Int. J. Nonlinear Anal. Appl. 15 (2024) 1, 361–371
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.29135.4069

The generalized solution of fractional differential equations sample with Brownian motion

Nabil Laiche[a,*], Laid Gasmi[b], Zouaoui Chikr Elmezouar[c], Özen Özer[d]

[a]Department of Mathematics, Oum El Bouaghi University, Algeria
[b]Department of Mathematics, Adrar University, Algeria
[c]Department of Mathematics, College of Science King Khalid University, Abha, Saudi Arabia
[d]Department of Mathematics, Kırklareli University, Faculty of Science and Arts, Turkey

(Communicated by Mugur Alexandru Acu)

Abstract

It is well known that the solution of fractional models proved to be a powerful tool in studying various problems which appear in the sciences of real life. In view of the fact that economic applications are accelerating at an amazing pace, and the large number of modeling in this speciality, it has expanded the number of problems. So, our contribution is based on finding generalized solutions of a fractional differential equation known for their applications in microeconomics and finance and creating an algorithm which allows us to estimate the coefficients of this type of equation. And to really illustrate our results we will choose a model known in the stochastic literature by COGARCH but with fractional derivative, to demonstrate the asymptotic behavior of the estimators, including the impact of fractional order on the space of stochastic differential equations.

Keywords: Fractional differential equations, Stochastic models, Caputo expression
2020 MSC: 26A33, 34A08

1 Introduction

Stochastic models have received increasing attention for their ability to represent in the domain of complex systems. This research axis has become necessary not just for mathematicians and staticians but also for people of chemistry and biology and more importantly for economists and physicists [12]. In economics, most phenomena take stochastic modeling, and the same for physics, there are several phenomena that require stochastic theory to model them [2]. And as a historical overview, Great credit goes to the biologist R. Brown in 1827, who had found the observation of a very irregular motion displayed by a pollen particle immersed in a fluid, and the applications of this field are not only applied in biology, it took about 80 years until Einstein and Smouluchowsk gave the logical interpretation of this phenomenon [3]. By a general definition we can say that Brownian motion is the archetypal problem of stochastic process theory [4].

*Corresponding author
Email addresses: nlaiche2020@gmail.com (Nabil Laiche), gasmi1983@gmail.com (Laid Gasmi), chikrtime@yahoo.fr (Zouaoui Chikr Elmezouar), ozenozer39@gmail.com (Özen Özer)

Received: November 2022 *Accepted:* September 2023

Fractional differential equations with Brownian motion have proven to be a powerful tool in the study of various problems in the life sciences, especially in economics and finance, the question that arises, why exactly in economics and finance. One of the main reasons for the success of fractional differential equations in science when the description of financial phenomena becomes locally simplified [11]. For example, most financial phenomena take shape of a differential equation and through unknown circumstances the phenomenon is pushed to take the form of a fractional derivative. Differential equations inaugurated modern theoretical quantitative science. which requires economists and theoretical mathematicians to understand the best numerical modeling and simulation methods in finance field [10]. According to the theory of influence of the research tools between them, it has been constituted what is called a coupling between the stochastic equations by the fractional derivative operator which is now called by the stochastic fractional differential equations. This idea gives a great opening to mathematicians to optimize the modelling of this type of models to use them in the in several applications. Through this study, we will shed light on the influence of fractional order on proposed sample of equations. the paper organized on the following plan

Second section is an extension to the theoretical study of a sample of stochastic processes with a fractional derivative and of exposing and generalizing some theorems of stochastic analysis. In **third section** we estimat model coefficients with moments method, the study is enriched by clear fractional theory. **Fourth section** based on the quote algorithm for estimating coefficients of fractional models as a generalized case. **Fifth section** illustrated by numerical simulations to prove the value of theoritical study.

2 Preliminaries

The fractional derivative is not considered as the classical derivative, for certainty we can say that the classical derivative is a particular case of the fractional derivative, in this chapter we will expose the great famous definitions of fractional differential calculus which help us to specify the solutions of our sample models. First, we pose the generalized definition of an integral known by Riemann integral, which is used to extract the fractional derivative.

