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Fuzzy fractional pantograph stochastic differential equations: Existence, uniqueness and averaging principle

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Abstract

Fuzzy fractional pantograph stochastic differential equations (FFPSDEs) is investigated here. The initial objective is to show the existence and uniqueness of solutions using Banach fixed point theorem. The second objective is discussing averaging principle of FFPSDEs, precisely, we will prove that the solutions of FFPSDEs can be approximated in the sense of mean square by the solutions of averaged fuzzy fractional stochastic system.

Keywords: fuzzy fractional pantograph stochastic differential equations, Banach fixed point, averaging principle
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1 Introduction

Pantograph equations are a sort of delay differential equation that may be encountered in physics, medicine, biology, and other domains. The word pantograph come from the work of Taylor and Ockendon [28]. Many academics have recently explored pantograph differential equations (PDEs) of fractional order, for example, we recommend the reader to [3, 14, 15, 17, 18, 20]. Furthermore, multiple writers have demonstrated the existence and uniqueness of solutions for various fractional pantograph differential equations (FPDEs) with distinct fractional derivatives, for example see [2, 13, 19, 32, 33]. Recently, PDEs have also been extended to pantograph stochastic differential equations (PSDEs), see [27, 24], in this context, Priyadharsini et al [30], extended PSDEs to fuzzy setting, they proposed a new type of equation nemely fuzzy fractional stochastic pantograph differential equations (FFSPDEs). On the other hand, the notion of averaging principle has a long history. It's a great way to look at the qualitative properties of a dynamical system. Then, the study of this method for stochastic differential equations (SDEs) has received a lot of attention as theory has progressed [4, 13, 16, 22, 25, 29, 34]. Arhrrabi et al [5] initiated the study of averaging principle of fuzzy SDEs, also, Arhrrabi et al [6, 7, 8, 9, 10, 11, 12] studied the existence and stability of solutions for fuzzy fractional SDEs (FFSDEs) with Brownian motions, existence and uniqueness results of FFSDEs with impulsive and Fuzzy fractional boundary value problem and other types of FFDEs. To our knowledge, no publication has looked at the averaging principle of fuzzy fractional PSDEs, instead, numerous studies have looked at the averaging principle of fractional PSDEs in a crisp case. To close this gap, we will investigate the existence, uniqueness, and averaging principle of

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solutions for a class of fuzzy fractional PSDEs defined by

$$\begin{cases} {}^{C}\mathbf{D}^{\gamma}\mathbf{z}(u) = f\big(u, \mathbf{z}(u), \mathbf{z}(\lambda t)\big) + \Big\langle \displaystyle\int_0^u g\big(s, \mathbf{z}(s), \mathbf{z}(\lambda s)\big) dw(s) \Big\rangle, u \in \mathcal{I} := [0, T]. \\ \mathbf{z}(0) = \mathbf{z}_0, \end{cases} \tag{1.1}$$

where ${}^{C}\mathbf{D}^{\gamma}$ is the Caputo fractional derivative of order $\gamma \in (0, 1)$, $w(u)$ is standard Brownian motion with m-dimensional and $\lambda \in (0, 1)$. The functions $f : \mathcal{I} \times \mathcal{F}^2_{\mathbb{R}^n} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ and $g : \mathcal{I} \times \mathcal{F}^2_{\mathbb{R}^n} \longrightarrow \mathbb{R}^{n\times m}$ are continuous on $\mathcal{I}$. The following are the innovations and main contributions of this paper:

- In the fuzzy stochastic setting, fuzzy fractional PSDEs are a novel concept.

- Under Lipschitz conditions, the averaging notion is employed to investigate the property of solution for a class of fuzzy fractional PSDEs.

The remainder of this work is structured as follows: Section 2 has some fundamental definitions, premises, and notes on fuzzy fractional PSDEs that will be useful later. Some relevant criteria for the existence and uniqueness of solutions for system (1.1) are derived in Section 3. We expand the averaging technique for system (1.1) in Section 4 under certain conditions. Section 5 includes an example to demonstrate the usefulness of our findings. The last section is where you come to a conclusion.

2 Preliminaries

Let $\mathcal{F}_{\mathbb{R}^n}$ indicates the fuzzy subsets on $\mathbb{R}^n$, defined as $\zeta : \mathbb{R}^n \longrightarrow [0, 1]$, which satisfies:

(1) ζ is normal, i.e $\exists z_0 \in \mathbb{R}^n$ such that $\zeta(z_0) = 1$,

(2) ζ is a convex fuzzy set, i.e for $0 \leq \beta \leq 1$

$$min\,\{\zeta(z_1), \zeta(z_2)\} \leq \zeta\big(\beta z_1 + (1-\beta) z_2\big), \forall z_1, z_2 \in \mathbb{R}^n,$$

(3) ζ is upper semicontinous on $\mathbb{R}^n$,

(4) $[\zeta]^0 = cl\{z \in \mathbb{R}^n : \zeta(z) > 0\}$ is compact, where cl represents the closure of a set.

