] & & A^{n-2} \sqcup B^{n-2} \arrow[rr, dotted] \arrow[dr, dotted] & & A^{n-1} \sqcup B^{n-1} \arrow[dr, dotted, "\theta^{n-1}"] & \\
& X^0 \arrow[rr] & & X^1 \arrow[rr] & & X^2 \arrow[r] & \cdots \arrow[rrr] & & & X^{n-2} \arrow[rr] & & X^{n-1} \\
B^0 \arrow[rr] \arrow[uu, "\beta^0"] \arrow[ur, "g^0"] & & B^1 \arrow[rr] \arrow[uu] \arrow[ur] & & B^2 \arrow[rr] \arrow[uu] \arrow[ur] & & \cdots \arrow[rr] & & B^{n-2} \arrow[rr] \arrow[uu] \arrow[ur] & & B^{n-1} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

(ii) the sequences

$$A^0 \xrightarrow{\alpha^0} A^0 \sqcup B^0 \xrightarrow{d_*^0} A^1 \sqcup B^1 \xrightarrow{d_*^1} ...A^{n-2} \sqcup B^{n-2} \xrightarrow{d_*^{n-2}} A^{n-1} \sqcup B^{n-1} \xrightarrow{\theta^{n-1}} X^{n-1}$$

and

$$B^0 \xrightarrow{\beta^0} A^0 \sqcup B^0 \xrightarrow{d_*^0} A^1 \sqcup B^1 \xrightarrow{d_*^1} ...A^{n-2} \sqcup B^{n-2} \xrightarrow{d_*^{n-2}} A^{n-1} \sqcup B^{n-1} \xrightarrow{\theta^{n-1}} X^{n-1}$$

exists such that the sequences $(d_*^0, ..., d_*^{n-2}, \theta^{n-1})$ is an n-coker of α^0 and $(d_*^0, ..., d_*^{n-2}, \theta^{n-1})$ is an n-coker of β^0 where we define

$$d_*^k := (A^{k+1} \sqcup B^{k+1}, \alpha^{k+1}\theta^k\alpha^k, \beta^{k+1}\theta^k\alpha^k) : A^k \sqcup B^k \longrightarrow A^{k+1} \sqcup B^{k+1}$$

for each $k \in \{0, 1, .., n-2\}$.

Proposition 4.2. Let $\mathcal{C}$ be an additive category of commutative k-algebra such that k is a commutative rings, $A, B \in Ch^{n-2}(\mathcal{C})$, then $A \bigotimes_k B$ is the n-coproduct in the category of commutative k-algebra.

Proof . We prove this by induction on n, the case in which $n = 1$ having been dealt with in proposion 5.2 [12] . In the case in which $n = 2$, for a 2-coproduct, we define $\alpha^i : A^i \longrightarrow A^i \bigotimes_k B^i$ by $a^i \mapsto a^i \otimes 1$ and define $\beta^i : B^i \longrightarrow A^i \bigotimes_k B^i$ by $b^i \mapsto 1 \otimes b^i$; note that both α^i and β^i are k-algebra in chain maps.

Now let $X \in Ch^0(\mathcal{C})$, $f^i : A^i \longrightarrow X^i$ and $g^i : B^i \longrightarrow X^i$ be k-algebra in chain maps. First, the diagram commute: by the diagram

$$\begin{tikzcd}
 & A^0 \arrow[rr] \arrow[dl, "\alpha^0"'] \arrow[dd] & & A^1 \arrow[dl, "\alpha^1"'] \arrow[dd, "f^1"] \\
A^0 \bigotimes B^0 \arrow[rr, dotted, "f^0"] \arrow[dr, dotted, "\Theta^0"] & & A^1 \bigotimes B^1 \arrow[dr, dotted, "\Theta^1"] & \\
 & X^0 \arrow[rr] & & X^1 \\
B^0 \arrow[uu, "\beta^0"] \arrow[ur, "g^0"] \arrow[rr] & & B^1 \arrow[uu, "\beta^1"] \arrow[ur, "g^1"] &
\end{tikzcd}$$

If $a^i \in A^i$ and $b^i \in B^i$ for $i = 1, 2$, so that $\Theta^i \beta^i(b^i) = \Theta^i(1 \otimes b^i) = g^i(b^i)$ and $\Theta^i \alpha^i(a^i) = \Theta^i(a^i \otimes 1) = f^i(a^i)$ for $i = 0, 1$.

The function $\varphi^i : A^i \times B^i \longrightarrow X^i$, given by $(a^i, b^i) \longmapsto f(a^i)g(b^i)$ for $i = 0, 1$, is easily seen to be k-bilinear, and so there is a unique map of k-modules $\Theta^i : A^i \bigotimes_k B^i \longrightarrow X^i$ with $\Theta^i(a^i \times b^i) = f(a^i)g(b^i)$ for $i = 0, 1$. To prove that Θ^i is an k-algebra in chain maps, it suffices to prove that $\Theta^i((a \otimes b)(a' \otimes b')) = \Theta^i(a \otimes b)\Theta^i(a' \otimes b')$ for $i = 0, 1$. Now for $i = 0, 1$

$$\Theta^i((a^i \otimes b^i)(a'^i \otimes b'^i)) = \Theta^i((a^i a'^i \otimes b^i b'^i) = f(a^i)f(a'^i)g(b^i)g(b'^i).$$

On the other hand, $\Theta^i(a^i \otimes b^i)\Theta^i(a'^i \otimes b'^i) = f(a^i)g(b^i)f(a'^i)g(b'^i)$ for $i = 0, 1$. Since X is commutative, $f(a'^i)g(b^i) = g(b^i)f(a'^i)$, and so Θ^i does preserve multiplication for $i = 0, 1$.

Second, Θ^i is unique for $i = 0, 1$. If $\Phi^i : A^i \otimes_k B^i \longrightarrow X^i$ be a k-algebra in chain maps making the diagram commutate for $i = 0, 1$. In $A^i \otimes_k B^i$, we have $(a^i \otimes b^i) = (a^i \otimes 1)(1 \otimes b^i) = \alpha^i(a^i)\beta^i(b^i)$, where $a^i \in A^i$ and $b^i \in B^i$ for $i = 0, 1$. Thus, for $i = 0, 1$

$$\Phi^i(a^i \otimes b^i) = \Phi^i[\alpha^i(a^i)\beta^i(b^i)] = \Phi^i(\alpha^i(a^i))\Phi^i(\beta^i(b^i)) = f(a^i)g(b^i) = \Phi^i(a^i \otimes b^i).$$

Since $A^i \otimes_k B^i$ is generated as a k-module by all $a^i \otimes b^i$, we have $\Psi^i = \Phi^i$ for $i = 0, 1$.

Third, the sequence (d^0_*, Θ^0) is 2-coker of α^1. By the definition α^0 is injection, then sequence $A^0 \xrightarrow{\alpha^0} A^0 \bigotimes B^0 \xrightarrow{d^0_*} A^1 \bigotimes B^1 \xrightarrow{\Theta^1} X^1$ is exact. Now let $Y \in (\mathcal{C})$ the induced sequence of abelian groups

$$0 \longrightarrow Hom(X^1, Y) \xrightarrow{\Theta^1 .?} Hom(A^1 \bigotimes B^1, Y) \xrightarrow{d^0_* .?} Hom(A^0 0 \bigotimes B^0, Y) \xrightarrow{\alpha^0 .?} Hom(A^0, Y),$$

so that is exact. For proof of (d^0_*, Θ^0) is an 2-coker of β^0 is dual.

