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New adapted spectral method for solving stochastic optimal control problem

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Abstract

Optimal control theory is a branch of mathematics. It is developed to find optimal ways to control a dynamic system. In 1957, R.Bellman applied dynamic programming to solve optimal control of discrete-time systems. His procedure resulted in closed-loop, generally nonlinear, and feedback schemes. Optimal control problems which will be tackled involve the minimization of a cost function subject to constraints on the state vector and the control. Lagrange multipliers provide a method of converting a constrained minimization problem into an unconstrained minimization problem of higher order. The necessary condition for optimality can be obtained as the solution of the unconstrained optimization problem of the Lagrange function and the bordered Hessian matrix is used for the second-derivative test. A spectral method for solving optimal control problems is presented. The method is based on Bernoulli polynomials approximation. By using the Bernoulli operational matrix of integration and the Lagrangian function, stochastic optimal control is transformed into an optimisation problem, where the unknown Bernoulli coefficients are determined by using Newton's iterative method. The convergence analysis of the proposed method is given. The simulation results based on the Monte-Carlo technique prove the performance of the proposed method. Some error estimations are provided and illustrative examples are also included to demonstrate the efficiency and applicability of the proposed method.

Keywords: Bernoulli polynomials, open loop, feedback, optimal control problem, operational matrix, Brownian motion
2020 MSC: 65CXX, 65RXX, 93EXX, 60HXX, 49MXX, 65KXX

1 Introduction

A range of application areas, including epidemiology, chemistry, biology, mechanics, economics, physics,... etc are modelled by stochastic differential equations [12, 33, 36]. So the study of this type of problem is very useful and there is increasing demand for studying the behavior of the number of dynamic systems which depend on more sources such as Gaussian white noise. Control optimal theory is a long field in which a lot of researchers are interested in solving various aspects. The aim is to find the control variable which minimizes a given performance index, where all the given constraints are satisfied [4, 10, 39]. Stochastic optimal control is one of the main sub-fields in control theory, it is the subject of study in industry and in cyber security systems in computer science [34]. When the system

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randomness is bounded and the bound is known the problem of finding a suitable control can be dealt with robust control or while the bound of uncertainty is known with the probability distribution of the noise is available, the stochastic framework can be used [13, 41, 46]. One of the ideas to convert stochastic optimal control to deterministic one is by using the Fokker equation which models the time evolution of probability density of the corresponding Fokker-Planck stochastic process. Several researches have been used to solve stochastic optimal control problems where the control function is only dependent on time, it is state-dependent as well as time-dependent [1, 2]. The use of spectral techniques for solving optimal control problems is one of the interesting research areas. One of the advantages of using this technique is its exponential convergence and it is an efficient approximation method for integration in the cost function. Also, it is one of the top methods for solving differential and partial differential equations. The stochastic optimal theory is a combination of optimal control theory and probability theory in which the indeterminate factors or objective is indeterminacy. There are few kinds of literature which focus on continuous time uncertain stochastic optimal control [1, 2, 18, 38]. In recent years there has been increasing activity in providing some numerical schemes based on orthogonal polynomials or the so-called spectral or pseudo-spectral methods. Many different techniques based on different basis functions such as block pulse functions [26, 27, 32], hat basis functions [28], Bernoulli polynomials [3, 5, 29, 43], have been used for solving deterministic and stochastic integral equations. In [44], the authors proposed a stochastic linear quadratic (LQ) optimal control problem with an expectation-type linear equality constraint on the terminal state. The constrained stochastic LQ problem is solved completely by the Lagrangian duality theory. A kind of stochastic optimal control problems where the cost functional is defined by a symmetric function where the authors applied the theories of BSDEs and PDEs to find the explicit form of the optimal control [8]. A recent work to characterize the optimal feedback controls for general linear quadratic optimal control problem of a stochastic evolution equation with random coefficients is given by Qi Lü et al [24]. Another improved iterative algorithm based on a Newton iterative algorithm, which is used to research the stochastic linear quadratic optimal tracking (SLQT) control for stochastic continuous-time systems is introduced in [42]. In [15], the authors represent necessary and sufficient conditions for the solvability of discrete time, mean-field, and stochastic linear-quadratic optimal control problems. Secondly, the optimal control within a class of linear feedback controls is investigated using a matrix dynamical optimization method.

In this study, main concern is focused on numerically solving optimal control problem by using spectral method. So, we concentrate on the following optimisation problem.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtrated probability space and $\mathcal{B}(t), t \geq 0$ is a Brownian motion $\mathcal{F}_t$ -measurable. Consider the following controlled stochastic Volterra integral equation

$$x^u(t) = x_0 + \int_0^t a(x^u(s), u(s))ds + \int_0^t \sigma(x^u(s), u(s))d\mathcal{B}(s), x(0) = x_0, \tag{1.1}$$

where $u(t)$ is the optimal control process belonging to the space of admissible controls denoted U which is a closed and convex subset of $\mathbb{R}$ and $x(t)$ is the behavior of dynamic system. The optimal control problem system consists in finding a control $u(t)$ that minimizes the performance function

$$J(u) = E\left[\int_0^T c(x^u(t), u(t))ds + g(x^u(T))\right], \tag{1.2}$$

where $a : \mathbb{R} \times U \mapsto \mathbb{R}$ and $c, \sigma : \mathbb{R} \times U \mapsto \mathbb{R}$ are $\mathcal{F}_t$-predictable and $g : \mathbb{R} \mapsto \mathbb{R}$ is $\mathcal{F}_T$-measurable, and T is a positive constant.
In [19], The authors introduced a set of sufficient conditions for the existence of unique optimal control and its corresponding Hamiltonian system [19] and also in [31].

2 Bernoulli polynomials

Bernoulli polynomials have been used by researchers to solve various problems for instant see [5, 3, 37].

Definition 2.1. Bernoulli polynomials of degree i noted by $B_i(t)$ satisfy the equation [23]

$$\sum_{j=0}^{i} \binom{i+1}{j} B_j(t) = (i+1)t^i, \quad i = 0, 1, 2, \cdots \tag{2.1}$$

The first six Bernoulli polynomials are

$$\begin{aligned}
B_0(t) &= 1,\\
B_1(t) &= t - \frac{1}{2},\\
B_2(t) &= t^2 - t + \frac{1}{6},\\
B_3(t) &= t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,\\
B_4(t) &= t^4 - 2t^3 + t^2 - \frac{1}{30},\\
B_5(t) &= t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t.
\end{aligned}$$

Bernoulli polynomials satisfy the following properties [11]:

- $B_i'(t) = iB_{i-1}(t), \quad i \geq 1,$
- $B_i(t+1) - B_i(t) = it^{i-1}, \quad i \geq 1,$
- $B_i(t) = \sum_{k=0}^{i} \binom{i}{k} B_k(0)t^{i-k}, \quad i \geq 1,$
- $\int_0^1 B_i(t)dt = 0, \quad i \geq 1,$
- $\int_0^1 B_i(t)B_j(t)dt = (-1)^{i+j} \frac{i!j!}{(i+j)!} B_{i+j}(0).$

Let $B(x) = [B_0(t), B_1(t), \cdots B_m(t)]^T$ be the Bernoulli vector, then we can write

$$B(t) = DT_m(t), \tag{2.2}$$

such that

$$D = \begin{pmatrix}
\binom{0}{0}B_0(0) & 0 & 0 & \cdots & 0\\
\binom{1}{1}B_1(0) & \binom{0}{1}B_0(0) & 0 & \cdots & 0\\
\binom{2}{2}B_2(0) & \binom{2}{1}B_1(0) & \binom{2}{0}B_0(0) & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
\binom{m}{m}B_m(0) & \binom{m}{m-1}B_{m-1}(0) & \binom{m}{m-2}B_{m-2}(0) & \cdots & \binom{m}{0}B_0(0)
\end{pmatrix},$$

and

$$T_m(x) = [1, t, t^2, \cdots, t^m]. \tag{2.3}$$

2.1 Function approximation

Let $\mathcal{L}^2([0,1])$ be the space of square integrable functions with respect to Lebesgue mesure on [0,1], each function $f \in \mathcal{L}^2([0,1])$ can be expanded by using Bernoulli polynomials as follows

$$f(t) = \sum_{i=0}^{m} f_i B_i(t) = F^T B(t), \tag{2.4}$$

where $F = (f_1, f_2, \cdots, f_m)^T$. The coefficients $f_1, f_2, \cdots, f_m$ can be determined by

$$f_m = \frac{1}{m!}\int_0^1 f^{(m)}(t)dt. \tag{2.5}$$

The coefficients f_m decrease as follows

$$f_m \leq \frac{F_m}{m!},$$

where F_m is the maximum of $f^{(m)}(t)$ in the interval [0,1]. Assume that $k(x,t) \in \mathcal{L}^2([0,1]\times[0,1])$, so we can approximate $k(x,t)$ as follows

$$k(x,t) \simeq \sum_{i=0}^{m}\sum_{j=0}^{m} k_{ij}B_i(x)B_j(t) = B^T(x)KB(t), \tag{2.6}$$

where $K = [k_{ij}]_{j,i=0}^m$ is an $(m+1)\times(m+1)$ matrix, and can be calculated by the following formula

$$k_{ij} = \frac{1}{i!j!}\int_0^1\int_0^1 \frac{\partial^{i+j}k(x,t)}{\partial x^i \partial t^j}dxdt, \quad i,j = 0,1,...,m, \tag{2.7}$$

the coefficients k_{ij} decrease as follows

$$k_{ij} \leq \frac{K'_{i,j}}{i!j!},$$

where $K'_{i,j}$ is the maximum of $\frac{\partial^{i+j}k(x,t)}{\partial x^i \partial t^j}$ in the unit square $[0,1]\times[0,1]$.

Proof . See [37] □

Theorem 2.2. Let $g(t)$ be an enough smooth function in the interval $[0,1]$ and $Q_m[g](t)$ is the approximate polynomial of $g(t)$ in terms of Bernoulli polynomials. Then we have

$$g(t) = Q_m[g](t) + R_m[g](t), \quad t \in [0,1], \tag{2.8}$$

$$Q_m[g](t) = \int_0^1 g(t) + \sum_{j=0}^{m}\frac{B_j(t)}{j!}(g^{m-1}(1) - g^{m-1}(0)),$$

$$R_m[g](t) = \frac{-1}{m!}\int_0^1 B_m^*(x-t)g^{(m)}(t)dt,$$

where $B_m^*(x) = B_m(x-[x])$.

