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Approximate analytical solutions to delay fractional differential equations with Caputo derivatives of fractional variable orders and applications

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Abstract

Fractional derivatives are suitable for describing several physical phenomena. The construction of efficient analytical and numerical methods for the solutions of ordinary and partial fractional differential equations is an active research area and it is of great interest to the researchers. The Caputo fractional derivative is of great use in the modelling and simulation of phenomena where consideration is given to the interactions within the past and problems with nonlocal properties. This study considers the use of a hybrid of the Sumudu Transform method for constructing the solution of nonlinear equations that describe the processes in the functional and structural materials. This study considers the models that are given by the integer-order derivatives, Caputo derivatives of fractional variable orders and Caputo derivatives of fractional variable orders that are associated with delays. The study applies a hybrid of Sumudu Transform to present solutions for each considered model and makes use of graphs to show the correlation among the models. The study is of great importance in the numerical and experimental characterization of the decay properties of functional and structural materials.

Keywords: Sumudu Transform, Caputo derivatives, Fractional, Decay, Model
2020 MSC: 34A08, 34A25, 81V35

1 Introduction

Nuclear reactors are the heart of a nuclear power plant (see, e.g, [24, 9]). Figure 1 is an example of typical nuclear reactor fueled with Uranium 235. The nuclear reactor is made up of the reactor itself and a heat exchanger. The reactor is fueled with uranium elements that are enclosed by graphite moderators and eclipsed by charge tubes through which the fuel elements and boron control rods are loaded. A pressure vessel that is walled by a concrete shield houses the whole reactor (see, e.g, [24, 9]). Xenon 135 (^{135}Xe) is an important product of Uranium 235 fission. Studies show that little of ^{135}Xe results directly from fission (see, e.g, [11, 19, 17]). The major souce of ^{135}Xe is the decay

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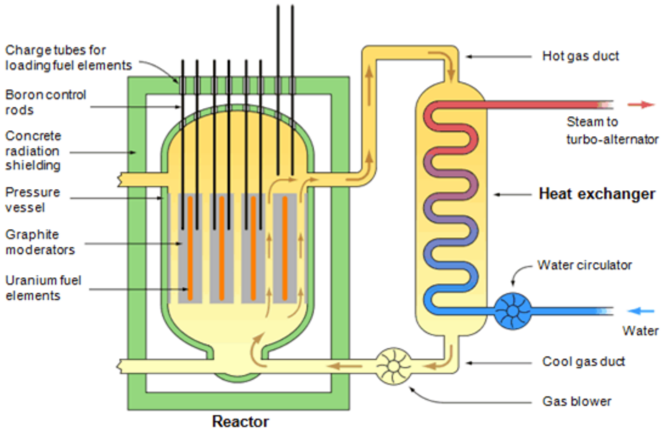

Figure 1: A nuclear reactor (see, e.g, [21]).

chain Tellurium-135 ($^{135}_{52}$Te with β^- decay, 19 sec) to Iodine 135 ($^{135}_{53}I$ with β^- decay 6.6 hr) to ^{135}Xe. The time rate of production rate of ^{135}Xe depends on the ^{135}I concentration and therefore on the local neutron flux history. The half-life of ^{135}Te is so short (19 sec), that ^{135}I can be considered as the primary fission product. ^{135}I is not a strong absorber of neutron but it decays to produce poison ^{135}Xe. It is an important role that the half-life of ^{135}I play in the production of ^{135}Xe, since 90% of available ^{135}Xe comes from the decay of ^{135}I. The decay of ^{135}I is characterized by the differential equation

$$I'(t) = \gamma - \lambda I(t), \tag{1.1}$$

where $I(t)$ denotes the number of ^{135}I $\left(\text{atoms}/cm^3\right)$ i.e. the atom density of iodine, λ is the decay constant for ^{135}I and the product of thermal neutron flux, macroscopic fission cross-section and effective fission yield of the isotope is γ, which is a constant (see, e.g, [16, 7, 8]). The solution of the linear differential equation (1.1) is obtained as

$$I(t) = \gamma/\lambda + (I(0) - \gamma/\lambda)\, e^{-\lambda t},$$

where $I(0)$ signifies ^{135}I at the time $t = 0$. An equilibrium state exits for ^{135}I when its rate of production is equal to its rate of removal. This equilibrium state is referred to as ^{135}Xe reservoir as ^{135}I undergoes complete decay to xenon. One can determine the ^{135}I equilibrium concentration from (1.1) by setting $I'(t) = 0$. At equilibrium, the ^{135}I concentration remains constant and it is determined as

$$I(t) = \gamma/\lambda.$$

The equilibrium ^{135}I concentration is proportional to the fission reaction rate and therefore to the reactor power level. Studies on ^{135}I are extremely essential as they can help the engineers and operators to have perfect knowledge of the behavior of a reactor with ^{135}Xe. Studies such as this can help to predict and respond to the transients induced by equilibrium or non-equilibrium xenon distribution.