We now turn to the inductive step. Assume, inductively, that $n = (k' + 1)$, where $k \geq 2$, and that the result has been proved in the case where $n = k'$. In the case in which $n = (k' + 1)$, for a $(k' + 1)$-coproduct requires, we define $\alpha^{k'} : A^{k'} \longrightarrow A^{k'} \bigotimes_k B^{k'}$ by $a^{k'} \mapsto a^{k'} \otimes 1$ and define $\beta^{k'} : B^{k'} \longrightarrow A^{k'} \bigotimes_k B^{k'}$ by $b^{k'} \mapsto 1 \otimes b^{k'}$; note that both $\alpha^{k'}$ and $\beta^{k'}$ are k-algebra in chain maps.

Now let $X \in Ch^{k'}(\mathcal{C})$, and let $f^{k'} : A^{k'} \longrightarrow X^{k'}$ and $g^{k'} : B^{k'} \longrightarrow X^{k'}$ be k-algebra in chain maps.

$$\begin{tikzcd}[column sep=small]
 & A^0 \arrow[rr] \arrow[dl, "\alpha^0"'] \arrow[dd] & & A^1 \arrow[rr] \arrow[dl] \arrow[dd] & & A^2 \arrow[r] \arrow[dl] \arrow[dd] & \dots \arrow[rr] & & A^{k'-1} \arrow[rr] \arrow[dl] \arrow[dd] & & A^{k'} \arrow[dl, "\alpha^{k'}"'] \arrow[dd, "f^{k'}"] \\
A^0 \bigotimes B^0 \arrow[rr, dotted, "f^0"] \arrow[dr, dotted, "\Theta^0"] & & A^1 \bigotimes B^1 \arrow[rr, dotted] \arrow[dr, dotted] & & A^2 \bigotimes B^2 \arrow[r, dotted] \arrow[dr, dotted] & \dots \arrow[r, dotted] & A^{k'-1} \bigotimes B^{k'-1} \arrow[rrr, dotted] \arrow[drr, dotted] & & & A^{k'} \bigotimes B^{k'} \arrow[dr, dotted, "\Theta^{k'}"] & \\
 & X^0 \arrow[rr] & & X^1 \arrow[rr] & & X^2 \arrow[r] & \dots \arrow[rr] & & X^{k'-1} \arrow[rr] & & X^{k'} \\
B^0 \arrow[uu, "\beta^0"] \arrow[ur, "g^0"] \arrow[rr] & & B^1 \arrow[uu] \arrow[ur] \arrow[rr] & & B^2 \arrow[uu] \arrow[ur] \arrow[r] & \dots \arrow[r] & B^{k'-1} \arrow[uu] \arrow[urr] \arrow[rrr] & & & B^{k'} \arrow[uu, "\beta^{k'}"] \arrow[ur, "g^{k'}"] &
\end{tikzcd}$$

If $a^{k'} \in A^{k'}$ and $b^{k'} \in B^{k'}$, so that $\Theta^{k'}\beta^{k'}(b^{k'}) = \Theta^{k'}(1 \otimes b^{k'}) = g^{k'}(b^{k'})$ and $\Theta^{k'}\alpha^{k'}(a^{k'}) = \Theta^{k'}(a^{k'} \otimes 1) = f^{k'}(a^{k'})$.

The function $\varphi^{k'} : A^{k'} \times B^{k'} \longrightarrow X^{k'}$, given by $(a^{k'}, b^{k'}) \longmapsto f(a^{k'})g(b^{k'})$, is easily seen to be k-bilinear, and so there is a unique map of k-modules $\Theta^{k'} : A^{k'} \bigotimes_k B^{k'} \longrightarrow X^{k'}$ with $\Theta^{k'}(a^{k'} \times b^{k'}) = f(a^{k'})g(b^{k'})$. To prove that $\Theta^{k'}$ is an k-algebra in chain maps, it suffices to prove that $\Theta^{k'}((a^{k'} \otimes b^{k'})(a'^{k'} \otimes b'^{k'})) = \Theta^{k'}(a^{k'} \otimes b^{k'})\Theta^{k'}(a'^{k'} \otimes b'^{k'})$. Now

$$\Theta^{k'}((a^{k'} \otimes b^{k'})(a'^{k'} \otimes b'^{k'})) = \Theta^{k'}((a^{k'}a'^{k'} \otimes b^{k'}b'^{k'}) = f(a^{k'})f(a'^{k'})g(b^{k'})g(b'^{k'}).$$

On the other hand, $\Theta^{k'}(a^{k'} \otimes b^{k'})\Theta^{k'}(a'^{k'} \otimes b'^{k'}) = f(a^{k'})g(b^{k'})f(a'^{k'})g(b'^{k'})$. Since X is commutative, $f(a'^{k'})g(b^{k'}) = g(b^{k'})f(a'^{k'})$, and so $\Theta^{k'}$ does preserve multiplication.

Second, $\Theta^{k'}$ is unique. If $\Phi^{k'} : A^{k'} \otimes_k B^{k'} \longrightarrow X^{k'}$ be a k-algebra in chain maps making the diagram commutate. In $A^{k'} \otimes_k B^{k'}$, we have $(a^{k'} \otimes b^{k'}) = (a^{k'} \otimes 1)(1 \otimes b^{k'}) = \alpha^{k'}(a^{k'})\beta^{k'}(b^{k'})$, where $a^{k'} \in A^{k'}$ and $b^{k'} \in B^{k'}$. Thus,

$$\Phi^{k'}(a^{k'} \otimes b^{k'}) = \Phi^{k'}[\alpha^{k'}(a^{k'})\beta^{k'}(b^{k'})] = \Phi^{k'}(\alpha^{k'}(a^{k'}))\Phi^{k'}(\beta^{k'}(b^{k'})) = f(a^{k'})g(b^{k'}) = \Phi^{k'}(a^{k'} \otimes b^{K'}).$$

Since $A^{k'} \otimes_k B^{k'}$ is generated as a k-module by all $a^{k'} \otimes b^{k'}$, we have $\Psi^{k'} = \Phi^{k'}$.

Third, the sequence $(d_*^0, d_*^1, ..., d_*^{k-1}, \Theta^k)$ is $(k+1)$-coker of α^0. By the definition α^0 is injection and the sequence $(d_*^0, d_*^1, ..., d_*^{k-2}, \Theta^{k-1})$ is k-coker of α^0, then we have the sequence $A^0 \xrightarrow{\alpha^0} A^0 \bigotimes B^0 \xrightarrow{d_*^0} A^1 \bigotimes B^1 \xrightarrow{d_*^1} ... \xrightarrow{d_*^{k'-2}} A^{k'-2} \bigotimes B^{k'-2} \xrightarrow{\Theta^{k'-1}} X^{k'-1}$ such that, for all $Y \in (\mathcal{C})$ the induced sequence of abelian groups

$$0 \longrightarrow Hom(X^{k'-1}, Y) \xrightarrow{\Theta^{k'-1}.?} Hom(A^{k'-2} \bigotimes B^{k'-2}, Y) \xrightarrow{d_*^{k'-2}.?} ... \xrightarrow{d_*^1.?} Hom(A^1 \bigotimes B^1, Y) \xrightarrow{d_*^0.?} Hom(A^0 \bigotimes B^0, Y) \xrightarrow{\alpha^0.?} Hom(A^0, Y)$$