Theorem 2.3. Suppose $g(t) \in \mathcal{C}^\infty(0,1)$, then the error between g(t) and its approximate polynomial Q_m by Bernoulli polynomials is given by

$$E(g) = ||g(t) - Q_m[g](t)||_\infty \leq \frac{1}{m!}B_mG_m, \tag{2.9}$$

where B_m and G_m are respectively the maximum values of $B_m(t)$ and $g^{(m)}(t)$ in the interval $[0,1]$. By the same technique we can see that if $K(x,t)$ is approximated by Bernoulli series then, we have

$$E(K) = ||K(x,t) - Q_m[K](x,t)||_\infty \leq \frac{1}{(m!)^2}B_m^2K_{m,m}, \tag{2.10}$$

where

$$B_m = \max_{t\in[0,1]} B_m(t) \qquad \text{and} \qquad K_{m,m} = \max_{(x,t)\in[0,1]\times[0,1]} \frac{\partial^{2m}}{\partial x^m \partial t^m}K(x,t).$$

2.2 Deterministic Bernoulli operational matrix

We have

$$\int_0^t B_m(s)ds = \frac{1}{m+1}(B_{m+1}(t) - B_{m+1}(0)), \;\; m \geq 0. \tag{2.11}$$

So, we can approximate the integration of the vector $B(t)$ as

$$\int_0^t B(s)ds \simeq PB(t), \tag{2.12}$$

where P is the $(m+1)\times(m+1)$ Bernoulli operational matrix of integration given by

$$P = \begin{pmatrix} -B_1(0) & 1 & 0 & \cdots & 0 \\ \frac{-B_2(0)}{2} & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{-B_m(0)}{m} & 0 & 0 & \cdots & \frac{1}{m} \\ \frac{-B_{m+1}(0)}{m+1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

From Eqs.(2.12) and (2.4), we can approximate the integral of function $f(t)$ by

$$\int_0^t f(s)ds = \int_0^t F^T B(s)ds \simeq F^T P B(t). \tag{2.13}$$

2.3 Stochastic Bernoulli operational matrix

We can approximate the integration of the vector $B(t)$ as

$$\int_0^t B(s)d\mathcal{B}(s) \simeq P_s B(t), \tag{2.14}$$

where P_s is the $(m+1)\times(m+1)$ stochastic Bernoulli operational matrix of integration given by $P_s = DD_sD^{-1}$ where

$$D_s = \begin{pmatrix} B(0.5) & 0 & 0 & \cdots & 0 \\ 0 & \frac{5}{6}B(0.5) - \frac{2}{3}B(0.25) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & (1-\frac{m}{6})B(0.5) - \frac{m}{3\times 2^{m-2}}B(0.25) \end{pmatrix}.$$

3 Stochastic linear quadratic (LQ) optimal control

The linear quadratic gaussian control optimal problem with quadratic cost is the well known solvable stochastic control in continuous time. One of the fondamental issues in control theory is to design feedback controls which are particulary important in practical applications. An optimal linear feedback control is determined by the solution of a stochastic Riccati equation. The main of introducing Riccati equations is the study of deterministic control problem which is an problem well know [35, 45]. For the stochastic case there is also equivalence between the existence of control optimal and the solvability of backward stochastic Riccati equations in a suitable case [6, 7, 9, 21, 22, 25, 40].

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete filtrated probability space and $\{\mathcal{B}(t), t \in [0,T]\}$ is a standard Brownian motion, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the natural filtration generated by $\mathcal{B}_t$. Let consider the stochastic optimal control

$$J = \min_{u_s \in H} E \int_0^T (A(s)x_s^2 + C(s)u_s^2)ds + S_T x_T^2, \tag{3.1}$$

subjet the stochastic differential equation

$$\begin{cases} dx_s = (a(s)x_s + b(s)u_s)ds + (c(s)x_s + d(s)u_s)d\mathcal{B}_s \\ x(0) = x_0 \end{cases}$$

where, u_t is the control variable, x_t represents the state variable, T represent the terminal time and $a(t)$, $b(t)$, $c(t)$ and $d(t) : [0,T] \mapsto \mathbb{R}$ are a given continuous functions. The stochastic linear quadratic problem is well posed if $J(t,x_t) > -\infty, \ \ \forall (t,x_t) \in [0,T]\times\mathbb{R}$.

Theorem 3.1. The control u_t^* is the optimal control of equation (3.1) if and only if

$$u_t^* = \frac{-b(t)k(t) - c(t)d(t)k(t)}{2c(t) + d^2(t)k(t)}, \tag{3.2}$$

where $k(t)$ satisfy the following Riccati equation

$$\begin{cases} \frac{dk(t)}{dt} = -2A(t) - [2a(t) + c^2(t)]k(t) + \frac{[b(t)k(t)+c(t)d(t)k(t)]^2}{2c(t)+d^2(t)k(t)} \\ k(T) = 2S_T \qquad \text{and} \qquad 2c(t) + d^2(t)k(t) > 0. \end{cases} \tag{3.3}$$

The optimal value $J(t,x_t)$ is given by

$$J(t,x_t) = \frac{1}{2}k(t)x_t^2.$$

If we assume that the state of the system is generated by a linear noisy system

$$\dot{x} = Ax(t) + Bu(t) + \mathcal{B}(t), \tag{3.4}$$

with $\mathcal{B}(t)$ is a Brownian motion, J is the performance index function defined by

$$J = E\left\{\int_0^T (Qx^2(s) + Ru^2(s))ds + Q_T x(T)^2\right\}, \tag{3.5}$$

where the expectation operation is taken with respect to the statistics of $\mathcal{B}(t), t \geq 0$.

Theorem 3.2. [16] The optimal control solution which minimize the performance function

$$\min_{u(t)=F(t)x(t)} E\left\{\int_0^T (Qx^2(s) + RF^2x^2(s))ds + Q_T x(T)^2\right\}, \tag{3.6}$$

is given by

$$u_{fb}(t) = -R^{-1}Bk(t)x(t), \tag{3.7}$$

where $k(t)$ is the solution of the following Riccati differential equation:

$$\dot{k}(t) = -2Ak(t) + Sk^2(t) - Qk(t) = Q_T, \;\; S = R^{-1}B^2. \tag{3.8}$$

The space of control feedback admissible is given by:

$$\Gamma^{fb} = \{u(t)\,/u(t) = F(t)x(t)\}, \tag{3.9}$$

where F(.) is a piecewise continuous function.

4 Bernoulli operational matrices for solving optimal control problem

The success of spectral methods in practical computations has led to an increasing interest in their theoretical and numerical aspects, so there is a challenge in developing accurate numerical methods by using spectral methods. In this paper, we use Bernoulli operational matrix to get approximate solution of stochastic optimal control. Let consider the stochastic Volterra control problem given by equation (1.1). We approximate $x(t)$ and $u(t)$ as follows

$$x(t) \simeq x_m(t) = \sum_{i=0}^{m} x_i B_i(t) = X^T B(t), \tag{4.1}$$

$$u(t) \simeq u_m(t) = \sum_{i=0}^{m} u_i B_i(t) = U^T B(t), \tag{4.2}$$

where $X = (x_0, x_1, ..., x_m)^T$ and $U = (u_0, u_1, ..., u_m)^T$. For a given function $P(x(t), u(t)) \in \mathbb{R}$ such that

$$P(x(t), u(t)) = \sum_{i=0}^{l'} \sum_{j=0}^{l} P_{ij} f_i(x(t)) w_j(u(t)), \tag{4.3}$$

we approximate $f_i(x(t))$ and $w_j(u(t))$ as follows

$$f_i(x(t)) = \sum_{j=0}^{m} f_{ij} B_j(t) = F_i^T B(t), \tag{4.4}$$

$$w_i(x(t)) = \sum_{j=0}^{m} w_{ij} B_j(t) = W_i^T B(t), \tag{4.5}$$

where $F_i = (F_{i,0}, F_{i,1}, ..., F_{i,m})^T$ and $W_i = (W_{i,0}, W_{i,1}, ..., W_{i,m})^T$, now we suppose that the functions a and σ in equation (1.1) are written in the forme

$$a(x(t), u(t)) = \sum_{i=0}^{l'} \sum_{j=0}^{l} a_{ij} f_i^1(x(t)) w_j^1(u(t)), \tag{4.6}$$

$$\sigma(x(t), u(t)) = \sum_{i=0}^{k'} \sum_{j=0}^{k} \sigma_{ij} f_i^2(x(t)) w_j^2(u(t)). \tag{4.7}$$

replacing Eqs. (4.1),(4.2),(4.6) and (4.7) in (1.1), we get

$$\begin{aligned} X^T B(t) &\simeq x_0 + \int_0^t \sum_{i=0}^{l'} \sum_{j=0}^{l} a_{ij} f_i^1(x(s)) w_j^1(u(s)) ds + \int_0^t \sum_{i=0}^{k'} \sum_{j=0}^{k} \sigma_{ij} f_i^2(x(s)) w_j^2(u(s)) d\mathcal{B}(s) \\ &\overset{(2.2)}{\simeq} x_0 + \sum_{i=0}^{l'} \sum_{j=0}^{l} a_{ij} (F_i^1)^T \left(\int_0^t (DT_m(s))(DT_m(s))^T ds \right) W_j^1 \\ &\quad + \sum_{i=0}^{k'} \sum_{j=0}^{k} \sigma_{ij} (F_i^2)^T \left(\int_0^t (DT_m(s))(DT_m(s))^T d\mathcal{B}(s) \right) W_j^2 \\ &\simeq x_0 + \sum_{i=0}^{l'} \sum_{j=0}^{l} a_{ij} (F_i^1)^T D \left(\int_0^t T_m(s) T_m^T(s) ds \right) D^T W_j^1 \\ &\quad + \sum_{i=0}^{k'} \sum_{j=0}^{k} \sigma_{ij} (F_i^2)^T D \left(\int_0^t T_m(s) T_m^T(s) d\mathcal{B}(s) \right) D^T W_j^2. \end{aligned} \tag{4.8}$$

Now, we calculate the integrals in Eqs.(4.8), we have

$$T_m(s) T_m^T(s) = \begin{pmatrix} 1 & s & s^2 & \cdots & s^m \\ s & s^2 & s^3 & \cdots & s^{m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s^m & s^{m+1} & s^{m+2} & \cdots & s^{2m} \end{pmatrix}.$$

Let

$$H = \int_0^t T_m(s) T_m^T(s) ds = \begin{pmatrix} t & \frac{t^2}{2} & \frac{t^3}{3} & \cdots & \frac{t^{m+1}}{m+1} \\ \frac{t^2}{2} & \frac{t^3}{3} & \frac{t^4}{4} & \cdots & \frac{t^{m+2}}{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{t^{m+1}}{m+1} & \frac{t^{m+2}}{m+2} & \frac{t^{m+3}}{m+3} & \cdots & \frac{t^{2m+1}}{2m+1} \end{pmatrix},$$

and

$$L = \int_0^t T_m(s) T_m^T(s) d\mathcal{B}(s),$$

so

$$L = \begin{pmatrix} \int_0^t 1 d\mathcal{B}(s) & \int_0^t s d\mathcal{B}(s) & \int_0^t s^2 d\mathcal{B}(s) & \cdots & \int_0^t s^m d\mathcal{B}(s) \\ \int_0^t s d\mathcal{B}(s) & \int_0^t s^2 d\mathcal{B}(s) & \int_0^t s^3 d\mathcal{B}(s) & \cdots & \int_0^t s^{m+1} d\mathcal{B}(s) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \int_0^t s^m d\mathcal{B}(s) & \int_0^t s^{m+1} d\mathcal{B}(s) & \int_0^t s^{m+2} d\mathcal{B}(s) & \cdots & \int_0^t s^{2m} d\mathcal{B}(s) \end{pmatrix}.$$