Fractional calculus generalizes differentiation to non-integer-orders. Fractional differential equations are highly esteemed for their application in modelling complex phenomena and they play important roles in the study of evolution of a system. The applications of fractional differential equations have been demonstrated in mechanics, physics, chemistry, control theory, to mention just a few (see, e.g, [14]). Applications of fractional differential equations in many fields fascinates scientists and engineers. It is preferrable in several occasion to describe physical phenomena by using fractional derivative operators (see, e.g, [6]). Many dynamical systems have delays that are associated with them as their natural components. Mathematical models that contain delays have potential for more vitality and suitability in describing physical systems. Delay Differential Equations (DDEs) of the form

$$I'(t) = f\left(t, I_i(t), I_i(\alpha t)\right), \tag{1.2}$$

where $i = 1, 2, 3, ..., n$ and $\alpha > 0$ is a constant delay, are referred to as pantograph differential equations. A device called 'pantograph' was used for the first time in the construction of an electric locomotive in 1851. In 1960, the British Railways opted to design a new type of electric locomotive, which can move the trains faster. The new fast-speed electric locomotive had pantograph as a noticable component. By the structure, an overhead wire supplies current to the pantograph. This structure of pantographs is essetial for the locomotive to move. Ockendon and Taylor employed the mechanism of the pantograph to propose DDEs of the form (1.2) [13]. Recent studies on pantograph differential equations and their applications include [15, 1, 2, 3].

^{135}I is a major source of an artificial element that has a tremendous impact on the operation of a nuclear reactor. Where the global correlation but not only local characteristics is desirable, fractional derivative operators are preferable as they can give more insight than the integer-order derivative operators [18]. This paper studies the models that describe the decay of ^{135}I and applies a hybrid of Sumudu Transform (ST) method to present solutions to the models that are given by the integer-order derivative operators, Caputo derivatives of fractional variable orders and Caputo derivatives of fractional variable orders that are associated with delays.

2 Preliminaries

This section introduces some definitions and proposition that are pertinent to the results which this paper presents. Throughout this paper, the set of real, natural and rational numbers will be denoted by $\mathbb{R}, \mathbb{N}$ and $\mathbb{Q}$, respectively.

Definition 2.1. Consider

$$\Omega = \left\{ I(t) : \exists\, Q, \tau_1, \tau_2 > 0, |I(t)| < Qe^{|t|/\tau_j}, \text{if } t \in (-1)^j \times [0, \infty) \right\},$$

which is a set of functions (see, e.g, [4]). $I(t) \in \Omega$ for all real $t \geq 0$. ST is an integral method and the ST for a given function $I(t)$, will be denoted by $\mathcal{S}[I(t)] = I(u)$, defined as

$$I(u) = \int_0^\infty I(tu)e^{-t}dt, \;\; u \in (-\tau_1, \tau_2). \tag{2.1}$$

In equation (2.1), the inverse ST of $I(u)$ is the function $I(t)$. The relation between a function $I(t)$ and its inverse $I(u)$, will be denoted by $I(t) = \mathcal{S}^{-1}[I(u)]$. Other well integral method is the Laplace transform, defined as

$$L(u) = \mathcal{L}[I(t)] = \int_0^\infty I(t)e^{-st}dt, \;\; s > 0, \tag{2.2}$$

for a given function $I(t)$. It can be observed from equations (2.1) and (2.2) that the duality relations between ST and Laplace transform are given as

$$I\left(1/s\right) = sL(s), \quad L\left(1/u\right) = uI(u).$$

ST is an impressive and a broad way to obtain a Lagrange multiplier. ST yields an accurate result quickly and it does not impose any restriction on the results. For arbitrary two given functions $I(t), F(t) \in \Omega$, and for arbitrary constants α and β,

$$\mathcal{S}\left[\alpha I(t) + \beta F(t)\right] = \alpha \mathcal{S}\left[I(t)\right] + \beta \mathcal{S}\left[F(t)\right],$$

which shows that ST satisfies linear property (see, e.g, [20, 5, 4, 10]). For an integer order derivative, its ST is expressed as

$$S\left[\frac{dI(t)}{dt}\right] = \frac{1}{u}\left[I(u) - I(0)\right], \tag{2.3}$$

and for the n-order derivative, the ST is given as

$$S\left[\frac{d^nI(t)}{dt^n}\right] = \frac{1}{u^n}\left[I(u) - \sum_{k=0}^{n-1} u^k \frac{d^kI(t)}{dt^k}|_{x=0}\right]. \tag{2.4}$$

Definition 2.2. Let $a > 0, b > 0$ be positive real numbers. The left and right sided Caputo-fractional derivatives of order μ are defined respectively as

$${}_a^C D^\mu I(t) = \frac{1}{\Gamma\left(1-\mu\right)} \int_a^t (t-\tau)^{-\mu} I'(\tau) d\tau$$

and

$${}^C D_b^\mu I(t) = \frac{-1}{\Gamma\left(1-\mu\right)} \int_t^b (\tau - t)^{\mu} I'(\tau) d\tau,$$

where $0 < \mu < 1$ (see, e.g, [4] Theorem 4.1 and 4.2). Consequently, the ST for Caputo-fractional derivatives of order μ has the form (see, e.g, [6])

$$\mathcal{S}\left[{}_0^C D^\mu I(t)\right] = u^{-\mu}\left(\mathcal{S}\left[I(t)\right] - I(0)\right). \tag{2.5}$$

Table 1: Frequently used Sumudu Transforms

$I(t)$	$I(u)=\mathcal{S}\left[I(t)\right]$
1	1
t	u
$\frac{t^n}{n!}$	u^n
e^{at}	$\frac{1}{1-au}$
$\frac{sin\ at}{a}$	$\frac{u}{1+a^2u^2}$
$cos\ at$	$\frac{1}{1+a^2u^2}$
$\frac{e^{bt}-e^{at}}{b-a}, b\neq a$	$\frac{1}{(1-bu)(1-au)}$

Table 1 gives the ST of some frequently used functions.