so that is exact. By $d_X^{k-1} : X^{k-1} \longrightarrow X^k$ is complex and by the definition (, for every object X) we put $X^k = A^k$ and we have $A^k \xrightarrow{\alpha^k} A^k \bigotimes B^k \xrightarrow{\theta^k} X^k$, then we have the sequence $A^0 \xrightarrow{\alpha^0} A^0 \bigotimes B^0 \xrightarrow{d_*^0} A^1 \bigotimes B^1 \xrightarrow{d_*^1} ... \xrightarrow{d_*^{k'-1}} A^{k'-1} \bigotimes B^{k'-1} \xrightarrow{\Theta^{k'}} X^{k'}$ and, for all $Y \in (\mathcal{C})$ the induced sequence of abelian groups

$$0 \longrightarrow Hom(X^{k'}, Y) \xrightarrow{\Theta^{k'}.?} Hom(A^{k'-1} \bigotimes B^{k'-1}, Y) \xrightarrow{d_*^{k'-1}.?} ... \xrightarrow{d_*^1.?} Hom(A^1 \bigotimes B^1, Y) \xrightarrow{d_*^0.?} Hom(A^0 \bigotimes B^0, Y) \xrightarrow{\alpha^0.?} Hom(A^0, Y)$$

so that is exact, and the sequence $(d_*^0, d_*^1, ..., d_*^{k-1}, d_*^k, \Theta^k)$ is $(k+1)$-coker of α^0. For proof of $(d_*^0, d_*^1, ..., d_*^{k-1}, \Theta^k)$ is $(k+1)$-coker of β^0 is dual. $\square$

Definition 4.3. let $\mathcal{C}$ be an additive category, A and X a complex in $Ch^n(\mathcal{C})$ is called an **n-initial object** if, for every object X in $\mathcal{C}$, there exists a unique chain maps $A \longrightarrow X$.

Lemma 4.4. let $\mathcal{C}$ an additive category, Any two n-initial objects A, A' a complex, in $Ch^n(\mathcal{C})$, should they exist, are n-isomorphic. In fact, the unique chain maps $f : A \longrightarrow A'$ is an n-isomorphism.

Proof . By hypothesis, there exist unique morphism in chain maps

$$\begin{array}{ccccccccccccc}
A & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & A^{n+1} \\
\downarrow f & & \downarrow f^0 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow f^{n+1} \\
A' & & A'^0 & \longrightarrow & A'^1 & \longrightarrow & A'^2 & \longrightarrow & ... & \longrightarrow & A'^n & \longrightarrow & A'^{n+1}
\end{array}$$

and

$$\begin{array}{ccccccccccccc}
A' & & A'^0 & \longrightarrow & A'^1 & \longrightarrow & A'^2 & \longrightarrow & ... & \longrightarrow & A'^n & \longrightarrow & A'^{n+1} \\
\downarrow g & & \downarrow g^0 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow g^{n+1} \\
A & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & A^{n+1}.
\end{array}$$

Since A is an n-initial object, the unique morphism in chain maps

$$\begin{array}{ccccccccccccc} A & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & A^{n+1} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h^0} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle h^{n+1}} \\ A & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & A^{n+1} \end{array}$$

must be the identity: $h = 1_A$. Thus the composition

$$\begin{array}{ccccccccccccc} A & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & A^{n+1} \\ \downarrow{\scriptstyle gf} & & \downarrow{\scriptstyle g^0 f^0} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle g^{n+1} f^{n+1}} \\ A & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & A^{n+1} \end{array}$$

and

$$\begin{array}{ccccccccccccc} A' & & A'^0 & \longrightarrow & A'^1 & \longrightarrow & A'^2 & \longrightarrow & ... & \longrightarrow & A'^n & \longrightarrow & A'^{n+1} \\ \downarrow{\scriptstyle fg} & & \downarrow{\scriptstyle f^0 g^0} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle f^{n+1} g^{n+1}} \\ A' & & A'^0 & \longrightarrow & A'^1 & \longrightarrow & A'^2 & \longrightarrow & ... & \longrightarrow & A'^n & \longrightarrow & A'^{n+1}. \end{array}$$

are identities, and so

$$\begin{array}{ccccccccccccc} A & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & A^{n+1} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle f^{n+1}} \\ A' & & A'^0 & \longrightarrow & A'^1 & \longrightarrow & A'^2 & \longrightarrow & ... & \longrightarrow & A'^n & \longrightarrow & A'^{n+1} \end{array}$$

is an n-isomorphism. $\square$

Proposition 4.5. Let $\mathcal{C}$ be an additive category, if A, B in $Ch^n(\mathcal{C})$, then any two n-coproducts of A and B, exists, are n-isomorphic.

Proof . If C is an n-coproduct of A and B, then there are morphisms in chain maps

$$\begin{array}{ccccccccccccc} A & & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & A^{n+1} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha^0} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \alpha^{n+1}} \\ C & & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & ... & \longrightarrow & C^n & \longrightarrow & C^{n+1} \end{array}$$

and

$$\begin{array}{ccccccccccccc} B & & B^0 & \longrightarrow & B^1 & \longrightarrow & B^2 & \longrightarrow & ... & \longrightarrow & B^n & \longrightarrow & B^{n+1} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta^0} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \beta^{n+1}} \\ C & & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & ... & \longrightarrow & C^n & \longrightarrow & C^{n+1} \end{array}$$

.

Define a new category $\mathcal{D}$ whose objects are diagrams

$$\begin{tikzcd}
A \arrow[d, "\gamma"] & A^0 \arrow[r] \arrow[d, "\gamma^0"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & ... \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "\gamma^{n+1}"] \\
X & X^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & ... \arrow[r] & X^n \arrow[r] & X^{n+1} \\
B \arrow[u, "\delta"] & B^0 \arrow[r] \arrow[u, "\delta^0"] & B^1 \arrow[r] \arrow[u] & B^2 \arrow[r] \arrow[u] & ... \arrow[r] & B^n \arrow[r] \arrow[u] & B^{n+1}, \arrow[u, "\delta^{n+1}"]
\end{tikzcd}$$

where X is a complex, in $Ch^n(\mathcal{C})$ and

$$\begin{tikzcd}
A \arrow[d, "\gamma"] & A^0 \arrow[r] \arrow[d, "\gamma^0"] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & ... \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[d, "\gamma^{n+1}"] \\
X & X^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & ... \arrow[r] & X^n \arrow[r] & X^{n+1}
\end{tikzcd}$$

and

$$\begin{tikzcd}
B \arrow[d, "\delta"] & B^0 \arrow[r] \arrow[d, "\delta^0"] & B^1 \arrow[r] \arrow[d] & B^2 \arrow[r] \arrow[d] & ... \arrow[r] & B^n \arrow[r] \arrow[d] & B^{n+1} \arrow[d, "\delta^{n+1}"] \\
X & X^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & ... \arrow[r] & X^n \arrow[r] & X^{n+1}
\end{tikzcd}$$

.