Each components in the matrix L is written as follows

$$\int_0^t s^i d\mathcal{B}(s), \quad i = 0, ..., 2m, \tag{4.9}$$

we use the integration by parts, we get

$$\int_0^t s^i d\mathcal{B}(s) = t^i \mathcal{B}(t) - i \underbrace{\int_0^t s^{i-1}\mathcal{B}(s)ds}_{(I)}. \tag{4.10}$$

Applying Simpson quadrature to calculate the integral (I), we get

$$\int_0^t s^{i-1}\mathcal{B}(s)ds = \frac{t}{6}\left[4\left(\frac{t}{2}\right)^{i-1}\mathcal{B}\left(\frac{t}{2}\right) + t^{i-1}\mathcal{B}(t),\right] \tag{4.11}$$

so we have

$$\int_0^t s^i d\mathcal{B}(s) = \begin{cases} \mathcal{B}(t) & \text{if} \quad i = 0, \\ t^i\mathcal{B}(t) - i\frac{t}{6}\left[4\left(\frac{t}{2}\right)^{i-1}\mathcal{B}\left(\frac{t}{2}\right) + t^{i-1}\mathcal{B}(t)\right] & \text{if} \quad i = 1, ..., 2m, \end{cases} \tag{4.12}$$

by replacing the values of deterministic and stochastic integrals in Eq.(4.8), we get

$$X^T B(t) - x_0 - \sum_{i=0}^{l'}\sum_{j=0}^{l} a_{ij}(F_i^1)^T DHD^T W_j^1 - \sum_{i=0}^{k'}\sum_{j=0}^{k} \sigma_{ij}(F_i^2)^T DLD^T W_j^2 = 0. \tag{4.13}$$

Eq.(4.13) can be written as follows

$$\psi(X, U) = 0, \tag{4.14}$$

where

$$\psi(X, U) = X^T B(t) - x_0 - \sum_{i=0}^{l'}\sum_{j=0}^{l} a_{ij}(F_i^1)^T DHD^T W_j^1 - \sum_{i=0}^{k'}\sum_{j=0}^{k} \sigma_{ij}(F_i^2)^T DLD^T W_j^2.$$

Now for the performance index function, we suppose that the function c in equation (1.2) is written in the forme

$$c(x(t), u(t)) = \sum_{i=0}^{n'}\sum_{j=0}^{n} c_{ij} f_i^3(x(t)) w_j^3(u(t)), \tag{4.15}$$

so we have

$$\begin{aligned} J(u) &= E\left[\int_0^T \sum_{i=0}^{n'}\sum_{j=0}^{n} c_{ij} f_i^3(x(s)) w_j^3(u(s))ds + g(x^u(T))\right] \\ &= E\left[\sum_{i=0}^{n'}\sum_{j=0}^{n} c_{ij}(F_i^3)^T \left(\int_0^T B(t)B^T(t)\right) W_j^3 + R^T B(t)\right] \\ &= E\left[\sum_{i=0}^{n'}\sum_{j=0}^{n} c_{ij}(F_i^3)^T D\left(\int_0^T T_m(t)T_m^T(t)\right) D^T W_j^3 + R^T B(t)\right] \\ &= E\left[\sum_{i=0}^{n'}\sum_{j=0}^{n} c_{ij}(F_i^3)^T DH'D^T W_j^3 + R^T B(t)\right] \end{aligned} \tag{4.16}$$

where R is the Bernoulli coefficients vector defined as

$$R = [g(x^u(T)), 0, 0, \cdots, 0]^T,$$

and

$$H' = \int_0^T T_m(s)T_m^T(s)ds = \begin{pmatrix} T & \frac{T^2}{2} & \frac{T^3}{3} & \cdots & \frac{T^{m+1}}{m+1} \\ \frac{T^2}{2} & \frac{T^3}{3} & \frac{T^4}{4} & \cdots & \frac{T^{m+2}}{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{T^{m+1}}{m+1} & \frac{T^{m+2}}{m+2} & \frac{T^{m+3}}{m+3} & \cdots & \frac{T^{2m+1}}{2m+1} \end{pmatrix}.$$

In the next section, we solve the optimisation problem (4.8)-(4.16). For each sample trajectory of Brownian motion, we have

$$J^i(X,U) = \int_0^T c^i(t, X^T B(t), U^T B(t))dt + g^i(X^T B(T)), \quad i = 1, 2, ..., r_1. \tag{4.17}$$

Now, we can approximate the expectation function (1.2) by the sample average

$$J_{r_1}(X,U) = \frac{1}{r_1}\sum_{i=1}^{r_1} J^i(X,U). \tag{4.18}$$

For simplicity, we use $J(X,U)$ rather to $J^i(X,U)$. The optimization problem is as follows

$$\begin{cases} \text{minimize } J(X,U), \\ \quad \text{s.t. } \psi(X,U) = 0, \end{cases} \tag{4.19}$$

If $(\check{X}, \check{U})$ is a solution of (4.19), then there exist a vector $\lambda = (\lambda_0, \lambda_1, ..., \lambda_m)^T$ such that

$$L(X,U,\lambda) = J(X,U) + \lambda^T \psi(X,U), \tag{4.20}$$

where $L(X,U,\lambda)$ is the Lagrangian function, and λ is the Lagrange multiplier associated with the constraint of equality. So to find the solution of problem (4.19), we have to solve the following system

$$\begin{aligned} &\frac{\partial L}{\partial X_i}(X,U,\lambda) = 0, \\ &\frac{\partial L}{\partial U_i}(X,U,\lambda) = 0, \qquad i = 0, 1, \cdots m, \\ &\frac{\partial L}{\partial \lambda_i}(X,U,\lambda) = 0, \end{aligned} \tag{4.21}$$

5 Convergence and error estimation

Let (a,b) be a bounded interval of the real line, and $M \geq 0$ be an integer, we define $H^M(a,b)$ to be the vector space of the functions $v \in L^2(a,b)$ such that all the distributional derivatives of v of order up to M can be represented by functions in L^2 that is

$$H^M(a,b) = \left\{ v \in L^2(a,b), \text{ for } 0 \leq k \leq M, \frac{d^k v}{dx^k} \in L^2(a,b) \right\}, \tag{5.1}$$

$H^M(a,b)$ is endowed with the inner product

$$(u,v)_M = \sum_{k=0}^{M} \int_a^b \frac{d^k u}{dx^k}(x) \frac{d^k v}{dx^k}(x) dx.$$

The space $H^M(a,b)$ is an Hilbert space with the associated norm

$$\|v\|_{H^M(a,b)} = \left(\sum_{k=0}^{M} \left\| \frac{d^k v}{dx^k} \right\|^2_{L^2(a,b)} \right)^{\frac{1}{2}}.$$

Properties

1. The space $H^M(a,b)$ verify
$$H^{M+1}(a,b) \subset H^M(a,b) \subset H^0(a,b) = L^2(a,b),$$
2. $C^M(a,b) \subset H^M(a,b)$,
3. $C^\infty(a,b)$ dense in $H^M(a,b)$.

Remark 5.1. For the bound error of the approximation of solution, it is suitable to introduce the semi norm

$$|v|_{H^{M,m}(a,b)} = \left(\sum_{j=\min(m,M)}^{M} \left\| v^{(j)} \right\|^2_{L^2(a,b)} \right)^{\frac{1}{2}}.$$

We can see that

$$|v|_{H^{M,m}(a,b)} \le \|v\|_{H^M(a,b)},$$

and if $M \le m$, we have

$$|v|_{H^{M,m}(a,b)} = \|v^M\|_{L^2(a,b)} = |v|_{H^M(a,b)},$$

Theorem 5.2. Let $v \in H^M(a,b)$ such that

$$v(t) = \sum_{i=0}^{m} \alpha_i B_i(t), \tag{5.2}$$

and $I_m v$ is the approximation by Bernoulli polynomials. Then the truncation error $v - I_m v$ satisfies

$$\|v - I_m v\|_{L^2(a,b)} \le c_M m^{-M} \|v\|_{H^{M,m}(a,b)}, \tag{5.3}$$

where

$$\|v\|_{H^{M,m}(a,b)} = \left[\sum_{j=\min(m,M)}^{M} \left(\frac{b-a}{2} \right)^{2j} \left\| v^{(j)} \right\|^2_{L^2(a,b)} \right]^{\frac{1}{2}}.$$

Moreover, in the maximum norm, we have

$$\|v - I_m v\|_{L^\infty(a,b)} \le \tilde{c}_M m^{\frac{1}{2}-M} \sqrt{\frac{2}{b-a}} \, \|v\|_{H^{M,m}(a,b)}\,, \tag{5.4}$$

where $\tilde{c}_M$ is a constant independent of m and depend of M and

$$\|v\|_{L^\infty(a,b)} = \sup_{t \in [a,b]} |v(t)|.$$

Theorem 5.3. Let $x^u_m(t)$ be the approximate solution of $x^u(t)$ by the proposed method. Suppose that a and σ satisfy the Lipshitz conditions (with constants L_1 and L_2 respectively) and linear growth conditions (these conditions ensures the existence and uniqueness of the solution). Then the numerical approximation (4.1) of the exact solution is convergent in the sense that $E|x^u(t) - x^u_m(t)|^2 \to 0$, as $m \to \infty$.

Proof . We have

$$x^u_m = x_0 + \int_0^t a(x^u_m, u_m) ds + \int_0^t \sigma(x^u_m, u_m) d\mathcal{B}_s. \tag{5.5}$$

Let, $e_m(t) = x^u(t) - x^u_m(t)$ so

$$e_m(t) = \int_0^t a(s, x^u_s, u) - a(s, x_m(s), u_m) ds + \int_0^t \sigma(s, x^u_s, u) - \sigma(s, x_m(s), u_m) d\mathcal{B}_s,$$

then

$$|x^u(t) - x^u_m(t)| \le \int_0^t |a(s, x^u_s, u) - a(s, x_m(s), u_m)| ds + \int_0^t |\sigma(s, x^u_s, u) - \sigma(s, x_m(s), u_m)| d\mathcal{B}_s,$$

so

$$|x^u(t) - x^u_m(t)|^2 \le \left(\int_0^t |a(s, x^u_s, u) - a(s, x_m(s), u_m)| ds + \int_0^t |\sigma(s, x^u_s, u) - \sigma(s, x_m(s), u_m)| d\mathcal{B}_s \right)^2.$$

Using the inequality

$$(a+b)^2 \leq 2a^2 + 2b^2 \quad a, b \in \mathbb{R}, \tag{5.6}$$

we get

$$|x^u(t) - x_m^u(t)|^2 \leq 2\left(\int_0^t |a(s, x_s^u, u) - a(s, x_m(s), u_m)|ds\right)^2 + 2\left(\int_0^t |\sigma(s, x_s^u, u) - \sigma(s, x_m(s), u_m)|d\mathcal{B}_s\right)^2.$$