We define left Riemann integral of order $\gamma > \mathbf{0}$ over the a finite interval $[a,b] \subset \mathbb{R}$ of function x by its expression

$$\mathbf{I}^{\gamma}_{(a,t)}x(t) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-\mathbf{1}}x(s)ds,\ t>a. \tag{2.1}$$

On the other hand, the right Riemann integral of same order γ defined by the following expression

$$\mathbf{I}^{\gamma}_{(t,b)}x(t) = \frac{1}{\Gamma(\gamma)}\int_t^b (s-t)^{\gamma-\mathbf{1}}x(s)ds,\ t<b. \tag{2.2}$$

Such as $\Gamma(\theta)$ is Gamma function defined by its expression

$$\Gamma(\theta) = \int_0^\infty t^{\theta-\mathbf{1}}\exp(-t)dt,\ Re(\theta) > \mathbf{0}.$$

holds, in particular if $\theta \in \mathbb{N}$, then

$$\Gamma(\theta+\mathbf{1}) = \theta\Gamma(\theta)$$

Definition 2.1. The left and right Riemann Liouville derivatives $\mathbf{d}^{\gamma}_{(a,t)}x(t)$ and $\mathbf{d}^{\gamma}_{(t,b)}x(t)$ of order γ of order are defined by

$$\begin{aligned}\mathbf{d}^{\gamma}_{(a,t)}x(t) &= \frac{d^N}{dt^N}\mathbf{I}^{(N-\gamma)}_{(a,t)}x(t) \\ &= \frac{1}{\Gamma(N-\gamma)}\frac{d^N}{dt^N}\left\{\int_a^t (t-s)^{N-\gamma-1}x(s)ds\right\},\ N\in\mathbb{N}.\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\mathbf{d}^{\gamma}_{(t,b)}x(t) &= (-1)^N\frac{d^N}{dt^N}\mathbf{I}^{(N-\gamma)}_{(t,b)}x(t) \\ &= \frac{(-1)^N}{\Gamma(N-\gamma)}\frac{d^N}{dt^N}\left\{\int_a^t (s-t)^{N-\gamma-1}x(s)ds\right\},\ N\in\mathbb{N}\end{aligned} \tag{2.4}$$

where $N = [\gamma] + 1$, $[.]$ means the integer part of γ. It can easily be noticed that

$$\mathbf{d}^{\mathbf{N}}_{(a,t)}x(t) = \mathbf{d}^{\mathbf{0}}_{(t,b)}x(t) = x(t).$$

In addition to

$$\begin{aligned} \mathbf{d}^{N}_{(a,t)}x(t) &= x^{(N)}(t) \\ \mathbf{d}^{N}_{(t,b)}x(t) &= (-1)^N x^{(N)}(t) \end{aligned}$$

Definition 2.2. The left and right Caputo fractional derivatives defined as follows

$$\mathbf{d}^{(C,\gamma)}_{(a,t)} = \mathbf{d}^{\gamma}_{(a,t)} \left\{ x(t) - \sum_{i=0}^{N-1} \frac{x^{(i)}(a)}{i!}(t-a)^i \right\} \tag{2.5}$$

and

$$\mathbf{d}^{(C,\gamma)}_{(t,b)} = \mathbf{d}^{\gamma}_{(t,b)} \left\{ x(t) - \sum_{i=0}^{N-1} \frac{x^{(i)}(b)}{i!}(b-t)^i \right\} \tag{2.6}$$

3 Some Basics of Stochastic Construction

This section is devoted to exposing and developing what is needed from the notions of stochastic equations, and the most famous theorems in the stochastic literature, and their impact in our study, and on the other hand, the preliminaries of fractional calculations, then we construct the sample which we will study, this sample which brings together the two stochastic and fractional derivative properties. First we will expose the preliminary bases for the maternal model

$$\left\{ \begin{array}{l} dx(t) = g(t, x(t)dt + h(t, x(t))dw(t) \\ x(t_0) = c_0, \; t_0 \le t \le M < \infty \end{array} \right. . \tag{3.1}$$

We start with the following theorem in reference [16].

Theorem 3.1. Let a stochastic process $(x(t))_{t\in\mathbb{R}}$ defined in a probability space (Ω, A, P) with expression (2.1), where $w(t)$ is Brownian motion, g and h tow measurable functions on the interval $[t_0, M]$, and these functions are checked on the following conditions. There exists a constant $C > 0$ such that
$\mathbf{A} : |g(t, x(t) - g(t, y(t)| + |h(t, x(t) - h(t, y(t)| \le C\,|x(t) - y(t)|$
This property is known by the Liptischizian condition.
$\mathbf{B} : |g(t, x(t))|^2 + |h(t, x(t))|^2 \le C^2(1 + x^2(t))$.
This condition is called restriction on growth. Then, the process accepts the unique solution.