are morphism in chain maps. Define a morphism in $\mathcal{D}$ to be a triple $(1_A^i, \theta^i, 1_B^i)$, for all $i \in \{1, 2, ..., n+1\}$, where θ is a morphism in chain maps in $\mathcal{C}$ making the following diagram commute:

$$\begin{tikzcd}
& A^0 \arrow[rr] \arrow[dl, "1_{A^0}"'] \arrow[dd, "\gamma^0"] & & A^1 \arrow[rr] \arrow[dl] \arrow[dd] & & A^2 \arrow[rr] \arrow[dl] \arrow[dd] & & ... \arrow[rr] & & A^n \arrow[rr] \arrow[dl] \arrow[dd] & & A^{n+1} \arrow[dl, "1_{A^{n+1}}"'] \arrow[dd, "\gamma^{n+1}"] \\
A^0 \arrow[rr] \arrow[dr] \arrow[dd, "\gamma'^0"'] & & A^1 \arrow[rr] \arrow[dr] \arrow[dd] & & A^2 \arrow[rr] \arrow[dr] \arrow[dd] & & ... \arrow[rr] & & A^n \arrow[rr] \arrow[dr] \arrow[dd] & & A^{n+1} \arrow[dr] \arrow[dd, "\gamma'^{n+1}"] & \\
& X^0 \arrow[rr] \arrow[dl] & & X^1 \arrow[rr] \arrow[dl] & & X^2 \arrow[rr] \arrow[dl] & & ... \arrow[rr] & & X^n \arrow[rr] \arrow[dl] & & X^{n+1} \arrow[dl] \\
X'^0 \arrow[rr] & & X'^1 \arrow[rr] & & X'^2 \arrow[rr] & & ... \arrow[rr] & & X'^n \arrow[rr] & & X'^{n+1} & \\
& B^0 \arrow[rr] \arrow[dl] \arrow[uu, "\delta^0"] & & B^1 \arrow[rr] \arrow[dl] \arrow[uu] & & B^2 \arrow[rr] \arrow[dl] \arrow[uu] & & ... \arrow[rr] & & B^n \arrow[rr] \arrow[dl] \arrow[uu] & & B^{n+1} \arrow[dl] \arrow[uu, "\delta^{n+1}"'] \\
B^0 \arrow[rr] \arrow[uu, "\delta'^0"] & & B^1 \arrow[rr] \arrow[uu] & & B^2 \arrow[rr] \arrow[uu] & & ... \arrow[rr] & & B^n \arrow[rr] \arrow[uu] & & B^{n+1} \arrow[uu, "\delta'^{n+1}"] &
\end{tikzcd}$$

Define composition in $\mathcal{D}$ by $(1_A^i, \psi^i, 1_B^i)(1_A^i, \theta^i, 1_B^i) = (1_A^i, \psi^i\theta^i, 1_B^i)$, for all $i \in \{1, 2, ..., n+1\}$. It is easy to check that $\mathcal{D}$ is a category and that an n-coproduct in $\mathcal{C}$ is an n-initial object in $\mathcal{C}$. By Lemma (4.4), n-coproduct in a category are unique to (unique) n-isomorphism if they exist. □

Definition 4.6. Let $\mathcal{C}$ be an additive category, if A and B are complexes in Ch^{n-2}, then their **n-product** is a triple $(A \sqcap B, p, q)$,where $A \sqcap B$ is an complex in Ch^{n-2} and $p : A \sqcap B \longrightarrow A$, $q : A \sqcap B \longrightarrow B$ are morphisms in chain maps, called **n-projections**, such that,

- **(i)** for every complex X in Ch^{n-2} and every pair of morphism in chain maps $f : X \longrightarrow A$ and $g : X \longrightarrow B$, there exists a unique morphism in chain maps $\theta : X \longrightarrow A \sqcap B$ making the diagram commute: $p\theta = f$ and $q\theta = g$.

$$
\begin{tikzcd}
 & A^0 \arrow[rr] & & A^1 \arrow[rr] & & A^2 \arrow[rr] & & \cdots \arrow[rr] & & A^{n-2} \arrow[rr] & & A^{n-1} \\
A^0 \sqcap B^0 \arrow[ur, "p^0"] \arrow[rr, dotted] \arrow[dd, "q^0"'] & & A^1 \sqcap B^1 \arrow[ur] \arrow[rr, dotted] \arrow[dd] & & A^2 \sqcap B^2 \arrow[ur] \arrow[rr, dotted] \arrow[dd] & & \cdots \arrow[rr, dotted] & & A^{n-2} \sqcap B^{n-2} \arrow[ur] \arrow[rr, dotted] \arrow[dd] & & A^{n-1} \sqcap B^{n-1} \arrow[ur, "p^{n-1}"] \arrow[dd] & \\
 & X^0 \arrow[uu, "f^0"] \arrow[ul, dotted, "\theta^0"] \arrow[dl, "g^0"] \arrow[rr] & & X^1 \arrow[uu] \arrow[ul, dotted] \arrow[dl] \arrow[rr] & & X^2 \arrow[uu] \arrow[ul, dotted] \arrow[dl] \arrow[rr] & & \cdots \arrow[rr] & & X^{n-2} \arrow[uu] \arrow[ul, dotted] \arrow[dl] \arrow[rr, "q^{n-1}"] & & X^{n-1} \arrow[uu, "f^{n-1}"'] \arrow[ul, dotted, "\theta^{n-1}"] \arrow[dl, "g^{n-1}"] \\
B^0 \arrow[rr] & & B^1 \arrow[rr] & & B^2 \arrow[rr] & & \cdots \arrow[rr] & & B^{n-2} \arrow[rr] & & B^{n-1} &
\end{tikzcd}
$$

(ii) the sequences

$$X^0 \xrightarrow{\theta^0} A^0 \sqcap B^0 \xrightarrow{d_*^0} A^1 \sqcap B^1 \xrightarrow{d_*^1} ...A^{n-2} \sqcap B^{n-2} \xrightarrow{d_*^{n-2}} A^{n-1} \sqcup B^{n-1} \xrightarrow{p^{n-1}} A^{n-1}$$

and

$$X^0 \xrightarrow{\theta^0} A^0 \sqcap B^0 \xrightarrow{d_*^0} A^1 \sqcap B^1 \xrightarrow{d_*^1} ...A^{n-2} \sqcap B^{n-2} \xrightarrow{d_*^{n-2}} A^{n-1} \sqcap B^{n-1} \xrightarrow{q^{n-1}} B^{n-1}$$

is exist such that the sequences $(\theta^0, d_*^0, ..., d_*^{n-2})$ is an n-ker of p^{n-1} and $(\theta^0, d_*^0, ..., d_*^{n-2})$ is an n-coker of q^{n-1} where we define

$$d_*^k := (A^{k+1} \sqcap B^{k+1}, p^{k+1}\theta^{k+1}p^k, q^{k+1}\theta^{k+1}q^k) : A^k \sqcap B^k \longrightarrow A^{k+1} \sqcap B^{k+1}$$

for each $k \in \{0, 1, .., n-2\}$.

Definition 4.7. let $\mathcal{C}$ an additive category, Ω and X a complex in $Ch^n(\mathcal{C})$ is called an **n-terminal object** if, for every object X in $\mathcal{C}$, there exists a unique morphism in chain maps $X \longrightarrow \Omega$.

Lemma 4.8. let $\mathcal{C}$ be an additive category, Any two n-terminal objects Ω, Ω' a complex, $Ch^n(\mathcal{C})$, should they exist, are n-isomorphic. In fact, the unique chain maps $f : \Omega \longrightarrow \Omega'$ is an n-isomorphism.

Proof . Just reveres all the arrows in the proof of Lemma (4.4); that is, apply Lemma (4.4) to the opposite category $\mathcal{C}^{op}$. $\square$

Proposition 4.9. let $\mathcal{C}$ be an additive category, Any two n-terminal objects A and B a complex, $Ch^n(\mathcal{C})$, then any two n-product of A and B, should they exist, are n-isomorphic.