Proposition 2.3. Let $\phi,\varphi:[0,\infty)\to\mathbb{R}$, then the classical convolution product is given by

$$(\phi\times\varphi)(t)=\int_0^t\phi(t-x)\varphi(x)dx.$$

The ST for the convolution product is given by

$$\begin{aligned}\mathcal{S}\left[(\phi\times\varphi)(t)\right] &= u\mathcal{S}[\phi(t)]\mathcal{S}[\varphi(t)]\\ &= u\phi(u)\varphi(u).\end{aligned}$$

Definition 2.4. One parameter Mittag-Leffler function $E_\mu(t)$ is defined as

$$E_\mu(t)=\sum_{n=0}^{\infty}t^n/(n\mu)!,\mu>0.$$

The following results about Mittag-Leffler functions and ST are well known (see, e.g, [12]):

(i) $\mathcal{S}\left[E_\mu\left(-at^\mu\right)\right]=\frac{1}{1+au^\mu}$;

(ii) $\mathcal{S}\left[1-E_\mu\left(-at^\mu\right)\right]=\frac{au^\mu}{1+au^\mu}$.

3 Main results

This study presents a hybrid of variational iterative method with ST for solving delay differential equations with Caputo derivatives of fractional variable order. Then the results are applied to obtain the solutions of a differential equations that characterize the decay of ^{135}I.

3.1 Hybrid Sumudu Variational (HSV) method

In this study, a blend of variational iterative method with ST will be referred to as Hybrid Sumudu Variational (HSV) method. Variational iterative method posseses adorable features such as flexibility, consistency and effectiveness (see, e.g, [22, 23] and references there in), which motivates its choice in this study as the most suitable companion for amalgamation with the ST.

3.1.1 Presentation of HSV method

The Caputo derivatives of fractional variable orders play important roles in the modelling of real-life phenomena where it is essential to give attention to the interactions within the past and also problems with nonlocal properties (see, e.g, [18]). The HSV method is presented for solving a given nonlinear universal equation that involves delay and Caputo derivative of fractional variable order μ,

$$ {}_a^C D^\mu I(t)+\Phi\left[I(t)\right]+\Psi\left[I(t/2)\right]=\omega(t), \tag{3.1}$$

subject to the initial conditions

$$I(0) = I_0, \tag{3.2}$$

where Φ is a linear operator, Ψ is a nonlinear operator and $\omega(t)$ is a given continuous function. Taking the ST of (3.1) gives

$$\mathcal{S}\left[{}_a^C D^\mu I(t)\right] = \mathcal{S}\left[\omega(t) - \Phi\left[I(t)\right] + \Psi\left[I(t/2)\right]\right].$$

Refer to (2.5) with $a = 0$, to obtain

$$u^{-\mu}\left(\mathcal{S}\left[I(t)\right] - I(0)\right) = \mathcal{S}\left[\omega(t) - \Phi\left[I(t)\right] - \Psi\left[I(t/2)\right]\right].$$

Since $I(0) = I_0$ by (3.2), we have

$$u^{-\mu}\left(I(u) - I_0\right) = \mathcal{S}\left[\omega(t) - \Phi\left[I(t)\right] - \Psi\left[I(t/2)\right]\right].$$

Therefore, the HSV formula is given as

$$I_{n+1}(u) = I_n(u) + \alpha(u)\left(\frac{I_n(u) - I_0}{u^\mu} - \mathcal{S}\left[\omega(t) - \Phi\left[I(t)\right] - \Psi\left[I(t/2)\right]\right]\right), n \in \mathbb{N}. \tag{3.3}$$

Considering $\mathcal{S}\left[\Phi\left[I(t)\right] - \Psi\left[I(t/2)\right]\right]$ as the restricted term in taking the classical variation operator on both sides of (3.3) leads to

$$\delta I_{n+1}(u) = \delta I_n(u) + \alpha(u)\frac{1}{u^\mu}\delta I_n(u),$$

from which a Lagrange multiplier is obtained as

$$\alpha(u) = -u^\mu. \tag{3.4}$$

Substituting (3.4) into (3.3) and taking its inverse ST gives the explicit iteration formula

$$\begin{aligned} I_{n+1}(t) &= I_n(t) + \mathcal{S}^{-1}\left[-u^\mu\left(\frac{I_n(u) - I_0}{u^\mu} - \mathcal{S}\left[\omega(t) - \Phi\left[I_n(t)\right] - \Psi\left[I_n(t/2)\right]\right]\right)\right] \\ &= I_1(t) + \mathcal{S}^{-1}\left[u^\mu \mathcal{S}\left[\omega(t) - \Phi\left[I_n(t)\right] - \Psi\left[I_n(t/2)\right]\right]\right], \end{aligned}$$

with the initial approximation which is given as $I_1(t) = \mathcal{S}^{-1}\left[-u^\mu\left(\frac{-I_0}{u^\mu}\right)\right] = I_0\mathcal{S}^{-1}\left[1\right] = I_0$.