Let $\gamma \in (0, 1]$, we define $[\zeta]^{\gamma} = \{z \in \mathbb{R}^n | \zeta(z) \geq \gamma\}$ and $[\zeta]^0 = \{z \in \mathbb{R}^n | \zeta(z) > 0\}$. From the conditions (1) to (4). The notation $[\zeta]^{\gamma} = [\underline{\zeta}(\gamma), \overline{\zeta}(\gamma)]$, denote the γ-cut set of ζ, for $\gamma \in [0, 1]$. We denote by $\underline{\zeta}$ and $\overline{\zeta}$ as the left and right end point of ζ, respectively. For $\zeta \in \mathcal{F}_{\mathbb{R}^n}$, we define the lengh of the γ-cut set of ζ as $len([\zeta]^{\gamma}) = \overline{\zeta}(\gamma) - \underline{\zeta}(\gamma)$. For addition and scalar multiplication in fuzzy set space $\mathcal{F}_{\mathbb{R}^n}$, we have $[\zeta_1 + \zeta_2]^{\gamma} = [\zeta_1]^{\gamma} + [\zeta_2]^{\gamma}$, $[\beta\zeta]^{\gamma} = \beta[\zeta]^{\gamma}$. The Hausdorff distance is given by

$$\begin{aligned} \mathbf{D}_{\infty}\big(\zeta_1, \zeta_2\big) &= \sup_{0\leq\gamma\leq1} \Big\{ |\underline{\zeta}_1(\gamma) - \underline{\zeta}_2(\gamma)|, |\overline{\zeta}_1(\gamma) - \overline{\zeta}_2(\gamma)| \Big\}, \\ &= \sup_{0\leq\gamma\leq1} \mathbf{D}_H\big([\zeta_1]^{\gamma}, [\zeta_2]^{\gamma}\big). \end{aligned}$$

We know that $(\mathcal{F}_{\mathbb{R}^n}, \mathbf{D}_{\infty})$ is complet metric space and satisfies:

$$\mathbf{D}_{\infty}\big(\zeta_1 + \zeta_3, \zeta_2 + \zeta_3\big) = \mathbf{D}_{\infty}\big(\zeta_1, \zeta_2\big),$$

$$\mathbf{D}_{\infty}\big(a\zeta_1, a\zeta_2\big) = |a|\mathbf{D}_{\infty}\big(\zeta_1, \zeta_2\big),$$

$$\mathbf{D}_{\infty}\big(\zeta_1, \zeta_2\big) \leq \mathbf{D}_{\infty}\big(\zeta_1, \zeta_3\big) + \mathbf{D}_{\infty}\big(\zeta_3, \zeta_2\big),$$

for all $\zeta_1, \zeta_2, \zeta_3 \in \mathcal{F}_{\mathbb{R}^n}$ and $a \in \mathbb{R}^n$.

Definition 2.1. [28] • The derivative $v'(u)$ of v is given by

$$[v'(s)]^\gamma = [(\underline{v}^\gamma)'(s), (\overline{v}^\gamma)'(s)],$$

as long as $v'(s) \in \mathcal{F}_{\mathbb{R}^n}$.
• The fuzzy integral $\int_c^d v(s)ds$, $c, d \in \mathcal{I}$ is given by

$$\left[\int_c^d v(s)ds\right]^\gamma = \left[\int_c^d \underline{v}^\gamma(s)ds, \int_c^d \overline{v}^\gamma(s)ds\right],$$

as long as the integral on the right hand side exist.

Definition 2.2. [31] Let $f : \mathcal{I} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$, the fuzzy Rieman-Liouville integral of f is defined by:

$$(\mathcal{J}^\gamma f)(v) = \frac{1}{\Gamma(\gamma)} \int_0^u (u-v)^{\gamma-1} f(v)dv.$$

Definition 2.3. [31] Let $Df \in C(\mathcal{I}, \mathcal{F}_{\mathbb{R}^n}) \cap L(\mathcal{I}, \mathcal{F}_{\mathbb{R}^n})$. The fuzzy fractional Caputo diffentiability of f is given by:

$${}^C\mathbf{D}^\gamma f(u) = \mathcal{J}_{c^+}^{1-\gamma}(Df)(u) = \frac{1}{\Gamma(1-\gamma)} \int_0^u \frac{(Df)(v)}{(u-v)^\gamma} dv.$$

The set $\mathbb{R}^n$ can be embedded into $\mathcal{F}_{\mathbb{R}^n}$ by using the following embedding $\langle . \rangle : \mathbb{R}^n \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ such that for $u \in \mathbb{R}^n$ we have

$$\langle u \rangle(b) = \begin{cases} 1, & b = u, \\ 0, & b \neq u. \end{cases}$$

Notations: Let $(\Omega, \mathcal{F}_{\mathbb{R}^n})$ be the complete probability space and $w(u)$ be a m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}_{\mathbb{R}^n})$. Let $L^2(\Omega, \mathcal{F}_{\mathbb{R}^n})$ be the collection of all strongly measurable square integrable $(\Omega, \mathcal{F}_{\mathbb{R}^n})$-valued random variable, which is a complete metric space equipped with the following metric

$$D^2(\zeta_1, \zeta_2) = \mathbb{E}\mathbf{D}^2_\infty(\zeta_1, \zeta_2).$$

Let $C\big(\mathcal{I}, L^2(\Omega, \mathcal{F}_{\mathbb{R}^n})\big)$ be the Banach space of all continuous process from $\mathcal{I}$ into $L^2(\Omega, \mathcal{F}_{\mathbb{R}^n})$ such that $\mathbb{E}\mathbf{D}^2_\infty(\zeta_1, \zeta_2) < \infty$. Denote by $\mathcal{B}_h := C\big(\mathcal{I}, L^2(\Omega, \mathcal{F}_{\mathbb{R}^n})\big)$ the closed bounded subspace of all continuous fuzzy process ζ in $L^2(\Omega, \mathcal{F}_{\mathbb{R}^n})$ consists of $\mathcal{A}_u$-adapted measurable process $\{\zeta(u), u \in \mathcal{I}\}$ equipped with the norm

$$\mathbb{E}\mathbf{D}^2_\infty(\zeta_1, \zeta_2) = \sup_{0 \le a \le T} \mathbb{E}\mathbf{D}^2_\infty(\zeta_1(u), \zeta_2(u)).$$

Remark 2.4. Note that $(\mathcal{B}_h, \mathbf{D}_\infty)$ is a complete metric space.