Proof . Adapt the proof of property, Proposition (4.5); n-products are n-terminal objects in suitable category. $\square$

5 n-Homology

Example 5.1. We show that n-ker is an n-pullback. Let $\mathcal{C}$ be an additive category, X a complex in $Ch^n(\mathcal{C})$, and $f^{n+1} : X^{n+1} \longrightarrow Y^{n+1}$ a morphism in complex.

Let $Z_0, Z_1 \in \mathcal{C}$ the induced sequence of abelian groups

$$\mathcal{C}(Z^0, X^0) \xrightarrow{?.d^0} \mathcal{C}(Z^1, X^1) \xrightarrow{?.d^1} ... \xrightarrow{?.d^{n-1}} \mathcal{C}(Z^n, X^n) \xrightarrow{?.d^n} \mathcal{C}(Z^{n+1}, X^{n+1}) \longrightarrow 0$$

is exact, equivalently, the sequence $(d_X^0, d_X^1, ..., d_X^{n-1})$ is an n-ker of d_Y^n. Such that in the mapping cone $C = C(f)$

$$X^0 = X^1 \coprod Y^0 \xrightarrow{d_c^0} X^2 \coprod Y^1 \xrightarrow{d_c^1} \ldots \xrightarrow{d_c^{n-2}} X^n \coprod Y^{n-1} \xrightarrow{d_c^{n-1}} X^{n+1} \coprod Y^n \xrightarrow{d_c^n} Y^{n+1}.$$

the sequence $(d_c^0, ..., d_c^{n-1})$ is an n-ker of d_c^n.

Definition 5.2. Let $\mathcal{C}$ be an additive category and given a partially ordered set I. An **n-inverse system** in $\mathcal{C}$ is an ordered pair $((M_i)_{i\in I}, (\psi_i^j)_{j\succeq i})$, abbreviated $\{M_i, \psi_i^j\}$, where $(M_i)_{i\in I}$ is an indexed family of complexes in $Ch^{n-2}(\mathcal{C})$ and $(\psi_i^j : M_i \longrightarrow M_j)_{j\succeq i}$ is an indexed family of morphisms in chain maps for which $\psi_i^i = 1_{M_i}$, , for all i, and such that the following diagrams commute where $k \succeq j \succeq i$.

$$\begin{tikzcd}
(M_k^0)_{k\in K_0} \arrow[r] \arrow[d, "(\psi^0)^i_k"'] \arrow[dd, "(\psi^0)^k_j"] & (M_k^1)_{k\in K_1} \arrow[r] \arrow[d] \arrow[dd] & (M_k^2)_{k\in K_2} \arrow[r] \arrow[d] \arrow[dd] & \ldots \arrow[r] & (M_k^{n-2})_{k\in K_{n-2}} \arrow[r] \arrow[d] \arrow[dd] & (M_k^{n-1})_{k\in K_{n-1}} \arrow[d, "(\psi^{n-1})^k_i"'] \arrow[dd, "(\psi^{n-1})^k_j"] \\
(M_i^0)_{i\in I_0} \arrow[r] & (M_i^1)_{i\in I_1} \arrow[r] & (M_i^2)_{i\in I_2} \arrow[r] & \ldots \arrow[r] & (M_i^{n-2})_{i\in I_{n-2}} \arrow[r] & (M_i^{n-1})_{i\in I_{n-1}} \\
(M_j^0)_{j\in J_0} \arrow[r] \arrow[u, "(\psi^0)^j_i"] & (M_j^1)_{j\in J_1} \arrow[r] \arrow[u] & (M_j^2)_{j\in J_2} \arrow[r] \arrow[u] & \ldots \arrow[r] & (M_j^{n-2})_{j\in J_{n-2}} \arrow[r] \arrow[u] & (M_j^{n-1})_{j\in J_{n-1}} \arrow[u, "(\psi^{n-1})^j_i"']
\end{tikzcd}$$

Definition 5.3. Let $\mathcal{C}$ be an additive category, I be a partially ordered set, $(M_i)_{i\in I}$ a complex in Ch^{n-2}, and let $\{M_i, \psi_j^i\}$ be an n-inverse system in $\mathcal{C}$ over I. The **n-inverse limit** (also called **n-projective limit** or **n-limit**) is an object $\varprojlim M_i$ and a family of **projection in chain maps** $(\alpha_i : \varprojlim M_i \longrightarrow M_i)_{i\in I}$ such that

- **(i)** $\psi_j^i \alpha_j = \alpha_i$ whenever $j \succeq i$
- **(ii)**, for every $X \in Ch^{n-2}(\mathcal{C})$ and all chain maps $f_i : X \longrightarrow M_i$ satisfying $\psi_j^i f_i = f_i$, for all $j \succeq i$, there exists a unique morphisms in chain maps $\theta : X \longrightarrow \varprojlim M_i$ making diagram commutes.

$$\begin{tikzcd}
(M_j^0) \arrow[r] \arrow[ddd] & (M_j^1) \arrow[r] \arrow[ddd] & (M_j^2) \arrow[r] \arrow[ddd] & \ldots \arrow[r] & (M_j^{n-2}) \arrow[r] \arrow[ddd] & (M_j^{n-1}) \arrow[ddd] \\
\varprojlim M_i^0 \arrow[r, dotted] \arrow[u] \arrow[dd] & \varprojlim M_i^1 \arrow[r, dotted] \arrow[u] \arrow[dd] & \varprojlim M_i^2 \arrow[r, dotted] \arrow[u] \arrow[dd] & \ldots \arrow[r, dotted] & \varprojlim M_i^{n-2} \arrow[r, dotted] \arrow[u] \arrow[dd] & \varprojlim M_i^{n-1} \arrow[u] \arrow[dd] \\
X^0 \arrow[r] \arrow[uu] \arrow[d] \arrow[u, dotted] & X^1 \arrow[r] \arrow[uu] \arrow[d] \arrow[u, dotted] & X^2 \arrow[r] \arrow[uu] \arrow[d] \arrow[u, dotted] & \ldots \arrow[r] & X^{n-2} \arrow[r] \arrow[uu] \arrow[d] \arrow[u, dotted] & X^{n-1} \arrow[uu] \arrow[d] \arrow[u, dotted] \\
(M_i^0) \arrow[r] & (M_i^1) \arrow[r] & (M_i^2) \arrow[r] & \ldots \arrow[r] & (M_i^{n-2}) \arrow[r] & (M_i^{n-1})
\end{tikzcd}$$

(iii) We have

$$X^0 \xrightarrow{\theta^0} \varprojlim M_i^0 \xrightarrow{d^0} \varprojlim M_i^1 \ldots \xrightarrow{d^{n-2}} \varprojlim M_i^{n-1} \xrightarrow{\alpha_i^{n-1}} M_i^{n-1}$$

the sequence $(\theta^0, d^0, ..., d^{n-2})$ is an n-ker of α^{n-1}, where defined

$$d^k := d_{M_i}^k \alpha_i^k \theta_i^{K+1} : \varprojlim M_i^k \longrightarrow \varprojlim M_i^{k+1}$$

for each $K \in \{0, 1, .., n-2\}$.