3.1.2 Variable coefficients fractional differential equations with delay

Suppose the given general nonlinear problem (3.1) contains variable coefficients such that the equation has the form

$${}_a^C D^\mu I(t) + \lambda\Phi_1[I(t)] + \gamma(t)\Phi_2[I(t)] + \Psi\left[I(t/2)\right] = \omega(t), \tag{3.5}$$

where λ is a constant, $\gamma(t)$ is a variable coefficient, Φ_1 and Φ_2 denote linear operators and other terms remain as defined in (3.1). Taking the ST of (3.5) and further computations give the HSV formula

$$I_{n+1}(u) = I_n(u) + \alpha(u)\left(\frac{I_n(u) - I_0}{u^\mu} - \mathcal{S}\left[\omega(t) - \lambda\Phi_1[I(t)] - \gamma(t)\Phi_2[I(t)] - \Psi\left[I(t/2)\right]\right]\right), n \in \mathbb{N}. \tag{3.6}$$

Considering $\mathcal{S}\left[\gamma(t)\Phi_2[I(t)] + \Psi\left[I(t/2)\right]\right]$ as the restricted term in taking the classical variation operator on both sides of (3.6) leads to

$$\delta I_{n+1}(u) = \delta I_n(u) + \alpha(u)\frac{1}{u^\mu}\delta I_n(u),$$

from which a Lagrange multiplier as

$$\alpha(u) = -u^\mu.$$

Substitute for $\alpha(u)$ in (3.6) and take its inverse ST to obtain the explicit iteration formula

$$\begin{aligned} I_{n+1}(t) &= I_n(t) + \mathcal{S}^{-1}\left[-u^\mu\left(\frac{I_n(u) - I_0}{u^\mu} - \mathcal{S}\left[\omega(t) - \lambda\Phi_1[I_n(t)] - \gamma(t)\Phi_2[I_n(t)] - \Psi\left[I_n(t/2)\right]\right]\right)\right] \\ &= I_1(t) + \mathcal{S}^{-1}\left[u^\mu \mathcal{S}\left[\omega(t) - \lambda\Phi_1[I_n(t)] - \gamma(t)\Phi_2[I_n(t)] - \Psi\left[I_n(t/2)\right]\right]\right], \end{aligned}$$

with the initial approximation which is given as $I_1(t) = \mathcal{S}^{-1}\left[-u^\mu\left(\frac{-I_0}{u^\mu}\right)\right] = I_0\mathcal{S}^{-1}\left[1\right] = I_0$.

3.2 New models for decay of Iodine 135

In this section, new mathematical models that involve Caputo derivatives of fractional variable orders are introduced for the decay of decay of ${}^{135}I$. We obtain the solutions of the newly introduced models for the decay of ${}^{135}I$ and compare them with solution of the existing model that is associated with integer order derivative. We apply the ST method to obtain the solution of the newly introduced model that is not associated with delay while we apply the HSV method which we presented in Section 3.1 to the newly introduced model that is associated with delay. We choose suitable parameters for the variables and use Matlab 2015 to plot the graphs of the solutions of the models to show their comparison.

3.2.1 Models with Caputo-fractional derivatives for decay of ${}^{135}I$

An analogueue of the differential equation (1.1), given with Caputo derivatives of fractional variable orders for the decay of ${}^{135}I$ has the form

$$ {}_a^C D^{\mu} I(t) = \gamma - \lambda I(t), \; I(0) = I_0, \tag{3.7}$$

where $\mu \in (0,1)$. We shall solve (3.7) by using the ST method. Taking the ST of (3.7) gives

$$\mathcal{S}\left[{}_a^C D^{\mu} I(t)\right] = \gamma - \lambda \mathcal{S}\left[I(t)\right],$$

which leads to

$$u^{-\mu}\left(\mathcal{S}\left[I(t)\right] - I(0)\right) = \gamma - \lambda \mathcal{S}\left[I(t)\right].$$

Since it is given that $I(0) = I_0$, it is obtained that

$$u^{-\mu}\left(I(u) - I_0\right) = \gamma - \lambda I(u). \tag{3.8}$$

Factorising (3.8) leads to

$$\begin{aligned} I(u) &= \frac{u^{-\mu} I_0 + \gamma}{u^{-\mu} + \lambda} \\ &= \frac{u^{-\mu} I_0}{u^{-\mu} + \lambda} + \frac{\gamma}{u^{-\mu} + \lambda}. \end{aligned} \tag{3.9}$$

Taking the inverse ST of (3.9)

$$\begin{aligned} I(t) &= \mathcal{S}^{-1}\left[\frac{u^{-\mu}}{u^{-\mu} + \lambda}\right] I_0 + \mathcal{S}^{-1}\left[\frac{\gamma}{u^{-\mu} + \lambda}\right] \\ &= \mathcal{S}^{-1}\left[\frac{1}{1 + \lambda u^{\mu}}\right] I_0 + \frac{\gamma}{\lambda}\mathcal{S}^{-1}\left[\frac{\lambda u^{\mu}}{1 + \lambda u^{\mu}}\right]. \end{aligned}$$