Proposition 2.5. [21] Let $\psi : \mathcal{I} \longrightarrow \mathbb{R}^n$, then for $u \in \mathcal{I}$;

$$\sup_{u \in [0,t]} \mathbb{E}\left\| \int_0^u \psi(s)dw(s) \right\|^2 \le C_T \int_0^u \|\psi(s)\|^2 ds.$$

Proposition 2.6. [26] Let $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^2\big(\mathcal{I} \times \Omega, \mathbf{N}; \mathbb{R}^n\big)$. Then for all $u \in \mathcal{I}$ we have

$$\mathbf{D}^2_\infty\left(\left\langle \int_0^u \mathbf{z}(s)dw(s) \right\rangle, \left\langle \int_0^u \mathbf{z}'(s)dw(s) \right\rangle\right) = \int_0^u \mathbf{D}^2_\infty\big(\langle \mathbf{z}(s) \rangle, \langle \mathbf{z}'(s) \rangle\big)ds.$$

3 Existence and uniqueness result

In this part, by using Banach's contraction mapping principle, we will show the existence and uniqueness of solution for FFPSDEs (1.1).

Definition 3.1. We say that $\{\mathbf{z}(u), u \in \mathcal{I}\}$ is a solution of problem (1.1) if

(*i*) $\mathbf{z}(\cdot) \in C\big(\mathcal{I}, \mathcal{F}_{\mathbb{R}^n}\big)$,

(*ii*) $\mathbf{z}(0) = \mathbf{z}_0$,

(*iii*) for $0 \leq u \leq T$, we have

$$\mathbf{z}(u) = \mathbf{z}_0 + \frac{1}{\Gamma(\gamma)} \int_0^u (u-s)^{\gamma-1} f\big(s, \mathbf{z}(s), \mathbf{z}(\lambda s)\big) ds + \frac{1}{\Gamma(\gamma)} \int_0^u (u-s)^{\gamma-1} \Big\langle \int_0^s g\big(v, \mathbf{z}(v), \mathbf{z}(\lambda v)\big) dw(v) \Big\rangle ds. \tag{3.1}$$

The following assumptions are being prepared in order to get the primary conclusion in this section:

($\mathcal{A}1$) f is continuous and $\exists L_1 > 0$ such that

$$\mathbb{E}\mathbf{D}^2_\infty\big(f(u, \mathbf{z}, \mathbf{w}), f(u, \mathbf{z}', \mathbf{w}')\big) \leq L_1 \Big(\mathbb{E}\mathbf{D}^2_\infty(\mathbf{z}, \mathbf{z}') + \mathbb{E}\mathbf{D}^2_\infty(\mathbf{w}, \mathbf{w}')\Big).$$

($\mathcal{A}2$) g is continuous and $\exists L_2 > 0$ such that

$$\mathbb{E}\big\|g(u, \mathbf{z}, \mathbf{w}) - g(u, \mathbf{z}', \mathbf{w}')\big\|^2 \leq L_2 \Big(\mathbb{E}\mathbf{D}^2_\infty(\mathbf{z}, \mathbf{z}') + \mathbb{E}\mathbf{D}^2_\infty(\mathbf{w}, \mathbf{w}')\Big).$$

($\mathcal{A}3$) We have

$$\mathbb{E}\mathbf{D}^2_\infty\big(f(u, \hat{\mathbf{0}}, \hat{\mathbf{0}}), \hat{\mathbf{0}}\big) \leq q_1 \qquad \text{and} \qquad \mathbb{E}\|g(u, \hat{\mathbf{0}}, \hat{\mathbf{0}})\|^2 \leq q_2.$$

Theorem 3.2. Suppose that the assumptions ($\mathcal{A}1$)-($\mathcal{A}3$) holds, then problem (1.1) has a unique solution provided that

$$\frac{4L_1 T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{2L_2 T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2} < 1.$$

Proof . We define the operator $\mathbf{T} : \mathcal{B}_h \longrightarrow \mathcal{B}_h$ by

$$(\mathbf{T}\mathbf{z})(u) = \mathbf{z}_0 + \frac{1}{\Gamma(\gamma)} \int_0^u \frac{f\big(s, \mathbf{z}(s), \mathbf{z}(\lambda s)\big)}{(u-s)^{1-\gamma}} ds + \frac{1}{\Gamma(\gamma)} \int_0^u \frac{\Big\langle \int_0^s g\big(u, \mathbf{z}(u), \mathbf{z}(\lambda u)\big) dw(u) \Big\rangle}{(u-s)^{1-\gamma}} ds.$$

For each positive number r, we define $\mathcal{B}_r = \Big\{\mathbf{z} \in \mathcal{B}_h : \mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}, \hat{\mathbf{0}}\big) \leq r\Big\}$.