Definition 5.4. Let $\mathcal{C}$ be an additive category and given a partially ordered set I. An **n-direct system** in $\mathcal{C}$ is an ordered pair $((M_i)_{i\in I}, (\varphi_j^i)_{i\preceq j})$, abbreviated $\{M_i, \varphi_j^i\}$, where $(M_i)_{i\in I}$ is an indexed family of complexes in $Ch^{n-2}(\mathcal{C})$ and $(\varphi_j^i : M_i \longrightarrow M_j)_{i\preceq j}$ is an indexed family of morphisms in chain maps for which, $\varphi_i^i = 1_{M_i}$, for all i, and such that the following diagrams commute where $i \preceq j \preceq k$.

$$\begin{tikzcd}
(M_i^0)_{i\in I_0} \arrow[r] \arrow[d, "(\varphi^0)^i_k"'] \arrow[dd, "(\varphi^0)^j_k"] & (M_i^1)_{i\in I_1} \arrow[r] \arrow[d] \arrow[dd] & (M_i^2)_{i\in I_2} \arrow[r] \arrow[d] \arrow[dd] & ... \arrow[r] & (M_i^{n-2})_{i\in I_{n-2}} \arrow[r] \arrow[d] \arrow[dd] & (M_i^{n-1})_{i\in I_{n-1}} \arrow[d, "(\varphi^{n-1})^i_k"'] \arrow[dd, "(\varphi^{n-1})^j_k"] \\
(M_k^0)_{k\in K} & M_k^1)_{k\in K_1} \arrow[r] & (M_k^2)_{k\in K_2} \arrow[r] & ... \arrow[r] & (M_k^{n-2})_{k\in K_{n-2}} \arrow[r] & (M_k^{n-1})_{k\in K_{n-1}} \\
(M_j^0)_{j\in J_0} \arrow[r] \arrow[u, "(\varphi^0)^i_j"] & (M_j^1)_{j\in J_1} \arrow[r] \arrow[u] & (M_j^2)_{j\in J_2} \arrow[r] \arrow[u] & ... \arrow[r] & (M_j^{n-2})_{j\in J_{n-2}} \arrow[r] \arrow[u] & (M_j^{n-1})_{j\in J_{n-1}} \arrow[u, "(\varphi^{n-1})^i_j"]
\end{tikzcd}$$

Definition 5.5. Let $\mathcal{C}$ be an additive category, I be a partially ordered set, $(M_i)_{i\in I}$ a complex in Ch^{n-2}, and let $\{M_i, \varphi_i^j\}$ be an n-direct system in $\mathcal{C}$ over I. The **n-direct limit** (also called **n-inductive limit** or **n-colimit**) is an object $\varinjlim M_i$ and **insertion morphisms in chain maps** $(\alpha_i : M_i \longrightarrow \varinjlim M_i)_{i\in I}$ such that

- (i) $\alpha_j\varphi_i^j = \alpha_i$ whenever $i \preceq j$
- (ii) Let $X \in Ch^{n-2}(\mathcal{C})$, and let there be given morphisms in chain maps $f_i : M_i \longrightarrow X$ satisfying $f_j\varphi_i^j = f_i$, for all $i \preceq j$. There exists a unique morphism in chain maps $\theta : \varinjlim M_i \longrightarrow X$ making diagram commutes.

$$\begin{tikzcd}
(M_j^0) \arrow[r] \arrow[d] \arrow[dd] & (M_j^1) \arrow[r] \arrow[d] \arrow[dd] & (M_j^2) \arrow[r] \arrow[d] \arrow[dd] & ... \arrow[r] & (M_j^{n-2}) \arrow[r] \arrow[d] \arrow[dd] & (M_j^{n-1}) \arrow[d] \arrow[dd] \\
\varinjlim M_i^0 \arrow[r, dotted] \arrow[d, dotted] & \varinjlim M_i^1 \arrow[r, dotted] \arrow[d, dotted] & \varinjlim M_i^2 \arrow[r, dotted] \arrow[d, dotted] & ... \arrow[r, dotted] & \varinjlim M_i^{n-2} \arrow[r, dotted] \arrow[d, dotted] & \varinjlim M_i^{n-1} \arrow[d, dotted] \\
X^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & ... \arrow[r] & X^{n-2} \arrow[r] & X^{n-1} \\
(M_i^0) \arrow[r] \arrow[u] \arrow[uu] \arrow[uuu] & (M_i^1) \arrow[r] \arrow[u] \arrow[uu] \arrow[uuu] & (M_i^2) \arrow[r] \arrow[u] \arrow[uu] \arrow[uuu] & ... \arrow[r] & (M_i^{n-2}) \arrow[r] \arrow[u] \arrow[uu] \arrow[uuu] & (M_i^{n-1}) \arrow[u] \arrow[uu] \arrow[uuu]
\end{tikzcd}$$

- (iii) We have

$$M_i^0 \xrightarrow{\alpha_i^0} \varinjlim M_i^0 \xrightarrow{d^0} \varinjlim M_i^1 ... \xrightarrow{d^{n-2}} \varinjlim M_i^{n-1} \xrightarrow{\theta^{n-1}} X^{n-1}$$

  the sequence $(d^0, ..., d^{n-2}, \theta^{n-1})$ is an n-coker of α_i^0, where defined

$$d^k := \alpha_i^{k+1} d^k_{M_i} \theta^k : \varinjlim M_i^k \longrightarrow \varinjlim M_i^{k+1}$$

  for each $K \in \{0, 1, .., n-2\}$.

Definition 5.6. Let $\mathcal{C}$ be an n-abelian category, X a complex in $Ch^n(\mathcal{C})$, define

$$(n) - i - \textbf{cycles} = n - Z^i(X^n) = n - \ker d^n = (d^0, d^1, ..., d^{n-1})$$

$$(n) - i - \textbf{boundaries} = n - B^i(X^n) = (n) - imd^{n-1}$$

for all $i \in \{0, 1, 2, ..., n\}$. Notice that $n - Z^i$ and $n - B^i$ all lie in $\mathcal{C}$.

Definition 5.7. Let $\mathcal{C}$ be an n-abelian category, X a complex in $Ch^n(\mathcal{C})$ and $d^n : X^n \longrightarrow X^{n+1}$ a morphism in $\mathcal{C}$. A ith **n-homology** of d^n is, for all $i \in \{0, 1, 2, ..., n\}$

$$n-H_i(X)=\frac{n-Z^i(X)}{n-B^i(X)}=\frac{n-\ker d^n}{n-imd^{n-1}}=\frac{(d^0,d^1,...,d^{n-1})}{(imd^0,imd^1,...,imd^{n-1})}$$

Now $n-H_i(X)$ lies in $n-obj(\mathcal{C})$ if n-quotient are viewed as object, as on definition (3.8). However, if we recognize $\mathcal{C}$ as a full subcategory of **n-Ab**, then an element of $n-H_i(\mathcal{C})$ is a coset $(z^0+imd^0,z^1+imd^1,...,z^{n-2}+imd^{n-2},z^{n-1}+imd^{n-1})$; we call this element a **n-homology class** , and often denote it by $cls(z^0,z^1,...,z^{n-2},z^{n-1})$.

Proposition 5.8. If $\mathcal{C}$ is an n-abelian category, then $n-H_i:Comp(\mathcal{C})\longrightarrow(\mathcal{C})$ is an additive category for each $i\in Z$.