By taking expectation and using Cauchy-Shwartz inequality, we get

$$E|x^u(t) - x_m^u(t)|^2 \leq 2\int_0^t E|a(s, x_s^u, u_s) - a(s, x_m(s), u_m)|^2 ds + 2\int_0^t E|\sigma(s, x_s^u, u_s) - \sigma(s, x_m(s), u_m)|^2 ds,$$

by using Lipschitz conditions, we obtain

$$E|x^u(t) - x_m^u(t)|^2 \leq 2L_1^2 \int_0^t E\left(|x_s^u - x_m| + |u_s - u_m|\right)^2 ds + 2L_2^2 \int_0^t E\left(|x_s^u - x_m| + |u_s - u_m|\right)^2 ds.$$

Using inequality (5.6), we get

$$\begin{aligned}
E|x^u(t) - x_m^u(t)|^2 &\leq 2L_1^2 \int_0^t E\left(2|x_s^u - x_m|^2 + 2|u_s - u_m|^2\right) ds + 2L_2^2 \int_0^t E\left(2|x_s^u - x_m|^2 + 2|u_s - u_m|^2\right) ds \\
&\leq 4(L_1^2 + L_2^2)\left(\int_0^t E|x_s^u - x_m|^2 ds + \int_0^t E|u_s - u_m|^2 ds\right) \\
&= 4(L_1^2 + L_2^2)||u_s - u_m||^2 + 4(L_1^2 + L_2^2)\left(\int_0^t E|x_s^u - x_m|^2 ds\right),
\end{aligned}$$

by Gronwal inequality, we obtain

$$E|x^u(t) - x_m^u(t)|^2 \leq 4(L_1^2 + L_2^2)||u_s - u_m||^2 \left(1 + 4(L_1^2 + L_2^2)\int_0^t e^{4(L_1^2 + L_2^2)(t-s)} ds\right),$$

by using Theorem 5.2, we get

$$E|x^u(t) - x_m^u(t)|^2 \leq 4(L_1^2 + L_2^2)c_M m^{-M} \|u\|_{H^{M,m}(0,1)} \left(1 + 4(L_1^2 + L_2^2)\int_0^t e^{4(L_1^2 + L_2^2)(t-s)} ds\right),$$

when $m \to \infty$, we get the result. □

Theorem 5.4. Let J and J^* be the exact and approximate optimal perfermance index for problem (1.1)-(1.2). Suppose that the functions g and c satisfy the Lipschitz conditions that is

$$|c(t, x_1, u_1) - c(t, x_2, u_2)| \leq k_1|x_1 - x_2| + k_2|u_1 - u_2|,$$

$$|g(t, x_1) - g(t, x_2)| \leq k_3|x_1 - x_2|.$$

Then we have

$$||J - J^*||_{L^2} \leq \max(4k_1^2, 4k_2^2)c_M m^{-M}(\|x\|_{H^{M,m}(0,1)} + \|u\|_{H^{M,m}(0,1)}) + 2k_3^2 E|x_m^u(T) - x^u(T)|^2. \tag{5.7}$$

Proof . For each fixed $\omega \in \Omega$, we have

$$\begin{aligned}
|J - J^*|^2 &= \left|\int_0^t c(s, x, u) - c(s, x_m, u_m)ds + g(x_m^u(T)) - g(x^u(T))\right|^2 \\
&\leq 2|\int_0^t c(s, x, u) - c(s, x_m, u_m)ds|^2 + 2|g(x_m^u(T)) - g(x^u(T))|^2,
\end{aligned}$$

by Cauchy Schwartz inequality, and Lipschitz condition, we get

$$
\begin{aligned}
|J(u)-J^*(u)|^2 &\leq 2\int_0^T [k_1|x-x_m|+k_2|u-u_m|]^2\,ds + 2|g(x_m^u(T))-g(x^u(T))|^2\\
&\leq 4k_1^2\int_0^T |x^u-x_m|^2 ds + 4k_2^2\int_0^T |u-u_m|^2 ds + 2|g(x_m^u(T))-g(x^u(T))|^2,
\end{aligned}
$$

taking the expectation, we get

$$
\begin{aligned}
E|J(u)-J^*(u)|^2 &\leq 4k_1^2\int_0^T E|x^u-x_m|^2 ds + 4k_2^2\int_0^T E|u-u_m|^2 ds + 2k_3^2 E|x_m^u(T)-x^u(T)|^2\\
&\leq 4k_1^2||x^u-x_m||^2 + 4k_2^2||u-u_m||^2 + 2k_3^2 E|x_m^u(T)-x^u(T)|^2.
\end{aligned}
$$

By using theorem 5.2, we get

$$
E|J(u)-J^*(u)|^2 = ||J(u)-J^*(u)||_{L^2}^2 \leq \max(4k_1^2, 4k_2^2)c_M m^{-M}(\|x\|_{H^{M,m}(0,1)} + \|u\|_{H^{M,m}(0,1)}) + 2k_3^2 E|x_m^u(T)-x^u(T)|^2
$$

by using Theorem 5.3 with $m\to\infty$, or $M \longrightarrow \infty$, in the sens that the functions u and x are enough smooth, we obtain the result. $\square$

6 Numerical examples

In this section, we consider some numerical examples to illustrate the applicability and accuracy of the proposed method. For numerical simulations, it is useful to consider a discretization of Brownian motion on the time interval $[0,1]$. We take $K=100$ simultations to approximate the exact solutions. In the most examples, we use open loop and feedback controls. Some numerical tests are compared with some existing results in literature. We can see from examples that this spectral technique is efficiency for solving optimal control problems.

Example 6.1. Consider the following problem

$$
J = \min_{u\in U} E\left\{\frac{1}{2}\int_0^1 2x^2(t)+u^2(t)dt\right\}, \tag{6.1}
$$

$$
x(t) = x(0) + \int_0^t \left(-\frac{1}{2}x(s)+u(s)\right)ds + \sigma\int_0^t d\mathcal{B}(s), \quad x(0)=v, \tag{6.2}
$$

where $\sigma>0$, and v is a constant. The analytical solution is unknown, so, we need approximate solution to the problem. The aim here is to compare between open loop and feed back control to get approximate solution for the stochastic problem (6.1)-(6.2). The agreement between the results of simulations of the two strategies open loop and feedback controls, and the simplicity of the proposed method, enables the method as an excellent approach to solve the problem.

Open loop control The approximation of functions in equation 6.2 by Bernoulli polynomials leads to the following equation

$$
\begin{aligned}
X^T B(t) &\simeq X_0^T B(t) + \int_0^t \left(\frac{-1}{2}X^T B(s) + U^T B(s)ds\right) + \sigma\int_0^t 1\,d\mathcal{B}(s)\\
&\simeq X_0^T B(t) - \frac{1}{2}X^T P B(t) + U^T P B(t) + \sigma\int_0^t q^T B(t)\,d\mathcal{B}(t)\\
&\simeq X_0^T B(t) + \left(\frac{-1}{2}X^T + U^T\right)PB(t) + \sigma q^T P_s B(t)
\end{aligned} \tag{6.3}
$$

where $\sigma>0$, $q=[1,0,\cdots,0]^T$ and X_0 is written as follows

$$
X_0 = [v,0,\cdots,0]^T.
$$

For the performance index function, we have

$$
\begin{aligned}
J(X,U) &= \frac{1}{2}\int_0^1 \left(2X^T B(t)B(t)^T X + U^T B(t)B(t)^T U\right) dt \\
&= X^T \left(\int_0^1 B(t)B(t)^T dt\right) X + \frac{1}{2}U^T \left(\int_0^1 B(t)B(t)^T dt\right) U \\
&= X^T DHD^T X + \frac{1}{2}U^T DHD^T U. \qquad (6.4)
\end{aligned}
$$

So we obtain

$$\psi(X,U) = X - X_0 - \sigma P_s^T q + \frac{1}{2}P^T X - P^T U, \qquad (6.5)$$

and

$$J(X,U) = X^T DHD^T X + \frac{1}{2}U^T DHD^T U. \qquad (6.6)$$

Feedback control Applying Theorem 3.2, the Riccati differential equation associated to this problem is

$$\dot{k}(t) = k(t) + 2k^2(t) - 1, \quad k(1) = 0, \qquad (6.7)$$

with $A = \frac{-1}{2}$, $B = 1$, $Q = 1$, T=1, $Q_T = 0$, $R = \frac{1}{2}$ and $S = 2$. We can find the solution of this differential equation by using the approach in [16], so let

$$Z = \begin{pmatrix} -\frac{1}{2} & -2 \\ -1 & \frac{1}{2} \end{pmatrix},$$

the eigenvalues of Z are $\{\frac{3}{2}, \frac{-3}{2}\}$ and corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The solution of the differential equation (6.7) is given by

$$k(t) = \frac{v(t)}{u(t)}, \qquad (6.8)$$

where

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{Z(1-t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (6.9)$$

so the solution of Riccati differential equation is

$$k(t) = \frac{e^{-3t+3} - 1}{1 + 2e^{-3t+3}}. \qquad (6.10)$$

Then the optimal control for the stochastic control problem is

$$u_{fb}(t) = -2k(t)x(t), \qquad (6.11)$$

for simplicity we denote $K(t) = -2k(t)$, then we approximate the state and the function $k(t)$ via Bernoulli polynomials, we obtain

$$\psi(X,U) = X - X_0 - \sigma P_s^T q + \frac{1}{2}P^T X - P^T \tilde{X} D^T K, \qquad (6.12)$$

and

$$J(X,U) = X^T DHD^T X + \frac{1}{2}K^T D\tilde{K}^T DHD^T \tilde{X} D^T X, \qquad (6.13)$$

where

$$K(t) = \sum_{i=0}^{m} K_i B_i(t) = K^T B(t), \quad K = [K_0, K_2, ..., K_m]^T$$

and $\tilde{X} = [\tilde{E}_0, \tilde{E}_1, ...\tilde{E}_m]$ is a matrix of size $(m+1) \times (m+1)$, where

$$\tilde{E}_i = E_k X = [e_{k,0}, e_{k,1}, ..., e_{k,m}]X, \quad i = 0, ..., m,$$

Table 1: Approximate solutions by open loop and feedback strategies for different times with m=3 and $\sigma = 0.1$ for Example 1.

t	$X_{\text{open loop}(t)}$	$X_{\text{feed back}(t)}$	$U_{\text{open loop}(t)}$	$U_{\text{feed back}(t)}$
0	0.10198589	0.10287844	-0.08930474	-0.09538202
0.1	0.08888342	0.08913076	-0.07985662	-0.08043775
0.2	0.07754017	0.07733824	-0.06942082	-0.06727094
0.3	0.06781306	0.06733815	-0.05845357	-0.0556828
0.4	0.05955903	0.05896777	-0.04741109	-0.04546297
0.5	0.05263500	0.05206436	-0.03674961	-0.03638764
0.6	0.04689791	0.04646521	-0.02692535	-0.0282202
0.7	0.04220469	0.04200758	-0.01839456	-0.02071710
0.8	0.03841227	0.03852875	-0.01161344	-0.01364065
0.9	0.03537758	0.03586600	-0.007038242	-0.00678324
1	0.03295754	0.03385660	-0.00512517	0

Table 2: Approximate solutions by open loop and feedback strategies for different times with m=3 and $\sigma = 1$ for Example 1.

t	$X_{\text{open loop}(t)}$	$X_{\text{feed back}(t)}$	$U_{\text{open loop}(t)}$	$U_{\text{feed back}(t)}$
0	0.01216786	0.01227435	-0.01065488	-0.01137996
0.1	0.01060462	0.01063413	-0.009527640	-0.00959697
0.2	0.009251265	0.00922717	-0.008282552	-0.008026050
0.3	0.008090730	0.008034070	-0.006974056	-0.006643478
0.4	0.007105947	0.007035405	-0.005656585	-0.005424157
0.5	0.00627984	0.006211764	-0.00438457	-0.00434138
0.6	0.005595359	0.005543733	-0.003212446	-0.003366941
0.7	0.00503541	0.005011896	-0.002194642	-0.002471743
0.8	0.004582942	0.00459683	-0.001385592	-0.001627457
0.9	0.004220875	0.004279148	-0.000839728	-0.0008093038
1	0.003932142	0.004039408	-0.000611481	0

Table 3: Absolute Errors between Open loop and feedback strategies for m=3, $\sigma = 1$ and $\sigma = 0.1$ for Example 1.