Step 1: We prove that $\mathbf{T}(\mathcal{B}_r) \subseteq \mathcal{B}_r$. We choose

$$r \geq \frac{3\mathbb{E}\mathbf{D}^2_\infty(\mathbf{z}_0, \hat{\mathbf{0}}) + J_1}{1 - J_2}.$$

By using the assumptions above, Propositions 2.5-2.6, Hölder inequality and Itô isometric, that for $\mathbf{z} \in \mathcal{B}_r$, we get

$$\begin{aligned}
&\mathbb{E}\mathbf{D}^2_\infty\big((\mathcal{T}\mathbf{z})(u), \hat{\mathbf{0}}\big) \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}_0, \hat{\mathbf{0}}\big) + 3\mathbb{E}\mathbf{D}^2_\infty\Big(\frac{1}{\Gamma(\gamma)} \int_0^u (u-s)^{\gamma-1} f\big(s, \mathbf{z}(s), \mathbf{z}(\lambda s)\big) ds, \hat{\mathbf{0}}\Big) + 3\mathbb{E}\mathbf{D}^2_\infty\Big(\frac{1}{\Gamma(\gamma)} \int_0^u \Big\langle \int_0^s (u-s)^{\gamma-1} g\big(u, \mathbf{z}(u), \mathbf{z}(\lambda u)\big) dw(u) \Big\rangle ds, \hat{\mathbf{0}}\Big), \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}_0, \hat{\mathbf{0}}\big) + \frac{6T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \int_0^u \mathbb{E}\mathbf{D}^2_\infty\Big(f\big(s, \mathbf{z}(s), \mathbf{z}(\lambda s)\big), f\big(s, \hat{\mathbf{0}}, \hat{\mathbf{0}}\big)\Big) ds + \frac{6T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \int_0^u \mathbb{E}\mathbf{D}^2_\infty\Big(f\big(s, \hat{\mathbf{0}}, \hat{\mathbf{0}}\big), \hat{\mathbf{0}}\Big) ds \\
&+ \frac{6T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \int_0^u \Big(\int_0^s \mathbb{E}\Big\| g\big(u, \mathbf{z}(u), \mathbf{z}(\lambda u)\big) - g\big(u, \hat{\mathbf{0}}, \hat{\mathbf{0}}\big)\Big\|^2 du\Big) ds + \frac{6C_T T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \int_0^u \Big(\int_0^s \mathbb{E}\Big\| g\big(u, \hat{\mathbf{0}}, \hat{\mathbf{0}}\big)\Big\|^2 du\Big) ds, \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}_0, \hat{\mathbf{0}}\big) + \frac{6L_1 T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2} \int_0^u \Big[\mathbb{E}\mathbf{D}^2_\infty(\mathbf{z}(s), \hat{\mathbf{0}}) + \mathbb{E}\mathbf{D}^2_\infty(\mathbf{z}(\lambda s), \hat{\mathbf{0}})\Big] ds + \frac{6q_1 T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} \\
&+ \frac{6T^{2\gamma-1} L_2}{(2\gamma-1)(\Gamma(\gamma))^2} \int_0^u \Big(\int_0^s \Big[\mathbb{E}\mathbf{D}^2_\infty(\mathbf{z}(u), \hat{\mathbf{0}}) + \mathbb{E}\mathbf{D}^2_\infty(\mathbf{z}(\lambda u), \hat{\mathbf{0}})\Big] du\Big) ds + \frac{3C_T T^{2\gamma+1} q_2}{(2\gamma-1)(\Gamma(\gamma))^2} \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}_0, \hat{\mathbf{0}}\big) + \frac{12L_1 T^{2\gamma} r}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{6q_1 T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{6T^{2\gamma+1} L_2 r}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{3C_T T^{2\gamma+1} q_2}{(2\gamma-1)(\Gamma(\gamma))^2} \\
&\leq 3\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}_0, \hat{\mathbf{0}}\big) + J_1 + J_2 r,
\end{aligned}$$

where
$$J_1 = \frac{6q_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{3C_T T^{2\gamma+1}q_2}{(2\gamma-1)(\Gamma(\gamma))^2} \quad \text{and} \quad J_2 = \frac{12L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{6T^{2\gamma+1}L_2}{(2\gamma-1)(\Gamma(\gamma))^2}.$$

Finally, we have
$$\mathbb{E}\mathbf{D}^2_\infty\big((\mathbf{T}\mathbf{z})(u), \hat{\mathbf{0}}\big) \leq r,$$
which implies that $\mathbf{T}(\mathcal{B}_r) \subseteq \mathcal{B}_r$.

Step 2: In this step, we will prove that $\mathbf{T}$ is a contraction operator. Using the assumptions $(\mathcal{A}1)-(\mathcal{A}3)$, Proposition 2.6, Hölder inequality and Itô isometric, we have for $\mathbf{z}, \mathbf{z}' \in \mathcal{B}_r$ and $u \in \mathcal{I}$