Proof . This theorem must be proven because of its great importance in the following remarks, first we shall prove the uniqueness of solution we would like to show that

$$E\,|x(t) - y(t)|^2 = 0 \quad \text{for all} \quad t \in [t_0, M],$$

and we suppose that

$$E\,|x(t) - y(t)| = 0.$$

Such as $y(t)$ and $x(t)$ are two continuous solutions, in this case must be define a function φ

$$\varphi_k(t) = \left\{ \begin{array}{l} 1, \text{ if } |x(t)| \le k \text{ and } |y(t)| \le k \\ 0, \text{ otherwise} \end{array} \right.$$

Since $\varphi_k(t) = \varphi_k(t)\varphi_k(s),\; s \le t$, we have

$$\varphi_k(t)(x(t) - y(t)) = \varphi_k(t) \int_{t_0}^{M} \varphi_k(s)\{g(s, x(s)) - g(s, y(s))\}ds + \int_{t_0}^{M} \varphi_k(s)\{h(s, x(s)) - h(s, y(s))\}dw(s).$$

We apply **Lipschitiz** condition to bound the integral

$$\Phi(s) = |g(s, x(s)) - g(s, y(s)| + |h(s, x(s)) - h(s, y(s)| .$$

$$\varphi_k(s)\Phi(s) \leq k\varphi_k(s) |x(s) - y(s)| \leq 2k^2.$$

By **Schwartz inequality** we find

$$\begin{aligned} E\left[\varphi_k(t) |x(t) - y(t)|^2\right] \leq & 2E\left|\int_{t_0}^{M} \varphi_k(s)E\{g(s, x(s)) - g(s, y(s))\}ds\right|^2 + 2E\left|\int_{t_0}^{M} \varphi_k(s)\{h(s, x(s)) - h(s, y(s))\}dw(s)\right|^2 \\ \leq & 2(M - t_0)\int_{t_0}^{t} E\varphi_k(s) |g(s, x(s)) - g(s, y(s))|^2 + |h(s, x(s)) - h(s, y(s))|^2 \\ \leq & C_0\int_{t_0}^{t} E\varphi_k(s) |x(s) - y(s)|^2 ds = 0. \end{aligned}$$

Which shows that

$$E |x(t) - y(t)|^2 = 0.$$

□

Remark 3.2. If $x(t)$ and $y(t)$ are two solutions for equation (2.1), then

$$P\left(\sup_{t_0 \leq t \leq M} |x(t) - y(t)| = 0)\right) = 1.$$

Proof. See refrence [9].

3.1 General theorem construction

Many complex in nature and technology are described by differential equations,

$$\begin{cases} d_{(0,t)}^{(C,\beta)} x(t) = F(t, x(t), w(t)) \\ x(0) = \xi, \ t \in [t_0, T] \end{cases} .$$

The conditions of the first theorem on a stochastic process hold if the process is stochastic and fractional. Acronym. we will call these models fractional Stochastic processes. Now, we construct the following theorem.

Theorem 3.3. Let $(x(t))_{t\in\mathbb{R}}$ Frac-stochastic process defined in a probability space (Ω, A, P) with the following expression

$$d_{(t_0,t)}^{(C,q)} x(t) = g(t, x(t)dt + h(t, x(t))dw(t), \tag{3.2}$$

where, g and h are defined in the previous theorem and are checked on the following conditions. Let a constant $C > 0$ such that

1 : $|g(t, x(t) - g(t, y(t)| + |h(t, x(t) - h(t, y(t)| \leq C |x(t) - y(t)|$

2 : $|g(t, x(t))|^2 + |h(t, x(t))|^2 \leq C^2(1 + x^2(t))$.

Then, the process (3.2) accepts the solution.