	$\sigma = 0.1$			$\sigma = 1$
t	State Error	Control Error	State Error	Control Error
0	$8.92e^{-4}$	$6.07e^{-3}$	$1.06e^{-4}$	$7.25e^{-4}$
0.1	$2.47e^{-4}$	$5.81e^{-4}$	$2.95e^{-5}$	$6.93e^{-5}$
0.2	$2.01e^{-4}$	$2.14e^{-3}$	$2.40e^{-5}$	$2.56e^{-4}$
0.3	$4.74e^{-4}$	$2.77e^{-3}$	$5.66e^{-5}$	$3.30e^{-4}$
0.4	$5.91e^{-4}$	$1.94e^{-3}$	$7.05e^{-5}$	$2.32e^{-4}$
0.5	$5.70e^{-4}$	$3.61e^{-4}$	$6.80e^{-5}$	$4.31e^{-5}$
0.6	$4.32e^{-4}$	$1.29e^{-3}$	$5.16e^{-5}$	$1.54e^{-4}$
0.7	$1.97e^{-4}$	$2.32e^{-3}$	$2.35e^{-5}$	$2.77e^{-4}$
0.8	$1.16e^{-4}$	$2.02e^{-3}$	$1.38e^{-5}$	$2.41e^{-4}$
0.9	$4.88e^{-4}$	$2.55e^{-4}$	$5.82e^{-5}$	$3.04e^{-5}$
1	$8.99e^{-4}$	$5.12e^{-3}$	$1.07e^{-4}$	$6.11e^{-4}$

Table 4: Approximate solutions by open loop and feedback strategies for different times with m=5 and $\sigma = 0.1$ for Example 1.

t	$X_{\text{open loop}(t)}$	$X_{\text{feed back}(t)}$	$U_{\text{open loop}(t)}$	$U_{\text{feed back}(t)}$
0	0.09784629	0.10286199	-0.10007600	-0.09536678
0.1	0.08545688	0.08932835	-0.07941320	-0.08061607
0.2	0.07666973	0.07780160	-0.06815421	-0.06767398
0.3	0.06975341	0.06801113	-0.05873993	-0.05623930
0.4	0.06348507	0.05973073	-0.04823578	-0.04605120
0.5	0.05712513	0.05277399	-0.03680254	-0.03688360
0.6	0.05039195	0.04698974	-0.02616722	-0.02853883
0.7	0.04343647	0.04225746	-0.01809389	-0.02084033
0.8	0.03681689	0.03848269	-0.01285456	-0.01362434
0.9	0.03147332	0.03559245	-0.007699995	-0.006731504
1	0.02870247	0.03353065	0.00466942	0

Table 5: Approximate solutions by open loop and feedback strategies for different times with m=5 and $\sigma = 1$ for Example 1.

t	$X_{\text{open loop}(t)}$	$X_{\text{feed back}(t)}$	$U_{\text{open loop}(t)}$	$U_{\text{feed back}(t)}$
0	0.03161785	0.03323862	-0.03233836	-0.03081663
0.1	0.02761436	0.02886538	-0.02566142	-0.02605012
0.2	0.02477490	0.02514065	-0.0220232	-0.02186803
0.3	0.02253998	0.02197698	-0.01898110	-0.01817305
0.4	0.02051444	0.01930127	-0.01558681	-0.01488089
0.5	0.01845930	0.01705328	-0.01189230	-0.0119184
0.6	0.01628355	0.01518417	-0.008455624	-0.009221983
0.7	0.01403597	0.01365499	-0.005846826	-0.00673430
0.8	0.01189693	0.01243522	-0.004153799	-0.004402543
0.9	0.01017022	0.01150127	-0.002488161	-0.002175205
1	0.00927486	0.01083502	0.001508869	0

Table 6: Absolute Errors between Open loop and feedback strategies for m=5, $\sigma = 1$ and $\sigma = 0.1$ for Example 1.

	$\sigma = 0.1$			$\sigma = 1$
t	State Error	Control Error	State Error	Control Error
0	$5.01e^{-3}$	$4.70e^{-3}$	$1.62e^{-3}$	$1.52e^{-3}$
0.1	$3.87e^{-3}$	$1.20e^{-3}$	$1.25e^{-3}$	$3.88e^{-4}$
0.2	$1.13e^{-3}$	$4.80e^{-4}$	$3.65e^{-4}$	$1.55e^{-4}$
0.3	$1.74e^{-3}$	$2.50e^{-3}$	$5.62e^{-4}$	$8.08e^{-4}$
0.4	$3.75e^{-3}$	$2.18e^{-3}$	$1.21\mathrm{e}^{-3}$	$7.05e^{-4}$
0.5	$4.35e^{-3}$	$8.10e^{-5}$	$1.40e^{-3}$	$2.61e^{-5}$
0.6	$3.40e^{-3}$	$2.37e^{-3}$	$1.09e^{-3}$	$7.66e^{-4}$
0.7	$1.17e^{-3}$	$2.74e^{-3}$	$3.80e^{-4}$	$8.87e^{-4}$
0.8	$1.66e^{-3}$	$7.69e^{-4}$	$5.38e^{-4}$	$2.48e^{-4}$
0.9	$4.11e^{-3}$	$9.68e^{-4}$	$1.33\mathrm{e}^{-3}$	$3.12e^{-4}$
1	$4.82e^{-3}$	$4.66e^{-3}$	$1.56e^{-3}$	$1.50e^{-3}$

and the $\{e_{k,i}\}$ are the Bernoulli coefficients vector of $t^k B_i(t)$, see [3]. The following results summarized in tables 1, 2, 4 and table 5 represents the state variable and the control with both strategies open loop and feedback controls with $m = 3$, $m = 5$ and $x(0) = 0.1$. The absolute errors between the open loop and feedback controls for $m = 3$, $m = 5$ are presented in tabels 3 and 6.

Example 6.2. Consider the stochastic optimal control problem [14]

$$J(u) = \frac{1}{2}\int_0^T E|x_t^* - x_t|^2 dt + \frac{1}{2}\int_0^T u_t^2 dt, \tag{6.14}$$

$$dx_t = u_t x_t dt + \sigma x_t d\mathcal{B}_t, \quad x(0) = v. \tag{6.15}$$

where $\sigma > 0$, v is a constant and the exact solutions are

$$x_t = ve^{\int_0^t u_s ds - \frac{\sigma^2}{2}t + \sigma\mathcal{B}_t}, \qquad u_t = \frac{T-t}{\frac{1}{v} - Tt + \frac{t^2}{2}}. \tag{6.16}$$

The function x_t^* is given in [14]

$$x_t^* = \frac{e^{\sigma^2 t} - (1-t)^2}{1 - t + \frac{t^2}{2}} + 1. \tag{6.17}$$

Let $T = 1$, and $x(0) = v = 1$.

Open loop control We approximate all functions oppeared in the problem (6.14)-(6.15) by Bernoulli polynomials, we get

$$\begin{aligned} X^T B(t) &\simeq X_0^T B(t) + \int_0^t U^T B(s)B(s)^T X ds + \sigma \int_0^t X^T B(s)\, d\mathcal{B}(s) \\ &\simeq X_0^T B(t) + U^T \int_0^t B(s)B(s)^T X ds + \sigma X^T \int_0^t B(s)\, d\mathcal{B}(s) \\ &\simeq X_0^T B(t) + U^T D\tilde{X}^T PB(t) + \sigma X^T P_s B(t). \end{aligned} \tag{6.18}$$

We have

$$\psi(X, U) = X - X_0 - P^T \tilde{X} D^T U - \sigma P_s^T X. \tag{6.19}$$

The approximation of performance index is as follows

$$J(X,U) = \frac{1}{2}(X^*)^T DHD^T X^* + \frac{1}{2}X^T DHD^T X - (X^*)^T DHD^T X + \frac{1}{2}U^T DHD^T U,$$

where X^* represents the Bernoulli coefficients vector of x_t^*.

Table 7 and table 10 represents the state variable and the control with open loop strategy with $m = 3$, $m = 5$ and $\sigma = 0$, and $\sigma = 0.3$. Table 8 represents the absolute error between the approximate and exact solution for $m = 3$. The approximate solution of optimal control problem (6.14)-(6.15) and the exact solution with $m = 3$ are illustrated in Figures 1-4 with $\sigma = 0$, $\sigma = 0.01$, $\sigma = 0.1$ and $\sigma = 0.3$ respectively. The exact and approximate optimal control solutions for $m = 5$ are presented in Figures 5-6 for $\sigma = 0$ and $\sigma = 0.01$. Table 9 represents the approximate optimal cost function J_{op}^* in different choices of σ, the exact cost function J_{exa} and the error between them for $m = 3$. For $m = 5$ with $\sigma = 0$ and $\sigma = 0.01$, the approximate optimal cost functions are $J_{op}^* = 0.50019565431$ and $J_{op}^* = 0.49945429789$ respectively with an error equal to $1.95e^{-3}$ and $6.93e^{-4}$ respectively.

From the experiment tests, we see that, There is no significant difference between exact and approximate solutions for different values of the diffusion parameter σ. The error gradually increases in proportion to the value of diffusion parameter σ.

Example 6.3. Consider the following optimal control problem [17]

$$J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \tag{6.20}$$

Table 7: Approximate solutions by open loop control strategy for different times with m=3 for Example 2.

	$\sigma=0$		$\sigma=0.3$	
t	$X_{\text{open loop}(t)}$	$U_{\text{open loop}(t)}$	$X_{\text{open loop}(t)}$	$U_{\text{open loop}(t)}$
0	0.98554555	0.97980248	1.06536456	0.80200833
0.1	1.09066353	0.98795670	1.14554643	0.89456451
0.2	1.21196779	0.97387065	1.25698605	0.93608244
0.3	1.34286845	0.93686953	1.38928650	0.93196985
0.4	1.47677562	0.87627853	1.53205087	0.88763446
0.5	1.60709941	0.79142286	1.67488223	0.80848401
0.6	1.72724994	0.68162772	1.80738367	0.69992621
0.7	1.83063733	0.54621829	1.91915826	0.56736880
0.8	1.91067168	0.38451977	1.99980910	0.41621950
0.9	1.96076312	0.19585737	2.03893925	0.25188603
1	1.97432176	-0.02044371	2.02615180	0.07977612

Table 8: Computed errors when m=3 for Example 2.