Then by Definition 2.4 (i) and (ii),

$$I(t) = I_0 E_{\mu}\left(-\lambda t^{\mu}\right) + \gamma/\lambda\left(1 - E_{\mu}\left(-\lambda t^{\mu}\right)\right). \tag{3.10}$$

The real values are assigned to the constants by setting $\lambda = 0.3, \gamma = 0.1, \mu = 0.65$ and $I_0 = 1$, which signifies the value of ${}^{135}I$ at the time $t = 0$. Figure 2 compares the solution given by the integer order differential equation (1.1) with the solution given by the differential equation (3.7) that is associated with Caputo derivatives of fractional variable orders. In addition, by setting $\lambda = 0.3$ and $\gamma = 0.1$, Figure 3 displays the effect of variation of fractional order μ, of Caputo derivative operators.

3.2.2 Models with Caputo-fractional derivatives and time delay for decay of ${}^{135}I$

We shall introduce a time delay into the differential equations with Caputo derivatives of fractional variable orders for the decay of ${}^{135}I$, to propose a new model. An analogue of the differential equation (3.7), associated with time delay is given as

$$ {}_a^C D^{\mu} I(t) = \gamma - \lambda I(t/2), \tag{3.11}$$

where $\mu \in (0,1)$ and $I(0) = I_0$.

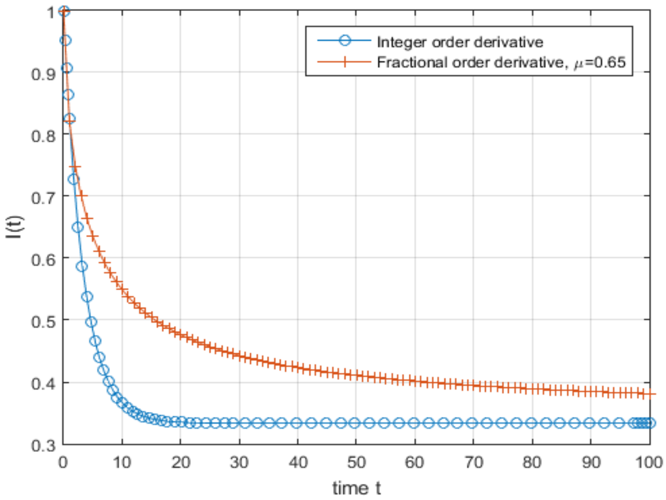

Figure 2: Comparison of solutions of differential equations with integer order derivatives with fractional order derivatives.

Remark 3.1. Notice that (3.11) is quite different from (3.7) as delay is present in (3.11). Generally, the presence of delays in a mathematical model can improve its vitality and suitability in describing several phenomena. However, the presence of delay in an equation makes it more difficult to obtain its solution. In this case, the ST method that was used to obtain the solution of (3.7) is not suitable to solve (3.11) due to its association with delay.

We shall apply HSV method (that was presented in Section 3.1) to solve (3.11). Starting with taking the ST of (3.11) to obtain

$$\mathcal{S}\left[{}_a^C D^\mu I(t)\right] = \gamma - \lambda \mathcal{S}\left[I(t/2)\right].$$

By applying (2.5) with $a = 0$, we obtain

$$u^{-\mu}\left(\mathcal{S}\left[I(t)\right] - I(0)\right) = \gamma - \lambda \mathcal{S}\left[I(t/2)\right],$$

which is equivalent to

$$u^{-\mu}\left(I(u) - I_0\right) + \lambda \mathcal{S}\left[I(t/2)\right] - \gamma = 0,$$

since $\mathcal{S}\left[I(t)\right] = I(u)$ and $I(0) = I_0$. Consequently, HSV iteration formula is given as

$$I_{n+1}(u) = I_n(u) + \alpha(u)\left(\frac{I_n(u) - I_0}{u^\mu} + \lambda \mathcal{S}\left[I_n(t/2)\right] - \gamma\right), n \in \mathbb{N}. \tag{3.12}$$

Considering $I_n(t/2)$ as the restricted term while taking the classical variation operator on both sides of (3.12), gives

$$\alpha(u) = -u^\mu.$$

Substituting for $\alpha(u)$ in 3.12 and taking its inverse-ST gives the explicit iteration formula

$$\begin{aligned} I_{n+1}(t) &= I_n(t) + \mathcal{S}^{-1}\left[-u^\mu\left(\frac{I_n(u) - I_0}{u^\mu} + \lambda \mathcal{S}\left[I_n(t/2)\right] - \gamma\right)\right] \\ &= I_1(t) - \mathcal{S}^{-1}\left[u^\mu\left(\lambda \mathcal{S}\left[I_n(t/2)\right] - \gamma\right)\right]. \end{aligned}$$

where the initial approximation $I_1(t) = \mathcal{S}^{-1}\left[I_0\right] = I_0 \mathcal{S}^{-1}\left[1\right] = I_0$. Therefore, the successive iteration formula is obtained as

$$\begin{cases} I_1(t) = I_0, \\ I_{n+1}(t) = I_0 - \mathcal{S}^{-1}\left[u^\mu\left(\lambda \mathcal{S}\left[I_n(t/2)\right] - \gamma\right)\right]. \end{cases} \tag{3.13}$$