Proof . The proof is very simple if we write process (3.2) under the following form

$$I_{(t_0,t)}^{q} d_{(t_0,t)}^{(C,q)}(x(t)) = I_{(t_0,t)}^{q} g(t, x(t)dt + I_{(t_0,t)}^{q} h(t, x(t))dw(t).$$

In addition to being

$${}_{(t_0,t)}^{(C,q)}(x(t)) = d_{(t_0,t)}^{q}(x(t) - x(t_0)), \ 0 < q < 1.$$

And

$$\begin{aligned} I^{q}_{(t_0,t)} d^{(C,q)}_{(t_0,t)}(x(t)) &= I^{q}_{(t_0,t)} d^{q}_{(t_0,t)}(x(t)-x(t_0)),\ 0<q<1 \\ &= x(t)-x(t_0) \end{aligned}$$

Also, the two functions $I^{q}_{(t_0,t)}g$ and $I^{q}_{(t_0,t)}h$ are conducive the tow propreties **1** and **2**. Thus, we will reach the following mathematical writing

$$x(t) = I^{q}_{(t_0,t)}g(t,x(t)dt + I^{q}_{(t_0,t)}h(t,x(t))dw(t) + x(t_0).$$

which shows that the sample of Frac-Stochastic models is a generalized case for stochastic models. Where $N=[\gamma]+1$ for $\gamma \notin \mathbb{N}$. We can extract the simple case which $\gamma \in]0,1[$ has to be applied several times in physics

$$\begin{cases} \mathbf{d}^{(C,\gamma)}_{(a,t)} = \mathbf{d}^{\gamma}_{(a,t)}\{x(t)-x(a)\} \\ \mathbf{d}^{(C,\gamma)}_{(t,b)} = \mathbf{d}^{\gamma}_{(t,b)}\{x(t)-x(b)\} \end{cases}.$$

If our function $x^{(i)}(t)$ is bounded $\{\left|\max x^{(i)}(t)\right| = K,\ i\in\mathbb{N}\}$, and N tends to infinity, in this case we have

$$\begin{aligned} \left|\mathbf{d}^{(C,\gamma)}_{(t,b)}\right| &\leq \left|\mathbf{d}^{\gamma}_{(t,b)}\left\{\max x(t) - \sum_{i=0}^{N-1}\frac{\max(x^{(i)}(b))}{i!}(b-t)^i\right\}\right| \\ &\leq \left|\max x^{(i)}(t)\right| \left|\mathbf{d}^{\gamma}_{(t,b)}\left\{1-\sum_{i=0}^{N-1}\frac{(b-t)^i}{i!}\right\}\right|,\ N\to\infty. \\ &\leq K\left|\mathbf{d}^{\gamma}_{(t,b)}\{1-\exp(b-t)\}\right| \\ &< \infty. \end{aligned}$$

□

We will recall some theoretical bases in the space of stochastic calculus, the stochastic process $w(t)$ is called a standard Brownian if the following conditions are satisfied

a) $w(0)=0$.

b) $w(t)$ is almost surely continuous in t.

c) $w(t)$ has independent increments.

d) $w(t)-w(s)$ obeys the normal distribution with mean zero and variance $t-s$.

There are fundamental properties that characterize brownian motion that need to be applied from the simulation section and among these properties we have

$$w(\frac{i+1}{N}) - w(\frac{i}{N}) = \frac{\varepsilon_i}{\sqrt{N}},$$

such that the sequence ε_i represents independent standard Gaussian random variables. Consider the sample of stochastic models with a first fractional derivative in the sense of Riemann Liouville with a Brownian motion $d^{\gamma}_{(a,t)}w(t)$

$$\mathbf{d}^{\gamma}_{(0,t)}x(t) = \beta(t)x(t)\mathbf{d}^{\rho}_{(0,t)}w(t).$$

We discuss in a simple way the model solutions for the case $\rho=\gamma=1$ we will find

$$\frac{\mathbf{d}x(t)}{x(t)} = \beta(t)\mathbf{d}w(t).$$

Then,

$$\int\frac{\mathbf{d}x(t)}{x(t)} = \int\beta(t)\mathbf{d}w(t).$$

Which shows that the solution is

$$x(t) = c \exp\left\{ \int \beta(t)\mathbf{d}w(t) \right\}.$$

Property

Suppose $g : [0,T] \to \mathbb{R}$ is continuously differentiable function, with $g(0) = g(T) = 0$, we défine

$$\int_0^T g dw(t) = -\int_0^T g'(t)w(t)dt.$$

This property known in the literature of stochastic calculus, by **PWZ** definition relative to the three mathematicians (Paley, Wiener and Zygmund). Here we need to represent Brownian motion so that we can construct the solution to the frational differential equation. So, we will take the famous brownian motion example see reference [8]. We can prove in the same way that the property remains correct in the case of fractional derivative

$$w(t) = \sum_{k=0}^{\infty} \sqrt{2}\varepsilon_k \frac{\sin\{(k-0.5)\pi t\}}{(k-0.5)\pi}. \tag{3.3}$$