	$\sigma=0$		$\sigma=0.3$	
t	State Error	Control Error	State Error	Control Error
0	$1.44e^{-2}$	$2.01e^{-2}$	$6.53e^{-2}$	$1.19e^{-1}$
0.1	$1.43e^{-2}$	$6.51e^{-3}$	$4.14e^{-2}$	$9.99e^{-2}$
0.2	$7.54e^{-3}$	$1.73e^{-3}$	$3.71e^{-2}$	$3.95e^{-2}$
0.3	$5.86e^{-4}$	$2.72e^{-3}$	$2.71e^{-2}$	$7.62e^{-3}$
0.4	$6.18e^{-3}$	$6.07e^{-3}$	$5.14e^{-2}$	$5.28e^{-3}$
0.5	$7.09e^{-3}$	$8.57e^{-3}$	$5.65e^{-2}$	$8.48e^{-3}$
0.6	$3.11e^{-3}$	$8.02e^{-3}$	$6.24e^{-2}$	$1.02e^{-2}$
0.7	$4.22e^{-3}$	$4.24e^{-3}$	$4.74e^{-2}$	$1.69e^{-2}$
0.8	$1.24e^{-2}$	$9.56e^{-5}$	$3.43e^{-2}$	$3.16e^{-2}$
0.9	$1.94e^{-2}$	$2.16e^{-3}$	$2.78e^{-2}$	$5.38e^{-2}$
1	$2.56e^{-2}$	$2.04e^{-2}$	$1.13e^{-2}$	$7.97e^{-2}$

Table 9: Optimal cost function for different choices of σ with m=3 in Example 2.

	J_{exa}	J^*_{op}	$\lvert J^*_{op}-J_{exa}\rvert$
$\sigma=0$	0.5	0.5000744766	$7.4e^{-5}$
$\sigma=0.01$	0.5001478657	0.5014163273	$1.2e^{-3}$
$\sigma=0.1$	0.5148980636	0.4901163308	$2.4e^{-2}$
$\sigma=0.3$	0.6425918186	0.5525869602	$9.0e^{-2}$

Table 10: Approximate solutions by open loop strategy and estimate errors for different times with m=5 and $\sigma=0$ for Example 2.

t	$X_{\text{open loop}(t)}$	$U_{\text{open loop}(t)}$	State Error	Control Error
0	0.96335843	0.95655876	$3.66e^{-2}$	$4.43e^{-2}$
0.1	1.06888356	0.97364998	$3.60e^{-2}$	$2.08e^{-2}$
0.2	1.19740152	0.96524833	$2.21e^{-2}$	$1.03e^{-2}$
0.3	1.34039511	0.93851170	$1.88e^{-3}$	$1.08e^{-3}$
0.4	1.48741892	0.89132711	$1.68e^{-2}$	$8.97e^{-3}$
0.5	1.62743599	0.81678371	$2.74e^{-2}$	$1.67e^{-2}$
0.6	1.75015454	0.70764575	$2.60e^{-2}$	$1.79e^{-2}$
0.7	1.84736464	0.56082561	$1.25e^{-2}$	$1.03e^{-2}$
0.8	1.91427490	0.38185677	$8.80e^{-3}$	$2.75e^{-3}$
0.9	1.95084916	0.18936682	$2.93e^{-2}$	$8.65e^{-3}$
1	1.96314316	0.01955043	$3.68e^{-2}$	$1.95e^{-2}$

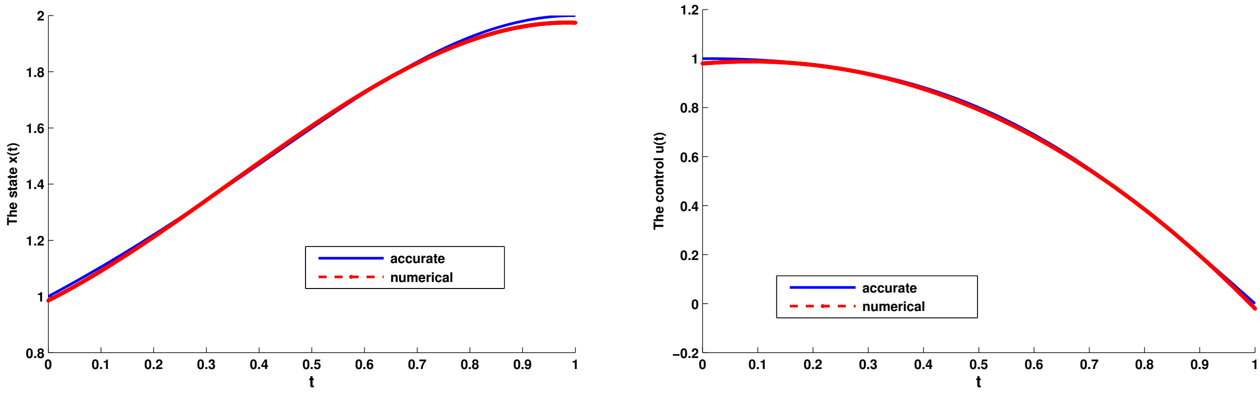

Figure 1: Exact and approximate solutions for m=3 and $\sigma = 0$ for Example 2.

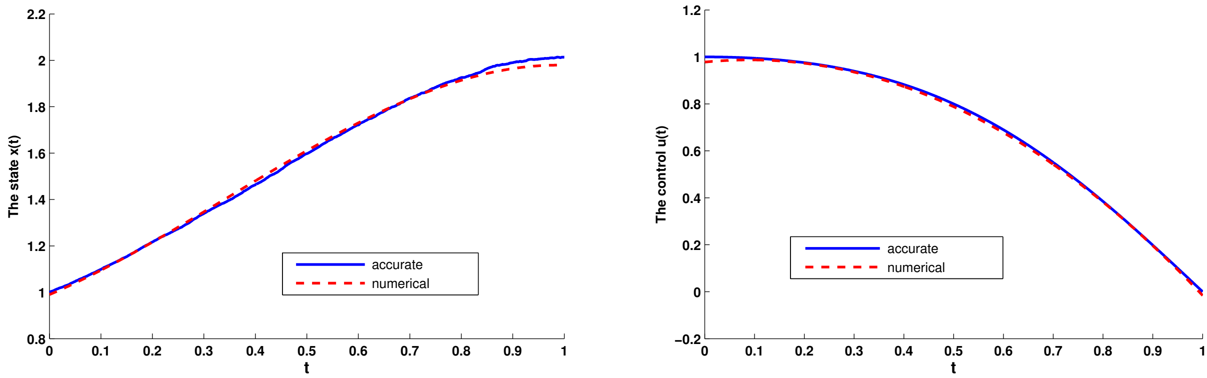

Figure 2: Exact and approximate solutions for m=3 and $\sigma = 0.01$ for Example 2.

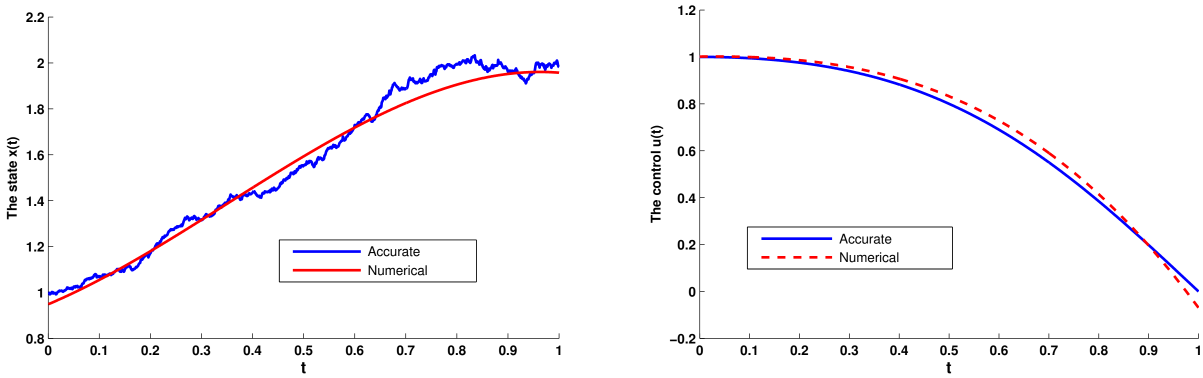

Figure 3: Exact and approximate solutions for m=3 and $\sigma = 0.1$ for Example 2.

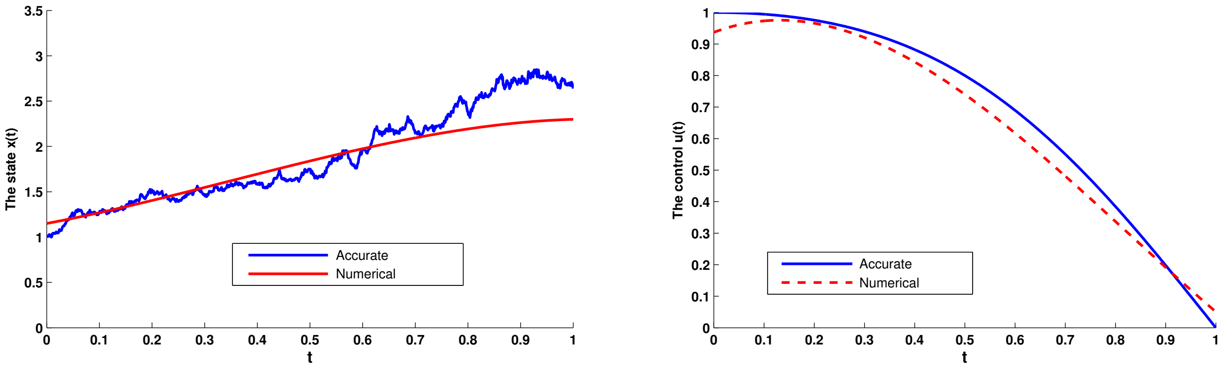

Figure 4: Exact and approximate solutions for m=3 and $\sigma = 0.3$ for Example 2.