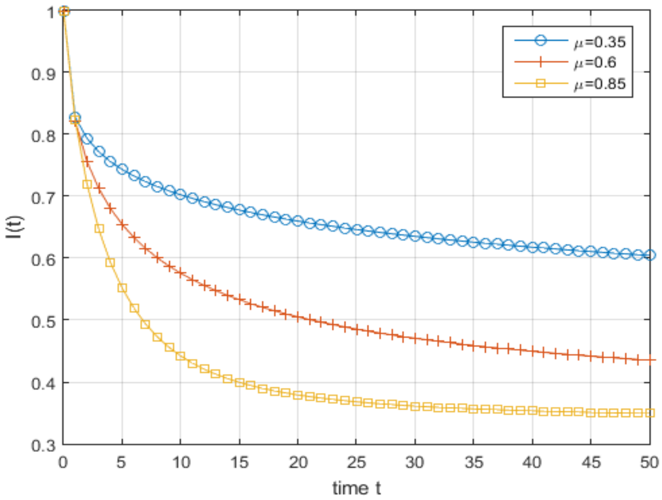

Figure 3: Effects of variation of fractional order of Caputo derivative operators.

Notice that $I_1(t/2) = I_0$, therefore

$$\begin{aligned} I_2(t) &= I_0 - \mathcal{S}^{-1}\left[u^\mu\left(\lambda\mathcal{S}\left[I_1(t/2)\right] - \gamma\right)\right] = I_0 - \mathcal{S}^{-1}\left[\left(\lambda I_0 - \gamma\right)u^\mu\right] = I_0 - \left(\lambda I_0 - \gamma\right)\mathcal{S}^{-1}\left[u^\mu\right] \\ &= I_0 - \left(\lambda I_0 - \gamma\right)\frac{t^\mu}{\mu!} = I_0\left(1 - \lambda\frac{t^\mu}{\mu!}\right) + \gamma\frac{t^\mu}{\mu!}. \end{aligned}$$

Notice that $I_2(t/2) = I_0\left(1 - \lambda\frac{t^\mu}{2^\mu\mu!}\right) + \gamma\frac{t^\mu}{2^\mu\mu!}$, therefore

$$\begin{aligned} I_3(t) &= I_0 - \mathcal{S}^{-1}\left[u^\mu\left(\lambda\mathcal{S}\left[I_2(t/2)\right] - \gamma\right)\right] = I_0 - \mathcal{S}^{-1}\left[u^\mu\left(\lambda\mathcal{S}\left[I_0\left(1 - \lambda\frac{t^\mu}{2^\mu\mu!}\right) + \gamma\frac{t^\mu}{2^\mu\mu!}\right] - \gamma\right)\right] \\ &= I_0 - \mathcal{S}^{-1}\left[u^\mu\left(\lambda\left(I_0\left(1 - \lambda\frac{u^\mu}{2^\mu}\right) + \gamma\frac{u^\mu}{2^\mu}\right) - \gamma\right)\right] \\ &= I_0 - \mathcal{S}^{-1}\left[I_0\left(\lambda u^\mu - \lambda^2\frac{u^{2\mu}}{2^\mu}\right) + \lambda\gamma\frac{u^{2\mu}}{2^\mu} - \gamma u^\mu\right] \\ &= I_0 - I_0\left(\lambda\frac{t^\mu}{\mu!} - \lambda^2\frac{t^{2\mu}}{2^\mu(2\mu)!}\right) - \lambda\gamma\frac{t^{2\mu}}{2^\mu(2\mu)!} + \gamma\frac{t^\mu}{\mu!} \\ &= I_0\left(1 - \lambda\frac{t^\mu}{\mu!} + \lambda^2\frac{t^{2\mu}}{2^\mu(2\mu)!}\right) + \gamma\left(\frac{t^\mu}{\mu!} - \lambda\frac{t^{2\mu}}{2^\mu(2\mu)!}\right). \end{aligned}$$

Notice that $I_3(t/2) = I_0\left(1 - \lambda\frac{t^\mu}{2^\mu\mu!} + \lambda^2\frac{t^{2\mu}}{2^{3\mu}(2\mu)!}\right) + \gamma\left(\frac{t^\mu}{2^\mu\mu!} - \lambda\frac{t^{2\mu}}{2^{3\mu}(2\mu)!}\right)$, therefore