$$\begin{aligned}
&\mathbb{E}\mathbf{D}^2_\infty\big((\mathbf{T}\mathbf{z})(u), (\mathbf{T}\mathbf{z}')(u)\big)\\
&\leq 2\mathbb{E}\mathbf{D}^2_\infty\bigg(\frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s, \mathbf{z}(s), \mathbf{z}(\lambda s)\big)ds, \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s, \mathbf{z}'(s), \mathbf{z}'(\lambda s)\big)ds\bigg)\\
&+ 2\mathbb{E}\mathbf{D}^2_\infty\bigg(\frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s g\big(u, \mathbf{z}(u), \mathbf{z}(\lambda u)\big)dw(u)\Big\rangle ds, \frac{1}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s g\big(u, \mathbf{z}'(u), \mathbf{z}'(\lambda u)\big)dw(u)\Big\rangle ds\bigg),\\
&\leq \frac{2T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \mathbb{E}\mathbf{D}^2_\infty\bigg(f\big(s, \mathbf{z}(s), \mathbf{z}(\lambda s)\big), f\big(s, \mathbf{z}'(s), \mathbf{z}'(\lambda s)\big)\bigg)ds\\
&+ \frac{2T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \bigg(\int_0^s \mathbb{E}\Big\| g\big(u, \mathbf{z}(u), \mathbf{z}(\lambda u)\big) - g\big(u, \mathbf{z}'(u), \mathbf{z}'(\lambda u)\big)\Big\|^2 du\bigg)ds,\\
&\leq \frac{2L_1T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \bigg(\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}(s), \mathbf{z}'(s)\big) + \mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}(\lambda s), \mathbf{z}'(\lambda s)\big)\bigg)ds\\
&+ \frac{2L_2T^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^u \bigg(\int_0^s \Big[\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}(s), \mathbf{z}'(s)\big) + \mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}(\lambda s), \mathbf{z}'(\lambda s)\big)\Big]du\bigg)ds,\\
&\leq \frac{4L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2}\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}, \mathbf{z}'\big) + \frac{2L_2T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}, \mathbf{z}'\big),\\
&\leq \bigg(\frac{4L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{2L_2T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\bigg)\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}, \mathbf{z}'\big).
\end{aligned}$$

Finally, we can get
$$\mathbb{E}\mathbf{D}^2_\infty\big((\mathbf{T}\mathbf{z})(u), (\mathbf{T}\mathbf{z}')(u)\big) \leq \bigg(\frac{4L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{2L_2T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\bigg)\mathbb{E}\mathbf{D}^2_\infty\big(\mathbf{z}, \mathbf{z}'\big).$$

Therefore, since $\frac{4L_1T^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2} + \frac{2L_2T^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2} < 1$, $\mathbf{T}$ is a contraction operator. Consequently, using Banach's contraction mapping principle, we get to the conclusion that $\mathbf{T}$ has a fixed point, which is the unique solution to (1.1). □

4 Averaging Principle result

The construction of an averaging concept for FFPSDEs is the focus of this section. First, we look at the standard form of Eq. (3.1).

$$\begin{aligned}
\mathbf{z}_\epsilon(u) = \mathbf{z}_0 &+ \frac{\epsilon}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1} f\big(s, \mathbf{z}_\epsilon(s), \mathbf{z}_\epsilon(\lambda s)\big)ds\\
&+ \frac{\sqrt{\epsilon}}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle \int_0^s g\big(u, \mathbf{z}_\epsilon(u), \mathbf{z}_\epsilon(\lambda u)\big)dw(u)\Big\rangle ds,
\end{aligned} \tag{4.1}$$

where $0 < \epsilon < \epsilon_0$ and ϵ_0 is a fixed integer. Moreover $\mathbf{z}_0$, f and g have the same requirements as in Eq. (3.1). For every fixed $0 < \epsilon < \epsilon_0$ and $u \in \mathcal{I}$, according to the existence and uniqueness findings, the Eq. (4.1) has a unique solution $\mathbf{z}_\epsilon(u)$. In order to determine if $\mathbf{z}_\epsilon(u)$ can be approximated by a small process to a simple process, we make certain assumptions about the coefficients. Let $\tilde{f} : \mathcal{F}_{\mathbb{R}^n} \times \mathcal{F}_{\mathbb{R}^n} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ and $\tilde{g} : \mathcal{F}_{\mathbb{R}^n} \times \mathcal{F}_{\mathbb{R}^n} \longrightarrow \mathbb{R}^{n\times m}$ be measurable functions satisfying $(\mathcal{A}1)$-$(\mathcal{A}3)$ and the other inequalities:

($\mathcal{A}4$) For $K \in \mathcal{I}$ and $\mathbf{z}, \mathbf{w} \in \mathcal{F}_{\mathbb{R}^n}$, we have

$$\frac{1}{K}\int_0^K \mathbf{D}_\infty^2\Big(f(s,\mathbf{z},\mathbf{w}),\tilde{f}(\mathbf{z},\mathbf{w})\Big)ds \le \gamma_1(K)\big(1+\mathbf{D}_\infty^2(\mathbf{z},\hat{\mathbf{0}})+\mathbf{D}_\infty^2(\mathbf{w},\hat{\mathbf{0}})\big),$$

$$\frac{1}{K}\int_0^K \big\|g(s,\mathbf{z},\mathbf{w})-\tilde{g}(\mathbf{z},\mathbf{w})\big\|^2 ds \le \gamma_2(K)\big(1+\mathbf{D}_\infty^2(\mathbf{z},\hat{\mathbf{0}})+\mathbf{D}_\infty^2(\mathbf{w},\hat{\mathbf{0}})\big),$$

where $\lim\limits_{K\longrightarrow\infty}\gamma_i(K)=0,\ i=1,2$. After making the necessary preparations, we will demonstrate that $\mathbf{z}_\epsilon$ converge as $\epsilon \longrightarrow 0$, to solution of the following averaged FFSPDEs

$$\mathbf{w}_\epsilon(u)=\mathbf{z}_0+\frac{\epsilon}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}f\big(\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s)\big)ds+\frac{\sqrt{\epsilon}}{\Gamma(\gamma)}\int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g\big(\mathbf{w}_\epsilon(u),\mathbf{w}_\epsilon(\lambda u)\big)dw(u)\Big\rangle ds. \quad (4.2)$$

Under the same presumptions as Eq. (4.1), it is obvious that Eq. (4.2) likewise has a unique solution $\mathbf{w}_\epsilon$. As the main outcome of this section, we now examine the connections between the processes $\mathbf{z}_\epsilon$ and $\mathbf{w}_\epsilon$.