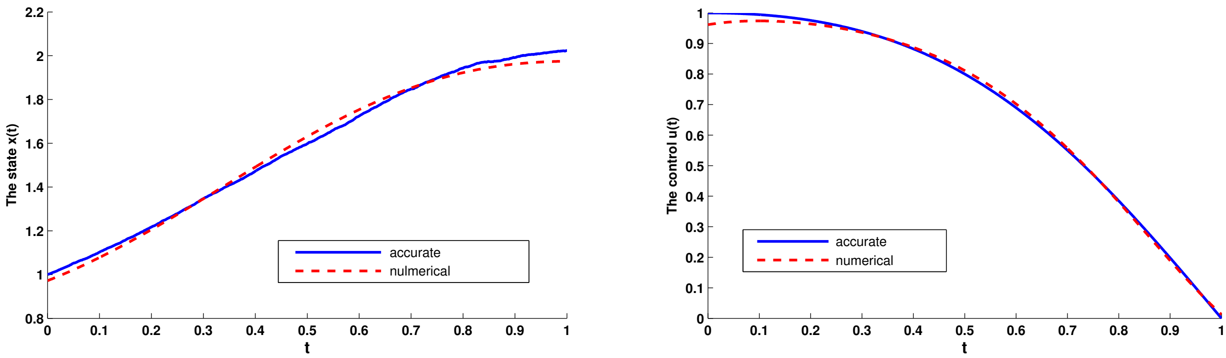

Figure 5: Exact and approximate solutions for m=5 and $\sigma = 0$ for Example 2.

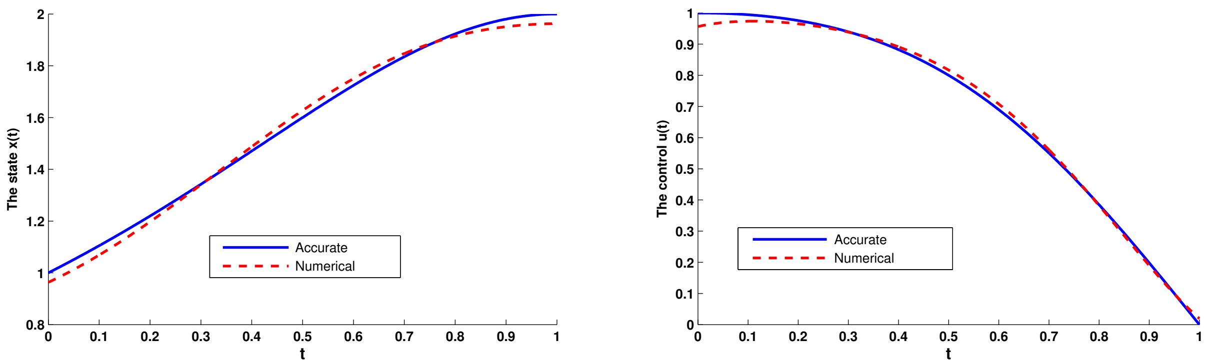

Figure 6: Exact and approximate solutions for m=5 and $\sigma = 0.01$ for Example 2.

$$x(t) = x(0) + \int_0^t sx(s)ds + \int_0^t u(s)ds, \quad x(0) = 1, \tag{6.21}$$

where the exact solution is unknown. In this example, we compare between the obtained results of the proposed method and some numerical results obtained by other numerical techniques [17]. From the results, we remark that all methods gives approximatively the same values for state variable and control.

Open loop control

$$\begin{aligned} X^T B(t) &\simeq X_0^T B(t) + \int_0^t q^T B(s) B(s)^T X ds + \int_0^t U^T B(s) ds \\ &\simeq X_0^T B(t) + q^T \int_0^t B(s) B(s)^T X ds + U^T P B(t) \\ &\simeq X_0^T B(t) + q^T D \tilde{X}^T P B(t) + U^T P B(t), \end{aligned} \tag{6.22}$$

where q is the Bernoulli coefficients vector of the function t given by

$$q = [\frac{1}{2}, 1, 0, ..., 0]^T.$$

We have

$$\psi(X, U) = X - X_0 - P^T \tilde{X} D^T q - P^T U. \tag{6.23}$$

For the performance index function, we have

$$J = \frac{1}{2} X^T D H D^T X + \frac{1}{2} U^T D H D^T U. \tag{6.24}$$

The approximate solutions of problem (6.20)-(6.21) for $m = 6$ is given by

$$\begin{aligned} x(t) &= 0.16121349t^6 + 1.00228234t^5 - 3.035104676t^4 \\ &\quad + 2.12328867t^3 + 0.84023095t^2 - 1.17150851t + 0.99999999 \\ u(t) &= 7.299826t^6 - 21.815416t^5 + 24.51146t^4 \\ &\quad - 12.7833604t^3 + 3.07268461t^2 + 0.68309244t - 0.96045177. \end{aligned}$$

The performance index function with $m = 6$ is $J = 0.483998$ and the value of objective functions of Bernstein polynomials, power series, shifted Chebyshev (1st kind) and shifted Chebyshev (2nd kind) with $m = 6$ are $J = 0.484228$,$J = 0.484072$, $J = 0.484265$ and $J = 0.484265$ respectively. Tables 11 -12 illustrates numerical results of state and control with different methods with m=6. The approximate solutions of optimal control problem with the proposed method and Bernstein method are plotted in Figure 7.

Table 11: Different approaches of $x(t)$ in different time for Example 3.

t	The proposed method	Bernstein approximation	Power series	Shifted Chebyshev (2nd kind)	Shifted Chebyshev (1st kind)
0	1	1	0.999999	0.999999	1
0.1	0.893081	0.912841	0.9125870	0.9128372	0.9128436
0.2	0.811768	0.844075	0.843908	0.844080	0.844087
0.3	0.759465	0.792566	0.792471	0.792579	0.792584
0.4	0.734949	0.75766	0.757555	0.757679	0.757683
0.5	0.733860	0.739162	0.739019	0.739185	0.739188
0.6	0.750317	0.7737388	0.737186	0.737366	0.737371
0.7	0.778636	0.752962	0.752840	0.752996	0.753002
0.8	0.815174	0.78739	0.787292	0.787431	0.787436
0.9	0.860287	0.84268	0.842556	0.842727	0.842731
1	0.920402	0.92174	0.921603	0.921792	0.921799

Table 12: Different approaches of $u(t)$ in different time for Example 3.

t	The proposed method	Bernstein approximation	Power series	Shifted Chebyshev (2nd kind)	Shifted Chebyshev (1st kind)
0	-0.960451	-0.968575	-0.968348	-0.968525	-0.968532
0.1	-0.871958	-0.868473	-0.868375	-0.868533	-0.868540
0.2	-0.770488	-0.768625	-0.768501	-0.768640	-0.768646
0.3	-0.673280	-0.669056	-0.668942	-0.669060	-0.669065
0.4	-0.579716	-0.570052	-0.569985	-0.570084	-0.570088
0.5	-0.484359	-0.47197	-0.471934	-0.472016	-0.472019
0.6	-0.384735	-0.375096	-0.375055	-0.375120	-0.375123
0.7	-0.283860	-0.279572	-0.279515	-0.279562	-0.279564
0.8	-0.187513	-0.185361	-0.185332	-0.185357	-0.185359
0.9	-0.096248	-0.092280	-0.092312	-0.092316	-0.092318
1	0.007841	$-8.24e^{-5}$	$-1.74e^{-8}$	$-5.25e^{-9}$	$-8.99e^{-8}$

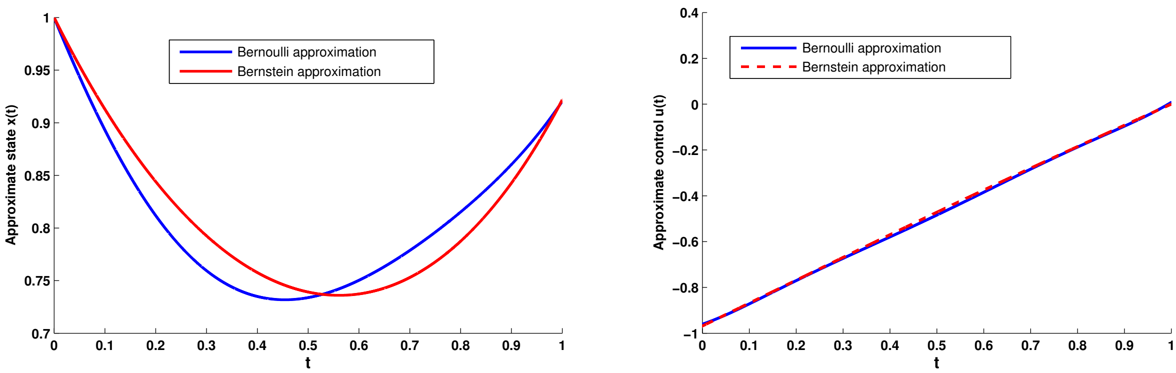

Figure 7: Approximate solutions x(t) and u(t) with both methods for Example 3.

Example 6.4. Consider the following problem [30]

$$J = \frac{1}{2}\int_0^1 (u^2(t) + 2x^2(t))dt \tag{6.25}$$

$$\dot{x}(t) = u(t) + \frac{x(t)}{2}, \quad x(0) = 1. \tag{6.26}$$

The analytical solutions are

$$x(t) = \frac{2e^{3t} + e^3}{e^{\frac{3t}{2}}(2 + e^3)}, \qquad u(t) = \frac{2(e^{3t} - e^3)}{e^{\frac{3t}{2}}(2 + e^3)}, \tag{6.27}$$

and the optimal value of performance index function is $J_{exa} = 0.86416449776911$.

Open loop control We have

$$\psi(X, U) = X - X_0 - P^T U - \frac{1}{2}P^T X, \tag{6.28}$$

$$J = X^T DHD^T X + \frac{1}{2}U^T DHD^T U. \tag{6.29}$$

Feedback control Using Theorem 3.2, the feedback optimal control is given by

$$u(t) = K(t)x(t), \tag{6.30}$$

where

$$K(t) = \frac{-2(e^{-3t+3} - 1)}{1 + 2e^{-3t+3}}, \tag{6.31}$$

and

$$\psi(X, U) = X - X_0 - P^T \tilde{X} D^T K - \frac{1}{2}P^T X, \tag{6.32}$$

$$J(X, U) = X^T DHD^T X + \frac{1}{2}K^T D\tilde{K}^T DHD^T \tilde{X} D^T X. \tag{6.33}$$

Numerical results by open and feedback control are summarized in Tables 13 and 14 with $m = 3$ and $m = 5$ respectively. Table 15 represents a comparison between the open loop and feedback strategies with $m = 3$ and $m = 5$. Figure 9 represents the approximate solution with both proposed strategies comparing with the exact solutions for $m = 3$. Tables 16-17 represents the main error between exact and approximate solutions with $m = 3$ and $m = 5$.

By taking $m = 2$, the optimal cost function obtained by open loop strategy is

$$J^*_{op} = 0.86434463794683776.$$

The error between the exact performance and J^*_{op} is

$$Error = |J^*_{op} - J_{exa}| = 1.8014e^{-4},$$

by comparing this result with the numerical results given in Table 1 in reference [20], the obtained error is less than all errors presented in table 1 of reference [20]. In [30], the approximate values of performance function are $J = 0.864374131$ and $J = 0.86444444238$ with errors $E_1 = 2.0963e^{-4}$ and $E_2 = 2.7999e^{-4}$ respectively. It is clear that, the obtained error by the proposed method is less than the errors E_1 and E_2, which confirm the good agreement between exact and approximate solutions.