$$\begin{aligned} I_4(t) &= I_0 - \mathcal{S}^{-1}\left[u^\mu\left(\lambda\mathcal{S}\left[I_3(t/2)\right] - \gamma\right)\right] \\ &= I_0 - \mathcal{S}^{-1}\left[u^\mu\left(\lambda\mathcal{S}\left[I_0\left(1 - \lambda\frac{t^\mu}{2^\mu\mu!} + \lambda^2\frac{t^{2\mu}}{2^{3\mu}(2\mu)!}\right) + \gamma\left(\frac{t^\mu}{2^\mu\mu!} - \lambda\frac{t^{2\mu}}{2^{3\mu}(2\mu)!}\right)\right] - \gamma\right)\right] \\ &= I_0 - \mathcal{S}^{-1}\left[u^\mu\left(\lambda I_0\left(1 - \lambda\frac{u^\mu}{2^\mu} + \lambda^2\frac{u^{2\mu}}{2^{3\mu}}\right) + \lambda\gamma\left(\frac{u^\mu}{2^\mu} - \lambda\frac{u^{2\mu}}{2^{3\mu}}\right) - \gamma\right)\right] \\ &= I_0 - \mathcal{S}^{-1}\left[\lambda I_0\left(u^\mu - \lambda\frac{u^{2\mu}}{2^\mu} + \lambda^2\frac{u^{3\mu}}{2^{3\mu}}\right) + \lambda\gamma\left(\frac{u^{2\mu}}{2^\mu} - \lambda\frac{u^{3\mu}}{2^{3\mu}}\right) - \gamma u^\mu\right] \\ &= I_0 - \lambda I_0\left(\frac{t^\mu}{\mu!} - \lambda\frac{t^{2\mu}}{2^\mu(2\mu)!} + \lambda^2\frac{t^{3\mu}}{2^{3\mu}(3\mu)!}\right) - \lambda\gamma\left(\frac{t^{2\mu}}{2^\mu(2\mu)!} - \lambda\frac{t^{3\mu}}{2^{3\mu}(3\mu)!}\right) + \gamma\frac{t^\mu}{\mu!} \\ &= I_0\left(1 - \lambda\frac{t^\mu}{\mu!} + \lambda^2\frac{t^{2\mu}}{2^\mu(2\mu)!} - \lambda^3\frac{t^{3\mu}}{2^{3\mu}(3\mu)!}\right) + \gamma\left(\frac{t^\mu}{\mu!} - \lambda\frac{t^{2\mu}}{2^\mu(2\mu)!} + \lambda^2\frac{t^{3\mu}}{2^{3\mu}(3\mu)!}\right). \end{aligned}$$

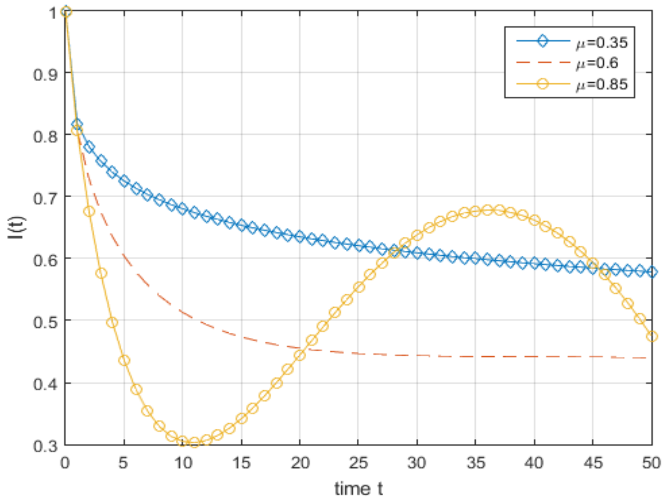

Figure 4: Graph of the models with Caputo-fractional derivatives that are associated with delay.

Notice that $I_4(t/2) = I_0\left(1-\lambda\frac{t^\mu}{2^\mu\mu!}+\lambda^2\frac{t^{2\mu}}{2^{3\mu}(2\mu)!}-\lambda^3\frac{t^{3\mu}}{2^{6\mu}(3\mu)!}\right)+\gamma\left(\frac{t^\mu}{2^\mu\mu!}-\lambda\frac{t^{2\mu}}{2^{3\mu}(2\mu)!}+\lambda^2\frac{t^{3\mu}}{2^{6\mu}(3\mu)!}\right)$, therefore