Theorem 4.1. If the conditions ($\mathcal{A}1$)-($\mathcal{A}4$) are verified. Then, for a given random tiny number $\delta > 0$ and a constant $k > 0$, $0 < \gamma < 1$, there exist $0 < \epsilon_1 \le \epsilon_0 \mid \forall\epsilon \in (0, \epsilon_1]$, we have

$$\sup_{u\in[0,k\epsilon^{-\gamma}]} \mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{z}_\epsilon(u),\mathbf{w}_\epsilon(u)\big) \le \delta.$$

Proof . For $0 < u \le v$, we have

$$\begin{aligned}
&\sup_{u\in[0,v]} \mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{z}_\epsilon(u),\mathbf{w}_\epsilon(u)\big)\\
&\le \frac{2\epsilon^2}{(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\bigg(\int_0^u (u-s)^{\gamma-1}f(s,\mathbf{z}_\epsilon(s),\mathbf{z}_\epsilon(\lambda s))ds,\int_0^u (u-s)^{\gamma-1}\tilde{f}(\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s))ds\bigg)\\
&+\frac{2\epsilon}{(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\bigg(\int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(s,\mathbf{z}_\epsilon(s),\mathbf{z}_\epsilon(\lambda s))dw(s)\Big\rangle ds,\int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s))dw(s)\Big\rangle ds\bigg).
\end{aligned}$$

Denote by

$$J_1=\frac{2\epsilon^2}{(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\Big(\int_0^u (u-s)^{\gamma-1}f(s,\mathbf{z}_\epsilon(s),\mathbf{z}_\epsilon(\lambda s))ds,\int_0^u (u-s)^{\gamma-1}\tilde{f}(\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s))ds\Big),$$

$$J_2=\frac{2\epsilon}{(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\Big(\int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(s,\mathbf{z}_\epsilon(s),\mathbf{z}_\epsilon(\lambda s))dw(s)\rangle ds,\int_0^u (u-s)^{\gamma-1}\Big\langle\int_0^s g(\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s))dw(s)\Big\rangle ds\Big).$$

Then, by utilizing the attributes of the metric $\mathbf{D}_\infty$, we obtain

$$\begin{aligned}
J_1 &\le \frac{4\epsilon^2}{(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\bigg(\int_0^u (u-s)^{\gamma-1}f(s,\mathbf{z}_\epsilon(s),\mathbf{z}_\epsilon(\lambda s))ds,\int_0^u (u-s)^{\gamma-1}f(s,\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s))ds\bigg)\\
&+\frac{4\epsilon^2}{(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\bigg(\int_0^u (u-s)^{\gamma-1}f(s,\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s))ds,\int_0^u (u-s)^{\gamma-1}\tilde{f}(\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s))ds,\bigg),\\
&:= J_{11}+J_{12}.
\end{aligned}$$

By using Hôlder inequality and assumption ($\mathcal{A}1$), we get

$$\begin{aligned}
J_{11} &\le \frac{4\epsilon^2 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\bigg(\int_0^u \mathbb{E}\mathbf{D}_\infty^2\Big(f(s,\mathbf{z}_\epsilon(s),\mathbf{z}_\epsilon(\lambda s)),f(s,\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s))\Big)ds\bigg),\\
&\le \frac{4\epsilon^2 L_1 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^v \Big(\mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{z}_\epsilon(s),\mathbf{w}_\epsilon(s)\big)+\mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{z}_\epsilon(\lambda s),\mathbf{w}_\epsilon(\lambda s)\big)\Big)ds,\\
&\le \frac{8\epsilon^2 L_1 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^v \mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{z}_\epsilon(s),\mathbf{w}_\epsilon(s)\big)ds.
\end{aligned}$$

For J_{12}, we use Hôlder inequality and assumption $(\mathcal{A}4)$, we obtain

$$\begin{aligned} J_{12} &\leq \frac{4\epsilon^2 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\bigg(\int_0^u \mathbb{E}\mathbf{D}_\infty^2\Big(f(s,\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s)),\tilde{f}(\mathbf{w}_\epsilon(s),\mathbf{w}_\epsilon(\lambda s))\Big)ds\bigg),\\ &\leq \frac{4\epsilon^2 v^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2}\gamma_1(v)\Big[1+\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{w}_\epsilon(u),\widehat{0}\big)+\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{w}_\epsilon(\lambda t),\widehat{0}\big)\Big],\\ &\leq \frac{4\epsilon^2 v^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2}\gamma_1(v)\big[1+2\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{w}_\epsilon(u),\widehat{0}\big)\big],\\ &:= 4\epsilon^2 v^{2\gamma}\beta_1, \end{aligned}$$

where $\beta_1 = \frac{\gamma_1(v)}{(2\gamma-1)(\Gamma(\gamma))^2}\big[1+2\sup_{u\in[0,v]}\mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{w}_\epsilon(u),\widehat{0}\big)\big]$. Therefore

$$J_1 \leq \frac{8\epsilon^2 L_1 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^v \mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{z}_\epsilon(s),\mathbf{w}_\epsilon(s)\big)ds + 4\epsilon^2 v^{2\gamma}\beta_1. \tag{4.3}$$