Proof . Let X and Y are complex in $Ch^n(\mathcal{C})$, if $f:X\longrightarrow Y$ is a chain map, define $n-H_i(f):n-H_i(X)\longrightarrow n-H_i(Y)$ by

$$n-H_i(f):cls(z^0,z^1,...,z^{n-1})\mapsto cls(f_iz^0,f_iz^1,...,f_iz^{n-1}).$$

We must show that f_iz^k is a cycle, for all $0\le k\le n-1$, and that $n-H_i(f)$ is independent of the choice of cycle z^k; both of these the following from f being a chain map; that is, from commutatively of following diagram:

$$\begin{array}{ccccccccccccc}
X & & X^0 & \xrightarrow{d_X^0} & X^1 & \longrightarrow & X^2 & \longrightarrow & ... & \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1}\\
\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f^0} & & \big\downarrow & & \big\downarrow & & & & \big\downarrow & & \big\downarrow{\scriptstyle f^{n+1}}\\
Y & & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \longrightarrow & Y^2 & \longrightarrow & ... & \longrightarrow & Y^n & \xrightarrow{d_Y^n} & Y^{n+1}.
\end{array}$$

First, let z be an i-cycle in $n-Z_i(\mathcal{X})$, so that $d_X^iz^k=0$, for all $0\le k\le n-1$. Then commutativity of the diagram gives $d_Y^kf^kz^k=f^{k+1}d_X^kz^k=0$, so that f_iz^k is an i-cycle.

Next, assume that $(z^0+imd^0,z^1+imd^1,...,z^{n-1}+imd^{n-1})=(y^0+imd^0,y^1+imd^1,...,y^{n-1}+imd^{n-1})$; hence, $z^k-y^k\in imd_X^k$, for all $0\le k\le n-1$; $z^ky^k=d_X^{k-1}x^{k-1}$ for some $x^{k-1}\in X^{k-1}$ and, for all $0\le k\le n$. Applying f_i gives

$$f_i^kz^k-f_i^ky^k=f_i^kd_X^{k-1}x^{k-1}=d_Y^{k-1}f^{k-1}x^{k-1}\in imd_Y^{k-1}$$

for all $0\le k\le n-1$. Thus, $cls(z^0,z^1,...,z^{n-1})=cls(y^0,y^1,...,y^{n-1})$, and $n-H_i(f)$ is well defined

Let us now see that $n-H_i$ is a functor. It obvious that $n-H_i(1_X)$ is the identity. If f and g are chain maps whose composite gf is defined, then, for every i-cycle z^k, wh have

$$\begin{aligned}
n-H_i(gf):cls(z^0,z^1,...,z^{n-1})\mapsto&(gf)_icls(z^0,z^1,...,z^{n-1})\\
=&g_if_icls(z^0,z^1,...,z^{n-1})\\
=&n-H_i(g)cls(f_i(z^0),f_i(z^1),...,f_i(z^{n-1}))\\
=&n-H_i(g)n-H_i(f)cls(z^0,z^1,...,z^{n-1}).
\end{aligned}$$

Finally $n-H_i$ is additive: if $f,g:(X,d_X)\longrightarrow(Y,d_Y)$ are chain maps, then

$$\begin{aligned}
H_i(f+g):cls(z^0,z^1,...,z^{n-1})\mapsto&(f_i+g_i)cls(z^0,z^1,...,z^{n-1})\\
=&cls((f_i+g_i)(z^0),(f_i+g_i)(z^1),...,(f_i+g_i)(z^{n-1}))\\
=&(H_i(f)+H_i(g))cls(z^0,z^1,...,z^{n-1}).
\end{aligned}$$

□

Proposition 5.9. Let $\mathcal{C}$ be an n-abelian category. If

$$X^0\xrightarrow{d^0}X^1\xrightarrow{d^1}...\xrightarrow{d^{n-1}}X^n\xrightarrow{d^n}X^{n+1}.$$

is an n-exact sequence in $Ch^n(\mathcal{C})$, then, for each $i \in Z$, there is a morphism in $\mathcal{C}$

$$\sigma_i : n - H_i(X^{n+1}) \longrightarrow H_{i-1}(X^0)$$

defined by

$$\begin{aligned}\sigma_i : cls(z^0_{n+1}, z^1_{n+1}, ..., z^{n-1}_{n+1}) \longrightarrow & cls((d^0_{X_{i-1}})^{-1}(d^1_{X_{i-1}})^{-1}(d^2_{X_{i-1}})^{-1}...(d^{n-1}_{X_{i-1}})^{-1}(d^{X^n}_i)(d^n_{x_i})^{-1}z^0_{n+1} \\ & ,(d^0_{X_{i-1}})^{-1}(d^1_{X_{i-1}})^{-1}(d^2_{X_{i-1}})^{-1}...(d^{n-1}_{X_{i-1}})^{-1}(d^{X^n}_i)(d^n_{x_i})^{-1}z^1_{n+1} \\ & ,... \\ & ,(d^0_{X_{i-1}})^{-1}(d^1_{X_{i-1}})^{-1}(d^2_{X_{i-1}})^{-1}...(d^{n-1}_{X_{i-1}})^{-1}(d^{X^n}_i)(d^n_{x_i})^{-1}z^{n-1}_{n+1})\end{aligned}$$

Proof . We will make many notation abbreviation in this proof. Consider the commutative diagram having n-exact rows:

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & & & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
X^0_{i+1} \arrow[r, "d^0_{X_{i+1}}"] \arrow[d, "d^{X^0}_{i+1}"'] & X^1_{i+1} \arrow[r, "d^1_{X_{i+1}}"] \arrow[d, "d^{X^1}_{i+1}"] & X^2_{i+1} \arrow[r, "d^2_{X_{i+1}}"] \arrow[d, "d^{X^2}_{i+1}"] & ... \arrow[r] & & X^{n-1}_{i+1} \arrow[r, "d^{n-1}_{X_{i+1}}"] \arrow[d, "d^{X^{n-1}}_{i+1}"] & X^n_{i+1} \arrow[r, "d^n_{X_{i+1}}"] \arrow[d, "d^{X^n}_{i+1}"] & X^{n+1}_{i+1} \arrow[d, "d^{X^{n+1}}_{i+1}"] \\
X^0_i \arrow[r, "d^0_{X_i}"'] \arrow[d, "d^{X^0}_i"'] & X^1_i \arrow[r, "d^1_{X_i}"'] \arrow[d, "d^{X^1}_i"] & X^2_i \arrow[r, "d^2_{X_i}"'] \arrow[d, "d^{X^2}_i"] & ... \arrow[r] & & X^{n-1}_i \arrow[r, "d^{n-1}_{X_i}"'] \arrow[d, "d^{X^{n-1}}_i"] & X^n_i \arrow[r, "d^n_{X_i}"'] \arrow[d, "d^{X^n}_i"] \arrow[d, bend right] & X^{n+1}_i \arrow[l, bend right] \arrow[d, "d^{X^{n+1}}_i"] \\
X^0_{i-1} \arrow[r, "d^0_{X_{i-1}}"'] \arrow[d] & X^1_{i-1} \arrow[l, bend right] \arrow[r, "d^1_{X_{i-1}}"'] \arrow[d] & X^2_{i-1} \arrow[l, bend right] \arrow[r, "d^2_{X_{i-1}}"'] \arrow[d] & ... \arrow[r] & & X^{n-1}_{i-1} \arrow[l, bend right] \arrow[r, "d^{n-1}_{X_{i-1}}"'] \arrow[d] & X^n_{i-1} \arrow[l, bend right] \arrow[r, "d^n_{X_{i-1}}"'] \arrow[d] & X^{n+1}_{i-1}. \arrow[d] \\
, & , & , & ... & & , & , & ,
\end{tikzcd}$$