7 Conclusion

In this paper, we have solved the stochastic optimal control problem from the view of open loop and feedback strategies. The proposed technique used operational matrices for the integration of Bernoulli polynomials. Then these operational matrices are used to reduce the considered problem to an optimisation problem and then has been reduced to a problem of solving a system of algebraic equations. The method is based on expanding the existing functions in terms of Bernoulli polynomials. Some advantages of the proposed methods are:

Table 13: Approximate solutions by open loop and feedback strategies in different times with m=3 for Example 4.

t	X_{exa}	$X_{\text{open loop}(t)}$	$X_{\text{feedback}(t)}$	U_{exa}	$U_{\text{open loop}(t)}$	$U_{\text{feedback}(t)}$
0	1.0	0.98807949	1.00669549	-1.72832899	-1.63233705	-1.73990101
0.1	0.88797706	0.88311942	0.89013900	-1.46031746	-1.45551431	-1.46387286
0.2	0.79597111	0.79527708	0.794036958	-1.22522472	-1.26224344	-1.22224751
0.3	0.72190810	0.72362006	0.71717611	-1.01775126	-1.06064101	-1.01108006
0.4	0.66411851	0.66721596	0.65834322	-0.83322018	-0.85882358	-0.82597435
0.5	0.62129962	0.62513235	0.61632504	-0.66747173	-0.664907715	-0.66212746
0.6	0.59248621	0.59643685	0.58990833	-0.51676957	-0.487009964	-0.51452114
0.7	0.57702875	0.58019703	0.57787985	-0.37771655	-0.333246892	-0.37827367
0.8	0.57457880	0.57548050	0.57902636	-0.24717810	-0.21173505	-0.24909139
0.9	0.58508112	0.58135485	0.59213460	-0.12221159	-0.13059102	-0.12368491
1	0.60877248	0.59688766	0.61599134	-3.6927e^{-18}	-0.097931344	0

Table 14: Approximate solutions by open loop and feedback strategies for different times with m=5 for Example 4.

t	$X_{\text{open loop}(t)}$	$X_{\text{feedback}(t)}$	$U_{\text{open loop}(t)}$	$U_{\text{feedback}(t)}$
0	0.90789344	1.00064883	1.804310211	-1.72945040
0.1	0.80924777	0.88723619	-1.43561218	-1.45909907
0.2	0.76250895	0.79431660	-1.23291225	-1.22267797
0.3	0.74101299	0.72000613	-1.06174073	-1.01506984
0.4	0.72459700	0.66258986	-0.87063649	-0.83130230
0.5	0.69997931	0.62056697	-0.66348403	-0.66668463
0.6	0.66113957	0.59269566	-0.47185048	-0.51695226
0.7	0.60969884	0.57803823	-0.32732266	-0.37837735
0.8	0.55529972	0.57600609	-0.23384406	-0.24779210
0.9	0.51598643	0.58640473	-0.14005192	-0.12248806
1	0.51858493	0.60947878	0.08838574	0

Table 15: Comparaison between open loop and feedback strategies for Example 4.

	$m=3$		$m=5$	
t	State Error	Control Error	State Error	Control Error
0	1.86e^{-2}	1.07e^{-1}	9.27e^{-2}	7.48e^{-2}
0.1	7.01e^{-3}	8.35e^{-3}	7.79e^{-2}	2.34e^{-2}
0.2	1.24e^{-3}	3.99e^{-2}	3.18e^{-2}	1.02e^{-2}
0.3	6.44e^{-3}	4.95e^{-2}	2.10e^{-2}	4.66e^{-2}
0.4	8.87e^{-3}	3.28e^{-2}	6.20e^{-2}	3.93e^{-2}
0.5	8.80e^{-3}	2.78e^{-3}	7.94e^{-2}	3.20e^{-3}
0.6	6.52e^{-3}	2.75e^{-2}	6.84e^{-2}	4.51e^{-2}
0.7	2.31e^{-3}	4.50e^{-2}	3.16e^{-2}	5.10e^{-2}
0.8	3.54e^{-3}	3.73e^{-2}	2.07e^{-2}	1.39e^{-2}
0.9	1.07e^{-2}	6.90e^{-3}	7.04e^{-2}	1.75e^{-2}
1	1.91e^{-2}	9.79e^{-2}	9.08e^{-2}	8.83e^{-2}

Table 16: Absolute error when m=3 for Example 4.

	Open loop		Feedback	
t	$\lvert X_{op} - X_{exa}\rvert$	$\lvert U_{op} - U_{exa}\rvert$	$\lvert X_{fb} - X_{exa}\rvert$	$\lvert U_{fb} - U_{exa}\rvert$
0	$1.19e^{-2}$	$9.59e^{-2}$	$6.69e^{-3}$	$1.15e^{-2}$
0.1	$4.85e^{-3}$	$4.80e^{-3}$	$2.16e^{-3}$	$3.35e^{-3}$
0.2	$6.94e^{-4}$	$3.70e^{-2}$	$1.19e^{-3}$	$2.97e^{-3}$
0.3	$1.71e^{-3}$	$4.28e^{-2}$	$4.73e^{-3}$	$6.67e^{-3}$
0.4	$3.09e^{-3}$	$2.56e^{-2}$	$5.77e^{-3}$	$7.24e^{-3}$
0.5	$3.83e^{-3}$	$2.56e^{-3}$	$4.97e^{-3}$	$5.34e^{-3}$
0.6	$3.95e^{-3}$	$2.97e^{-2}$	$2.57e^{-3}$	$2.24e^{-3}$
0.7	$3.16e^{-3}$	$4.44e^{-2}$	$8.51e^{-4}$	$5.57e^{-4}$
0.8	$9.01e^{-4}$	$3.54e^{-2}$	$4.44e^{-3}$	$1.91e^{-3}$
0.9	$3.72e^{-3}$	$8.37e^{-3}$	$7.05e^{-3}$	$1.47e^{-3}$
1	$1.18e^{-2}$	$9.79e^{-2}$	$7.21e^{-3}$	$3.69e^{-18}$

Table 17: Absolute error when m=5 for Example 4.

	Open loop		Feedback	
t	$\lvert X_{op} - X_{exa}\rvert$	$\lvert U_{op} - U_{exa}\rvert$	$\lvert X_{fb} - X_{exa}\rvert$	$\lvert U_{fb} - U_{exa}\rvert$
0	$9.21e^{-2}$	$7.59e^{-2}$	$6.48e^{-4}$	$1.12e^{-3}$
0.1	$7.87e^{-2}$	$2.47e^{-2}$	$7.40e^{-4}$	$1.21e^{-3}$
0.2	$3.34e^{-2}$	$7.68e^{-3}$	$1.65e^{-3}$	$2.25e^{-3}$
0.3	$1.91e^{-2}$	$4.39e^{-2}$	$1.90e^{-3}$	$2.68e^{-3}$
0.4	$6.04e^{-2}$	$3.74e^{-2}$	$1.52e^{-3}$	$1.91e^{-3}$
0.5	$7.87e^{-2}$	$3.98e^{-3}$	$7.32e^{-4}$	$7.87e^{-4}$
0.6	$6.86e^{-2}$	$4.49e^{-2}$	$2.09e^{-4}$	$1.82e^{-4}$
0.7	$3.26e^{-2}$	$5.03e^{-2}$	$1.00e^{-3}$	$6.60e^{-4}$
0.8	$1.92e^{-2}$	$1.33e^{-2}$	$1.42e^{-3}$	$6.14e^{-4}$
0.9	$6.90e^{-2}$	$1.78e^{-2}$	$1.32e^{-3}$	$2.76e^{-4}$
1	$9.01e^{-2}$	$8.83e^{-2}$	$7.06e^{-4}$	$3.69e^{-18}$

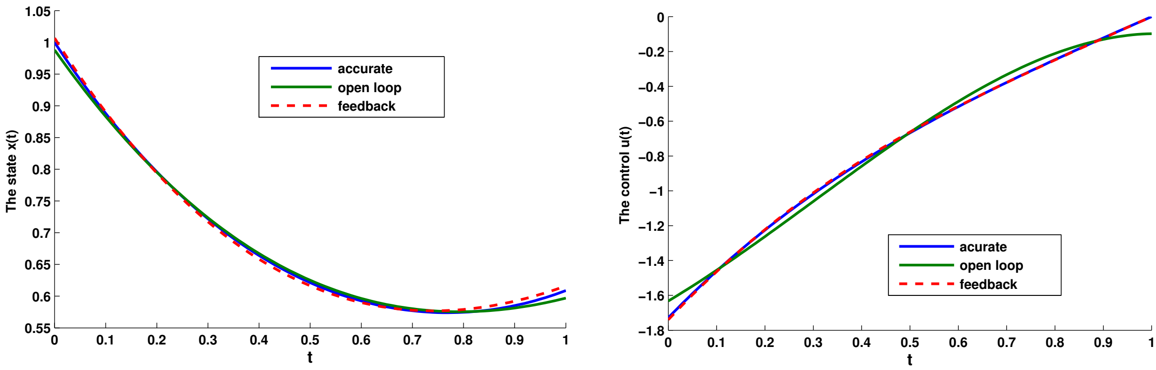

Figure 8: The approximate solutions compared with exact solution when m=3 for Example 4.

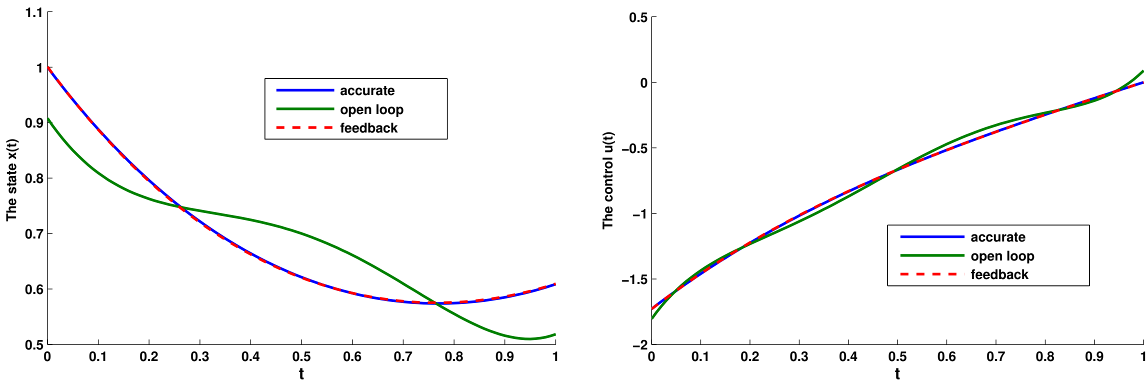

Figure 9: The approximate solutions compared with exact solution when m=5 for Example 4.

- The proposed method is a new technique for solving nonlinear optimal control problem and there is a few methods in the literature for solving this type of problem.
- The proposed method has high accuracy and little computational complexity for solving the considered problem.
- The effort required to implement the method is very low.
- It is also implementable and can be extended to higher dimensional control systems.

In the end, we note that the method can be easily extended and applied to stochastic optimal control problems governed by fractional Brownian motion. We also believe that it shall not be difficult to extend this approach to controlled stochastic delay equations of general form, which will be the subject of future research.

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