For the second term J_2, by using Proposition 2.6 and Hôlder inequality, we have

$$\begin{aligned} J_2 &\leq \frac{2\epsilon v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\int_0^u\bigg(\int_0^s \mathbb{E}\big\|g(v',\mathbf{z}_\epsilon(v'),\mathbf{z}_\epsilon(\lambda v'))-\tilde{g}(\mathbf{w}_\epsilon(v'),\mathbf{w}_\epsilon(\lambda v'))\big\|^2 dv'\bigg)ds,\\ &\leq \frac{4\epsilon v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\int_0^u\bigg(\int_0^s \mathbb{E}\big\|g(v',\mathbf{z}_\epsilon(v'),\mathbf{z}_\epsilon(\lambda v'))-g(v',\mathbf{w}_\epsilon(v'),\mathbf{w}_\epsilon(\lambda v'))\big\|^2 dv'\bigg)ds\\ &+\frac{4\epsilon v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\int_0^u\bigg(\int_0^s \mathbb{E}\big\|g(v',\mathbf{w}_\epsilon(v'),\mathbf{w}_\epsilon(\lambda v'))-\tilde{g}(\mathbf{w}_\epsilon(v'),\mathbf{w}_\epsilon(\lambda v'))\big\|^2 dv'\bigg)ds,\\ &:= J_{21}+J_{22}. \end{aligned}$$

Using assumption $(\mathcal{A}2)$, we get

$$\begin{aligned} J_{21} &\leq \frac{4\epsilon L_2 v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^v\bigg(\int_0^s \mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{z}_\epsilon(v'),\mathbf{w}_\epsilon(v')\big)+\mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{z}_\epsilon(\lambda v'),\mathbf{w}_\epsilon(\lambda v')\big)dv'\bigg)ds,\\ &\leq \frac{4\epsilon L_2 v^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^v \mathbb{E}d_\infty^2\big(\mathbf{z}_\epsilon(s),\mathbf{w}_\epsilon(s)\big)ds. \end{aligned}$$

Also, we use assumption $(\mathcal{A}4)$, we have

$$\begin{aligned} J_{22} &\leq \frac{4\epsilon v^{2\gamma-1}}{(2\gamma-1)(\Gamma(\gamma))^2}\sup_{u\in[0,v]}\bigg(\int_0^u\bigg(s\frac{1}{s}\int_0^s \mathbb{E}\big\|g(v',\mathbf{w}_\epsilon(v'),\mathbf{w}_\epsilon(\lambda v'))-\tilde{g}(\mathbf{w}_\epsilon(v'),\mathbf{w}_\epsilon(\lambda v'))\big\|^2 dv'\bigg)ds,\\ &\leq \frac{4\epsilon v^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\gamma_2(v)\Big[1+\sup_{u\in[0,v]}\mathbb{E}d_\infty^2\big(\mathbf{w}_\epsilon(u),\widehat{0}\big)+\sup_{u\in[0,v]}\mathbb{E}d_\infty^2\big(\mathbf{w}_\epsilon(\lambda t),\widehat{0}\big)\Big],\\ &\leq \frac{4\epsilon v^{2\gamma+1}}{(2\gamma-1)(\Gamma(\gamma))^2}\gamma_2(v)\big[1+2\sup_{u\in[0,v]}\mathbb{E}d_\infty^2\big(\mathbf{w}_\epsilon(u),\widehat{0}\big)\big],\\ &:= 4\epsilon v^{2\gamma+1}\beta_2, \end{aligned}$$

where $\beta_2 = \frac{\gamma_2(v)}{(2\gamma-1)(\Gamma(\gamma))^2}\big[1+2\sup_{u\in[0,u]}\mathbb{E}d_\infty^2\big(\mathbf{w}_\epsilon(u),\widehat{0}\big)\big]$. Therefore

$$J_2 \leq \frac{4\epsilon L_2 v^{2\gamma}}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^v \mathbb{E}d_\infty^2\big(\mathbf{z}_\epsilon(s),\mathbf{w}_\epsilon(s)\big)ds + 4\epsilon v^{2\gamma+1}\beta_2. \tag{4.4}$$

Hence, combining (4.3) and (4.4) together, we get

$$\begin{aligned} \sup_{u\in[0,v]}\mathbb{E}d_\infty^2\big(\mathbf{z}_\epsilon(u),\mathbf{w}_\epsilon(u)\big) &\leq 4\epsilon v^{2\gamma}\big(\epsilon\beta_1+v\beta_2\big)+\frac{4\epsilon v^{2\gamma}\big(\epsilon L_1 v^{-1}+L_2\big)}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^v \mathbb{E}d_\infty^2\big(\mathbf{z}_\epsilon(s),\mathbf{w}_\epsilon(s)\big)ds,\\ &\leq 4\epsilon v^{2\gamma}\big(\epsilon\beta_1+v\beta_2\big)+\frac{4\epsilon v^{2\gamma}\big(\epsilon L_1 v^{-1}+L_2\big)}{(2\gamma-1)(\Gamma(\gamma))^2}\int_0^v \sup_{v'\in[0,s]}\mathbb{E}d_\infty^2\big(\mathbf{z}_\epsilon(v'),\mathbf{w}_\epsilon(v')\big)dv'. \end{aligned}$$