So, we have

$$\mathbf{d}^{\gamma} x(t) = \beta(t)x(t)\mathbf{d}^{\rho} w(t). \tag{3.4}$$

First we have

$$\mathbf{d}^{\rho} w(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} w(s)ds.$$

We can write the model in the form

$$\begin{cases} \mathbf{d}^{(c,\gamma)}_{(0,t)} x(t) = \beta(t)x(t)\mathbf{d}^{\rho} w(t) = F(t, x(t), w(t)) \\ x(0) = x(T) = \varphi \end{cases}. \tag{3.5}$$

Lemma 3.4. The fractional equation with a Brownian motion (3.4) accepts a solution of the form

$$x(t) = x(0) + \frac{1}{\Gamma(\gamma)} \int_0^t \left\{ (t-s)^{\gamma-1} x(s) \frac{1}{\Gamma(\rho)} \int_0^s (t-s)^{\rho-1} w(s)ds \right\} dt.$$

Proof . It is easy to obtain that

$$\mathbf{d}^{\rho} w(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\rho-1} w(t)dt$$

We study the equations in the case $\rho = 1$ et γ remains containing in the interval $]0,1[$, and we keep the coefficient $\beta(t)$ as constant. Then, the equation becomes

$$\begin{cases} \mathbf{d}^{(C,\gamma)}_{(0,T)} x(t) = \beta(t)x(t)\mathbf{d}w(t) \\ x(0) = x(T). \end{cases} \tag{3.6}$$

We insert the Riemann-Liouville integral on both sides of the equation, we find

$$\mathbf{I}^{\gamma} \mathbf{d}^{(C,\gamma)}_{(0,T)} x(t) = \mathbf{I}^{\gamma} \beta(t)x(t)\mathbf{d}w(t).$$

$$x(t) - x(0) = \frac{1}{\Gamma(\gamma)} \beta(t) \int_0^T (T-s)^{\gamma-1} w(t)dx(t)dt.$$

By application **PWZ** property we find

$$x(t) = x(0) + \frac{1}{\Gamma(\gamma)} \int_0^t \left\{ (t-s)^{\gamma-1} x(s) \frac{1}{\Gamma(\rho)} \int_0^s (t-s)^{\rho-1} w(s)ds \right\} dt.$$

We can also obtain the solution of the equation numerically, where among the approximate properties of Brownian motion the Polygonal approximation.

$$w^n = w_{t_i} + (w_{t_i+1} - w_{t_i})\frac{t - t_i}{t_i - t_{i+1}} \; t \in [t_i, t_{i+1}]$$

$$d^q x(t) = g(t, x(t))dt + h(t, x(t))dw(t), \; 0 < q \leq 1 \tag{3.7}$$

□

4 Estimation

There are many approaches to estimation, and the samples of different stochastic models are innumerable and very complicated. Then, the nature of the approach is not always valid for estimation. In the stochastic literature, there are several methods for estimating nonlinear models in the continuous case, but the best known is the method of moments. For our sample, the COGARCH models were compiled by the statisticians Haug, Kluperllberg, Linder and Zapp in 2005 [4]. Moreover, this type of model is a special case of continuous stochastic models. And besides, the moment approach gives better estimation results. And to apply this method, it is necessary to specify a private category of fractional stochastic models. This model has acquired quite some attention in the physics literature relatively to its probabilistic properties and asymptotic behavior of its statistical inference

$$\begin{cases} d^{(C,q)}_{(0,t)} x(t) = \gamma_0(t)x(t) + \gamma_1(t)dt + \gamma_2(t)dw(t) \\ \quad x(0) = x_0, \; 0 < q < 1 \end{cases} \tag{4.1}$$

$(x(t))_{t>0}$ defined on some probability space (Ω, A, P) denoted by a fractional derivative in the Caputo sense, where $\{(\gamma_i(t))_{t\in\mathbb{R}}, i = 0, 1 \; or \; 2\}$ the coefficients part of the frac-stochastic model, such as $\gamma_0(t)\gamma_1(t)\gamma_2(t) \neq 0$, $w(t)$ represents brownian motion process.