$$
\begin{aligned}
I_5(t) =& I_0 - \mathcal{S}^{-1}\left[u^\mu\left(\lambda\mathcal{S}\left[I_4(t/2)\right]-\gamma\right)\right]\\
=& I_0 - \mathcal{S}^{-1}\Bigg[u^\mu\Bigg(\lambda\mathcal{S}\Bigg[I_0\left(1-\lambda\frac{t^\mu}{2^\mu\mu!}+\lambda^2\frac{t^{2\mu}}{2^{3\mu}(2\mu)!}-\lambda^3\frac{t^{3\mu}}{2^{6\mu}(3\mu)!}\right)+\gamma\left(\frac{t^\mu}{2^\mu\mu!}-\lambda\frac{t^{2\mu}}{2^{3\mu}(2\mu)!}+\lambda^2\frac{t^{3\mu}}{2^{6\mu}(3\mu)!}\right)\Bigg]-\gamma\Bigg)\Bigg]\\
&+\lambda\gamma\left(\frac{u^\mu}{2^\mu}-\lambda\frac{u^{2\mu}}{2^{3\mu}}+\lambda^2\frac{u^{3\mu}}{2^{6\mu}}\right)-\gamma\Bigg)\Bigg]\\
=& I_0 - \mathcal{S}^{-1}\left[\lambda I_0\left(u^\mu-\lambda\frac{u^{2\mu}}{2^\mu}+\lambda^2\frac{u^{3\mu}}{2^{3\mu}}-\lambda^3\frac{u^{4\mu}}{2^{6\mu}}\right)+\lambda\gamma\left(\frac{u^{2\mu}}{2^\mu}-\lambda\frac{u^{3\mu}}{2^{3\mu}}+\lambda^2\frac{u^{4\mu}}{2^{6\mu}}\right)-\gamma u^\mu\right]\\
=& I_0 - \lambda I_0\left(\frac{t^\mu}{\mu!}-\lambda\frac{t^{2\mu}}{2^\mu(2\mu)!}+\lambda^2\frac{t^{3\mu}}{2^{3\mu}(3\mu)!}-\lambda^3\frac{t^{4\mu}}{2^{6\mu}(4\mu)!}\right)-\lambda\gamma\left(\frac{t^{2\mu}}{2^\mu(2\mu)!}-\lambda\frac{t^{3\mu}}{2^{3\mu}(3\mu)!}+\lambda^2\frac{t^{4\mu}}{2^{6\mu}(4\mu)!}\right)+\gamma\frac{t^\mu}{\mu!}\\
=& I_0\left(1-\lambda\frac{t^\mu}{\mu!}+\lambda^2\frac{t^{2\mu}}{2^\mu(2\mu)!}-\lambda^3\frac{t^{3\mu}}{2^{3\mu}(3\mu)!}+\lambda^4\frac{t^{4\mu}}{2^{6\mu}(4\mu)!}\right)+\gamma\left(\frac{t^\mu}{\mu!}-\lambda\frac{t^{2\mu}}{2^\mu(2\mu)!}+\lambda^2\frac{t^{3\mu}}{2^{3\mu}(3\mu)!}-\lambda^3\frac{t^{4\mu}}{2^{6\mu}(4\mu)!}\right).
\end{aligned}
$$

Hence, it can be deduced that

$$
\begin{cases}
I_1(t) = I_0,\\
I_n(t) = I_0\sum_{k=0}^{n-1}(-\lambda)^k\,\dfrac{t^{k\mu}}{2^{\frac{k}{2}(k-1)\mu}(k\mu)!}+\gamma\sum_{k=1}^{n-1}(-\lambda)^{k-1}\,\dfrac{t^{k\mu}}{2^{\frac{k}{2}(k-1)\mu}(k\mu)!}, n\in\mathbb{N}, n>1,\\
I(t) = \lim_{n\to\infty} I_n(t).
\end{cases}
\tag{3.14}
$$

Figure 4 shows the solution of the model (3.11) for different values of μ. Figure 4 is obtained by setting the parameters in the differential equation (3.11) to be $\lambda = 0.3, \gamma = 0.1$ and $I_0 = 1$. Figure 4 displays how the solutions vary as μ varies. Setting $\mu = 0.75$, Figure 5 shows the iterations of the model with Caputo-fractional derivatives and time delay for the decay of ^{135}I.

Conclusion

This paper presents fractional differential equations that are analogue of an integer-order differential equation, which describes the decay of ^{135}I. The paper introduces new mathematical models that involve Caputo derivatives of fractional variable orders for the decay of ^{135}I. The paper presents a hybrid of ST and applied it to obtain the solutions of three different models that charaterises the decay if ^{135}I. The paper considers the models that are given

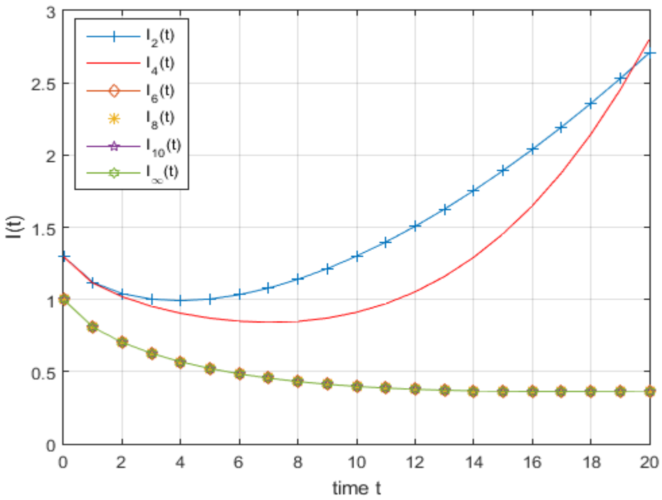

Figure 5: Iterations of the model with Caputo-fractional derivatives and time delay for decay of ^{135}I.

by the integer order derivatives, Caputo derivatives of fractional variable orders and Caputo derivatives of fractional variable orders that are associated with delays. The Caputo fractional derivative is of great use in the modelling and simulation of phenomena where consideration is given to the interactions within the past and problems with nonlocal properties. The paper present the graphs of the solutions for the models to show the correlation among them. In general, this study is of great importance in the numerical and experimental characterization of the decay property of the functional and structural materials.

Abbreviations:
DDEs: Delay Differential Equations
HSV: Hybrid Sumudu Variational
ST: Sumudu Transform

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