Thus, using Gronwall inequality, we obtain

$$\sup_{u\in[0,v]} \mathbb{E}d_\infty^2\big(\mathbf{z}_\epsilon(u),\mathbf{w}_\epsilon(u)\big) \leq 4\epsilon v^{2\gamma}\big(\epsilon\beta_1+v\beta_2\big)\big) \exp\left(\frac{4\epsilon v^{2\gamma}\big(\epsilon L_1 v^{-1}+L_2\big)}{(2\gamma-1)(\Gamma(\gamma))^2}\right).$$

Choose $0<\gamma<1$ and $L>0$ such that for every $u\in[0,L\epsilon^{-\gamma}]\subseteq\mathcal{I}$, we get

$$\sup_{u\in[0,L\epsilon^{-\gamma}]} \mathbb{E}d_\infty^2\big(\mathbf{z}_\epsilon(u),\mathbf{w}_\epsilon(u)\big) \leq k\epsilon^{1-\gamma},$$

where

$$k = 4L^{2\gamma}\epsilon^{1-2\gamma\gamma}\big(\epsilon\beta_1+L\epsilon^{-\gamma}\beta_2\big) \exp\left(\frac{4L^{2\gamma}\epsilon^{1-2\gamma\gamma}(L_1L^{-1}\epsilon^{1+\gamma}+L_2}{(2\gamma-1)(\Gamma(\gamma))^2}\right)$$

is a constant. Therefore, for any given number δ, $\exists\epsilon_1\in(0,\epsilon_0]$ such that for each $\epsilon\in(0,\epsilon_1]$ and $u\in[0,L\epsilon^{-\gamma}]$, we get

$$\sup_{u\in[0,L\epsilon^{-\gamma}]} \mathbb{E}\mathbf{D}_\infty^2\big(\mathbf{z}_\epsilon(u),\mathbf{w}_\epsilon(u)\big) \leq \delta.$$

□

5 Example

We give an example to illustrate our findings in this section. Consider the following FFPSDEs

$$\begin{cases} {}^C\mathbf{D}^\gamma\mathbf{z}(u) = \mathbf{z}(u)+\mathbf{z}(u)(\frac{u}{2}-1)^2+\big\langle\mathbf{z}(u)dw(u)\big\rangle, & 0\leq u\leq 1, \quad \frac{1}{2}<\gamma<1. \\ \mathbf{z}(0)=0, \end{cases} \tag{5.1}$$

Thus, the appropriate standard form of the FFPSDEs mentioned above is

$${}^C\mathbf{D}^\gamma\mathbf{z}^\epsilon = \mathbf{z}^\epsilon+\mathbf{z}^\epsilon(\frac{u}{2}-1)^2+\big\langle\mathbf{z}^\epsilon dw(u)\big\rangle.$$

Then, $f(u,\mathbf{z}^\epsilon(u),\mathbf{z}^\epsilon(\lambda u)) = \mathbf{z}^\epsilon+\mathbf{z}^\epsilon(\frac{u}{2}-1)^2$ and $g(u,\mathbf{z}^\epsilon(u),\mathbf{z}^\epsilon(\lambda u)) = \mathbf{z}^\epsilon$. Hence

$$\begin{aligned} \tilde{f}(\mathbf{z}^\epsilon(u),\mathbf{z}^\epsilon(\lambda u)) &= \int_0^1 f(s,\mathbf{z}^\epsilon(s),\mathbf{z}^\epsilon(\lambda s))ds, \\ &= \frac{19\mathbf{z}^\epsilon}{12}, \end{aligned}$$

and

$$\tilde{g}(\mathbf{z}^\epsilon(u),\mathbf{z}^\epsilon(\lambda u)) = \int_0^1 g(s,\mathbf{z}^\epsilon(s),\mathbf{z}^\epsilon(\lambda s))ds = \mathbf{z}^\epsilon.$$

As a result, the average form of (5.1) may be expressed as

$${}^C\mathbf{D}^\gamma\mathbf{w}^\epsilon = \frac{19\mathbf{z}^\epsilon}{12}du+\sqrt{\epsilon}\big\langle\mathbf{z}^\epsilon dw(u)\big\rangle. \tag{5.2}$$

We can see that the coefficients f and g satisfy the assumptions $(\mathcal{A}1)$-$(\mathcal{A}3)$. Then, according to Theorem 3.2 the FFPSDEs (5.1) has a unique fuzzy solution. On the other hand, we can naturally see that the coefficient $\tilde{f}$ and $\tilde{g}$ satisfy the assumption $(\mathcal{A}4)$, then, according to Theorem 4.1, as $\epsilon\longrightarrow 0$, the solution $\mathbf{z}^\epsilon$ and $\mathbf{w}^\epsilon$ to Eqs. (5.1) and (5.2) are equivalent in the sense of mean square. Clearly, the reduced system (5.2) is much easier to understand than the standard system (5.1). Even better, Theorem 4.1 ensures that just a minor mistake is introduced throughout the substitution procedure.

6 Conclusion

In this work, we have proved the existence and uniqueness results for FFPSDEs via Banach fixed point analysis. Also, the averaging principle for this type of equation is studied. Precisely, we proved that the solution of the simplified system converges to the solution of the original system in the mean square sense.

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