Let $z^k_{n+1} \in X^{n+1}_i$ and $x^k_n \in X^n_i$, for all $0 \leq k \leq n-1$, then by definition n-abelian category $d^n_{X_i}$ is epimorphism, by 2.3 coker $d^n_{X_i} = 0$, then $d^n_{X_i}$ is surjective and $d^n_{X_i} x^k_n = z^k_{n+1}$. Now push x^k_n down to $d^{X^n}_i x^k_n \in X^n_{i-1}$. By commutatively, $d^{n-1}_{X_{i-1}} d^{X^{n-1}}_i x^k_{n-1} = d^{X^n}_i d^{n-1}_{X_i} x^k_{n-1}$, so that $d^{n-1}_{X_{i-1}} d^{X^{n-1}}_i x^k_{n-1} \in imd^{n-1}_{X_{i-1}}$ and $d^{X^n}_i d^{n-1}_{X_i} x^k_{n-1} \in X^n_{i-1}$. Thus $d^{X^n}_i x^n \in imd^{n-1}_{X_{i-1}}$. Since $(d^0, d^1, ..., d^{n-1})$ is n-ker of d^n, for $z^k_0 \in X^0_{i-1}$, $d^{n-1}d^{n-2}...d^1d^0z^k_0 \subseteq n - \ker d^n_{X_{i-1}}$. Given that $imd^{n-1}_{X_{i-1}} \subseteq \ker d^n_{X_{i-1}} \subseteq n - \ker d^n_{X_{i-1}}$, then $d^{X^n}_i x^k_n = d^{n-1}d^{n-2}...d^1d^0x^k_0$, for all $0 \leq k \leq n-1$, for $d^0_{X_{i-1}}$ is injection, because $\ker d^0_{X_{i-1}} = 0$ by definition n-abelian category. Thus

$$(d^0_{X_{i-1}})^{-1}(d^1_{X_{i-1}})^{-1}(d^2_{X_{i-1}})^{-1}...(d^{n-1}_{X_{i-1}})^{-1}(d^{X^n}_i)(d^n_{x_i})^{-1}z^k_{n+1}$$

make senses; that is, the claim is that

$$cls(z^0_{n+1}, z^1_{n+1}, ..., z^{n-1}_{n+1}) = cls(x^0_0, x^1_0, ..., x^{n-1}_0)$$

is a well-defined homomorphism. First, let us show independence of the choice of lifting. Suppose that $d^n_{X_i} \check{x}^k_n = z^k_{n+1}$, where $\check{x}^k_n \in X_n$. Then $\check{x}^k_n = x^k_n$. By the definition of complex,

$$(d^0)^{-1}(d^1)^{-1}...(d^{n-2})^{-1}(d^{n-1})^{-1}(d^{X^n})\check{x}^k_n = (d^0)^{-1}(d^1)^{-1}...(d^{n-2})^{-1}(d^{n-1})^{-1}(d^{X^n})x^k_n \in B^{X^0}_{i-1};$$

that is,

$$cls((d^0)^{-1}(d^1)^{-1}...(d^{n-2})^{-1}(d^{n-1})^{-1}(d^{X^n})\check{x}^k_n) = cls((d^0)^{-1}(d^1)^{-1}...(d^{n-2})^{-1}(d^{n-1})^{-1}(d^{X^n})x^k_n).$$

Thus, the formula gives a well-defined function

$$Z_{X^{n+1}} \longrightarrow \frac{X^0}{B^{X^0}_{i-1}}$$

Second, the function $Z_{X^{n+1}} \longrightarrow \frac{X^0}{B_{i-1}^{X^0}}$ is a homomorphism. If $z_{n+1}^k, z'^k_{n+1} \in Z_{X^{n+1}}$, let $d^n x_n^k = z_{n+1}^k$ and $d^n x'^k_n = z'^k_{n+1}$, for all $0 \le k \le n-1$. Since the definition of σ independent of the choice of lifting, choose $x_n^k + x'^k_n$ as a lifting of $z_{n+1}^k + z'^k_{n+1}$, for all $0 \le k \le n-1$. This step may now be completed in a routine way.

Third, we show that x_0^k is a cycle: by $d_i^{X^n} x_n^k = d^{n-1}d^{n-2}...d^1 d^0 x_0^k = 0$, for all $0 \le k \le n-1$, d^0 is injective. Hence, the formula gives a homomorphism

$$Z_{X^{n+1}} \longrightarrow \frac{Z_{X^0}}{B_{i-1}^{X^0}} = 1 - H_{i-1}$$

Finally, the subgroup $B_{i-1}^{X^0}$ goes into $B_{i-1}^{X^{n+1}}$. Suppose that $z_{n+1}^k = d^{X^{n+1}} x_{n+1}^k$, where $x_{n+1}^k \in X_{i+1}^{n+1}$, and let $d^n x_n^k = x_{n+1}^k$, where $x_{n+1}^k \in X_{i+1}^n$, for all $0 \le k \le n-1$. Commutatively gives $d^n d^{X^n} x_n^k = d^{X^{n+1}} d^n x_n^k = d^{X^{n+1}} x_{n+1}^k = z_{n+1}^k$, for all $0 \le k \le n-1$. Since $\sigma(z_{n+1}^k)$ is independent of the choice of lifting, we choose $d^{X^n} x_n^k$ whit $d^n d^{X^n} x_n^k = z_{n+1}^k$, for all $0 \le k \le n-1$, and so

$$\begin{aligned}\sigma(cls(z_{n+1}^0, z_{n+1}^1, ..., z_{n+1}^{n+1})) =& cls((d^0)^{-1}(d^1)^{-1}...(d^{n-2})^{-1}(d^{n-1})^{-1}(d^{X^n})(d^{X^n})(x_0^0)) \\ &, cls((d^0)^{-1}(d^1)^{-1}...(d^{n-2})^{-1}(d^{n-1})^{-1}(d^{X^n})(d^{X^n})(x_0^1)) \\ &, ... \\ &, cls((d^0)^{-1}(d^1)^{-1}...(d^{n-2})^{-1}(d^{n-1})^{-1}(d^{X^n})(d^{X^n})(x_0^{n-1})) \\ &= cls(0,0,...,0).\end{aligned}$$

Thus, the formula gives a homomorphism $\sigma_i : n - H_i(X^{n+1}) \longrightarrow H_{i-1}(X^0)$. $\square$

Theorem 5.10. ((n+1)-Long Exact sequence) Let $\mathcal{C}$ be an n-abelian category. If

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}. \tag{5.1}$$

is an n-exact sequence in $Ch^n(\mathcal{C})$, then there is an $(n+1)$-exact sequence in C

$$\begin{aligned}&0 - H_i(X^0) \xrightarrow{d_*^0} 1 - H_i(X^1) \xrightarrow{d_*^1} 2 - H_i(X^2) \xrightarrow{d_*^2} ... \\ &\xrightarrow{d_*^{n-1}} n - H_i(X^n) \xrightarrow{d_*^n} (n+1) - H_i(X^{n+1}) \xrightarrow{\delta_i} 0 - H_{i+1}(X^0).\end{aligned} \tag{5.2}$$

Proof . We have to prove that, the sequence (5.1) have a $(n+1)$-ker and $(n+1)$-coker. By (5.1) is n-exact sequence, then $(d^0, d^1, ..., d^{n-1})$ is n-ker of d^n, so that $d^n d^{n-1} d^{n-2}...d^1 d^0 = 0$. Thus $\delta_i d_*^n d_*^{n-1} d_*^{n-2}...d_*^1 d_*^0 = \delta_i (d^n d^{n-1} d^{n-2}...d^1 d^0)_* = \delta_i 0_* = 0$, then $(d_*^0, d_*^1, ..., d_*^{n-1}, d_*^n)$ is n-ker of δ_i. This is the sequence (5.2) is n-coker now follows by duality. $\square$

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