Assumption 1: [4]. Under the following conditions, for any $T > 0$

$$\textbf{a)} \int_0^T |\gamma_i(t)| \, dt < \infty, \; i = 0, 1.$$

and

$$2\gamma_0(t) < 0.$$

Theorem 4.1. Under **Assumption 1,** we have the mean defined $E(x(t)) = m(t)$, and the variance $v(t)$ and

$$C(t, s) = E\left\{(x(t) - m(t))(x(s) - m(t))\right\}, t > s.$$

Functions of process (4.1) generated by its frac-stochastic expression are written respectively by

$$\begin{aligned} m(t) &= \varphi(t)m(0) \\ v(t) &= \varphi(t)\left\{v(0) + \int_0^t \varphi^{-1}(s)\gamma_1^2(s)dt\right\} \\ C(t, s) &= \varphi(t)\varphi^{-1}(s)v(s), \; t \geq s \geq 0. \end{aligned}$$

where

$$\varphi(t) = \exp\left\{\int_0^t 2\gamma_0(z)dz\right\}.$$

To demonstrate this theorem it is necessary to write the model (4.1) in the form of a product between a classical derivative and a function. Then, we will apply the theorem 2.1 of reference [14].

$$\begin{aligned} d^{(C,q)}_{(0,t)} x(t) &= \theta(t)dx(t) \\ &= \gamma_0(t)x(t)dt + \gamma_1(t)dw(t). \end{aligned}$$

In this case, the model will be written in a pure stochastic way

$$dx(t) = \theta^{-1}(t)\left\{\gamma_0(t)x(t)dt + \gamma_1(t)dw(t)\right\}.$$

The coefficients which we will estimate and we will designate by the following vector

$$\delta = (\gamma_0(t), \gamma_1(t), \gamma_2(t)).$$

Corollary 4.2. Under assumption 01 we have the following results

$$\begin{cases} m\left(0\right) = -\frac{\gamma_1(t)}{\gamma_0(t)} \\ v(0) = \frac{\gamma_2^2(t)}{\left|2\gamma_0^2(t)\right|} \\ v(h) = v(0)\exp\left\{\gamma_0(t)\left|h\right|\right\} \end{cases}$$

Corollary 4.3. It is assumed here that $\hat{\delta} = (\hat{\gamma}_{0,N}(t), \hat{\gamma}_{1,N}(t), \hat{\gamma}_{2,N}(t))$, where N represents the sample size. Then we have

$$\begin{cases} \hat{m}\left(0\right) = -\frac{\hat{\gamma}_{1,N}(t)}{\hat{\gamma}_{0,N}(t)} \\ \hat{v}(0) = \frac{(\hat{\gamma}_{2,N}(t))^2}{2\left|\hat{\gamma}_{0,N}(t))\right|^2} \\ \hat{v}(h) = v(0)\exp\left\{\hat{\gamma}_{0,N}(t))\left|h\right|\right\} \end{cases}$$

We seek the estimators through subconditions $\hat{v}(0) \to 0$ and $\hat{v}(h) \to 0$.

4.1 Consistency and Asymptotic Properties of Estimators

In the case of nonlinear models of discrete time series, the problem consistency and asymptotic properties of the estimators is solved by the famous theorem of Klimko and Nilsen [1]. But in the continuous stochastic case it suffices to verify the algirithem of the corollaries. To extract the asymptotic behavior of the estimators it is necessary to establish in the generalized sense the following theorem.

Theorem 4.4. Let $(x(t))_{t\geq 0}$ a fractional stochastic differential equation generated by the expression (4.1), with its coefficients, and when the fractional order tends to zero. The conditions $\hat{v}(0) \to 0$ and $\hat{v}(h) \to 0$ are checked. Then

H-1) $\hat{\delta} = (\hat{\gamma}_{0,N}(t), \hat{\gamma}_{1,N}(t), \hat{\gamma}_{2,N}(t)) \to \delta = (\gamma_0(t), \gamma_1(t), \gamma_2(t))$, when $N \to \infty$.

H-2) $\sqrt{N}(\hat{\delta} - \delta) \overset{G}{\to} N(0, 1)$,
where G indicates convergence in law.

Proof . The description of the equation with Euler approach's and applying the theorem of Klimko and Nilsen, the proof is obtained simply. □

5 Simulation

Simulation plays a fundamental role in demonstrating the efficiency of sample process estimation proposed in our study.

5.1 Model Simulation

This section is devoted to the simulation of the model (4.1) with real coefficients, the case where $q = 1$ represents the simulation of a classic model, which shows that fractional derivation is a generalization of normal derivation. We keep the same values for coefficients γ_0, γ_1 and γ_2 in order to make a constructive comparison between the two situations. we use here some convergence criteria noted as follows, γ_i, $i = 0, 1$ and 2 true values, but compared to the estimated values noted $\hat{\gamma}_{i,N}$, $i = 0, ..., 2$, where N sample size, NS (number of simulations) , $RMSE$ noted the root mean square error, here we consider the true values to be $\gamma_0 = 0.015, \gamma_1 = 0.05$ and $\gamma_2 = 0, 1$. To make a simulation it is necessary to take an example of Brownian movement defined by the expression (3.3) in reference [15].

$$w(t) = \sum_{k=0}^{\infty} \sqrt{2}\varepsilon_k \frac{\sin\left\{(k - 0.5)\pi t\right\}}{(k - 0.5)\pi},$$

	N	NS	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$
$q = 0.90$	1000	250	0.0245	0.0396	0.1456
	1500	250	0.0185	0.0511	0.0986
	3000	250	0.0162	0.0587	0.1234

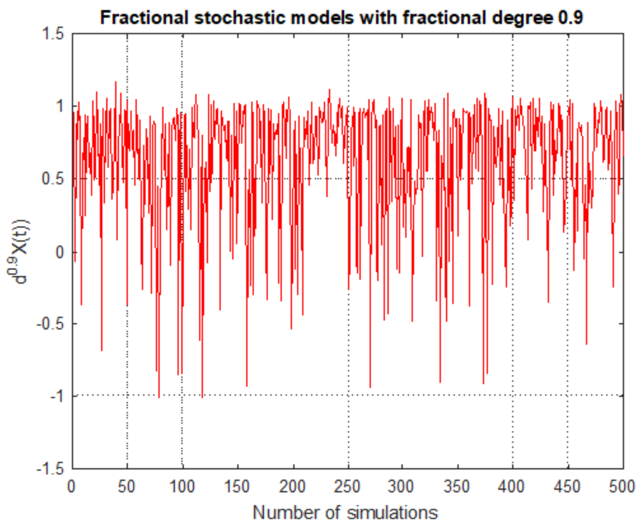
Table 1: Estimation for Frac-stochastic (4.1) model with true values $\gamma_0 = 0.015, \gamma_1 = 0.05$ and $\gamma_2 = 0, 1$

	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{2,N}$
$q = 0.5$	1000	500	0.0164	0.0548	1.4857
	1500	500	0.0245	0.0785	1.0142
	3000	500	0.0137	0.0864	0.9765
	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{1,N}$
$\beta = 0.2$	1000	500	0.1009	0.0458	1.2227
	1500	500	0.0354	0.2012	1.0165
	3000	500	0.0199	0.1025	0.1253
	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{1,N}$
$\beta = 0.1$	1000	500	0.0175	0.0589	1.1245
	1500	500	0.0402	0.0412	1.7856
	3000	500	0.0114	0.0415	0.9896

Table 2: Estimation for Frac-stochastic (4.1) model with true values $\gamma_0 = 0.015, \gamma_1 = 0.05$ and $\gamma_2 = 0, 1$

where the sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ is mutually independent standard Gaussian random variables. and by the polygonal approximation we have

$$w^n = w_{t_i} + (w_{t_i+1} - w_{t_i})\frac{t - t_i}{t_i - t_{i+1}}, \; t \in [t_i, t_{i+1}].$$



$RMSE$					
β	N	NS	$\hat{\gamma}_{0,N}$	$\hat{\gamma}_{1,N}$	$\hat{\gamma}_{1,N}$
0.7	1000	1000	0.0125	0.0102	0.0245
0.8	2000	2000	0.3125	0.0012	0.0290
0.9	5000	5000	0.0123	0.0223	0.0103

Table 3: RMSE for all situations simulation

5.2 Conclusion

We can notice that the classical case of COGARCH(1,1) presents a better approximation between the estimated values and the real values, which substantially shows the asymptotic behavior of the estimators in Table 1, where we notice that increasing the sample size and the number of simulations guarantees the convergence of the estimators to the real and almost identical values in the case N=3000 and NS=500. The frac-stochastique case keeps the same asymptotic behavior of the estimated values in terms of increasing NS and N which is illustrated in Table 2. Where numerical illustration in the Table 2 of our model simulation shows that the increase, the simulation in the second table proves that the increase in the value of the fractional derivative q makes the convergence clear between the model coefficients and their estimators each time ($q = \{0.2, 0.5 \text{ or } 0.9\}$, since if we compare Tables 3 and 2, we observe that the estimation by the moment approach very efficient in the models which take a fractional form because the estimated values of case q=0.9 almost closer to the estimated values of model. In Table 1 with the increase in the sample size N and NS. In most physicists the derivative according to Caputo is the best approximation, we observe when q tends to 1 then we will find the best approximation which shows that the Frac-stochastic model is the best approximation in our simulation. In Table 3, we observe that the RMSE criterion tends to zero where q approaches 1 with some perturbations. According to some experienced papers and estimation in this genre it can be deduced that fractional stochastic differential models are generalizations of stochastic models.

References

[1] A. Bibi and A.J. Oyet, *Estimation of some bilinear time series models with time varying coefficients*, Stochastic Anal. Appl. **22** (2004), no. 2, 355–376.

[2] L.C. Evans, *An Introduction to Stochastic Differential Equations*, American Mathematical Society, 2012.

[3] C. Francq and J.M. Zakoian, *GARCH Models: Structure, Statistical Inference and Financial Applications*, John Wiley & Sons, 2019.

[4] S. Haug, C. Klüppelberg, A. Lindner, and M. Zapp, *Method of moment estimation in the COGARCH (1, 1) model*, Economet. J. **10** (2007), no. 2, 320–341.

[5] R, Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, 2000.

[6] J. Kallsen and B. Vesenmayer, *COGARCH as a continuous-time limit of GARCH (1, 1)*, Stochastic Process. Appl. **119** (2009), no. 1, 74–98.

[7] M.B. Khan, G. Santos-García, M.A. Noor, and M. S. Soliman, *Some new concepts related to fuzzy fractional calculus for up and down convex fuzzy-number valued functions and inequalities*, Chaos Solitons Fractals **164** (2022), 112692.

[8] C. Klüppelberg, R. Maller, and A. Szimayer, *The COGARCH: a review, with news on option pricing and statistical inference*, Surveys Stochastic Processes, 2011, pp. 29–58.

[9] I. Koca and A. Atangana, *Some chaotic mathematical models with stochastic resetting*, Fractals **30** (2022), no. 8, 2240212.

[10] L. Nabil, G. Laid, and Z. Halim, *New perspectives in space of COGARCH (1, 1) models with fractional derivative*, Turk. J. Comput. Math. Educ. **14** (2023), no. 2, 877–887.

[11] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Elsevier, 1998.

[12] T.S. Rao and M.M. Gabr, *An introduction to Bispectral Analysis and Bilinear Time Series Models*, Springer Science & Business Media, 2012.

[13] E. Scalas, R. Gorenflo, and F. Mainardi, *Fractional calculus and continuous-time finance*, Phys. A: Statist. Mech. Appl. **284** (2000), no. 1-4, 376–384.

[14] J.M. Zakoian, *Threshold heteroskedastic models*, J. Econ. Dyn. Control **18** (1994), no. 5, 931–955.

[15] Z. Zhang and G. Karniadakis, *Numerical Methods for Stochastic Partial Differential Equations with White Noise*, Berlin, Springer, 2017.

[16] Y. Zhou, L. Shangerganesh, J. Manimaran, and A. Debbouche, *A class of time-fractional reaction-diffusion equation with nonlocal boundary condition*, Math. Meth. Appl. Sci. **41** (2018), no. 8, 2987–2999.

[17] Y. Zhou, J.R. Wang, and L. Zhang, *Basic Theory of Fractional Differential Equations*, World Scientific, 2016.