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Corrigendum to "η-admissible mappings in C^*-algebra-valued $\mathcal{MP}$-metric spaces with an application"

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Abstract

This article is a revision and correction of the chapter book [S. Hadi Bonab, V. Parvaneh, Z. Bagheri, $\eta_{\mathcal{A}}$-Admissible Mappings for Four Maps in $C*$-Algebra-Valued MP-Metric Spaces with an Application, In: P. Debnath, Delfim F. M. Torres, Yeol Je Cho, Advanced Mathematical Analysis and its Applications, CRC Press, 2023, 97-113.]. In this article, we first introduce the concept of η-admissible mapping in C^*-algebra valued $\mathcal{MP}$-metric spaces, which is a generalization and combination of "modular metric spaces", "parametric metric spaces" and "C^*-algebra-valued metric spaces". Then, for four mappings in these spaces, we prove several fixed-point theorems. We give an example and an application regarding the solvability of operator equations and integral equations, respectively, to support the new findings.

Keywords: Metric space, parametric metric space, modular metric space, η-admissible mappings, C^*-algebra-valued metric space, C^*-algebra-valued $\mathcal{MP}$-metric space
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1 Introduction

In this article, due to errors in the chapter book published in the CRC Press [12], we have reviewed and corrected the required items. To correct and edit the previous version, some definitions and a Remark (Definitions 2.1-2.6 and 2.10 and Remark 2.4) that are needed to prove the main results have been added. Also, some required assumptions in the presented theorems in the main results section have been revised in order to improve and modify the proof of the claims.

In recent years, many researchers have generalized the Banach fixed point theorem [2] in many directions and frameworks, for example: in cone metric spaces [9], G-metric spaces [10], vector-valued metric spaces [8, 11], b-metric spaces [17], b-rectangular metric spaces [29], generalized parametric metric spaces [31], modular b-metric spaces [7, 28] etc. Also, many mathematicians have presented different extensions of contraction mappings on complete

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metric spaces and developed them in different ways and turned it into a general rule. For more information, see [3, 5, 6, 13, 15, 22, 27, 30, 32].

Nakano [25] introduced modular spaces in connection with the theory of ordered spaces, which was later generalized in [24]. In [16], Hossein et al. introduced the concept of parametric metric spaces. Several authors investigated fixed point theorems for multivalued contractions. Kotbi and Latif [18] studied fixed points of multivalued maps in modular function spaces. Also, see [4, 14] for more information.

In 2014, using the set of all positive elements of a unit C^*-algebra instead of the set of real numbers, the concept of a C^*-algebra valued metric spaces [20] was introduced. Later, many authors studied in this field and presented many results (See [19, 23, 26, 33]).

In 2012, the concept of η-admissible mapping was presented by Samet et al [30]. In this chapter, we introduce this concept in a C^*-algebra valued modular parametric metric space (CAVMPMS) for four mappings, which is a generalization and combination of modular metric space, parametric metric space and C^*-algebra-valued metric space. In the following, these concepts are used to prove some fixed point theorems through C^*-contractions and also Kannan-Ćirić C^*-contractions.

Throughout this paper, $\mathcal{A}$ denotes a unital algebra with unit I, and θ is the zero element. An involution on $\mathcal{A}$ corresponds to the conjugate linear map $\kappa \mapsto \kappa^*$ on $\mathcal{A}$ if $\jmath^{**} = \jmath$. If $(\jmath + \wp)^* = \jmath^* + \wp^*$ and $(\jmath\wp)^* = \wp^*\jmath^*$ for all $\jmath, \wp \in \mathcal{A}$, then the pair $(\mathcal{A}, *)$ is called an $*$-algebra. A Banach $*$-algebra is an $*$-algebra $\mathcal{A}$ with the complete submultiplicative norm so that $\| \jmath^* \| = \| \jmath \|$ for all $\jmath \in \mathcal{A}$. A C^*-algebra is a Banach $*$-algebra so that $\| \jmath^*\jmath \| = \| \jmath \|^2$ for all $\jmath \in \mathcal{A}$. Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{B}(\mathfrak{H})$ be the set of bounded linear operators on $\mathfrak{H}$, then $\mathfrak{B}(\mathfrak{H})$ is a C^*-algebra with the operator norm. Let $\mathcal{A}_{sa}$ be the family of all self-adjoint elements in $\mathcal{A}$. An element $\jmath \in \mathcal{A}$ is positive ($\jmath \succeq \theta$) if $\jmath \in \mathcal{A}_{sa}$ and spectrum $\sigma(\jmath) = \{\lambda \in C \mid \lambda I - \jmath \, \textit{is not invertible}\} \subseteq \mathbb{R}_+$. Set $\mathcal{A}^+ = \{\jmath \in \mathcal{A} : \jmath \succeq \theta\}$, then $\mathcal{A}^+ = \{\jmath^*\jmath : \jmath \in \mathcal{A}\}$ [23] and $(\jmath^*\jmath)^{\frac{1}{2}} = |\jmath|$. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. Note that a partial ordering $\preceq$ on $\mathcal{A}_{sa}$ is as follows: $\jmath \preceq \wp \Leftrightarrow \wp - \jmath \succeq \theta$. If $\jmath, \wp \in \mathcal{A}_{sa}$ and $q \in \mathcal{A}$, then $\jmath \preceq \wp \Rightarrow q^*\jmath q \preceq q^*\wp q$, and if $\jmath, \wp \in \mathcal{A}_+$ are invertible, then $\jmath \preceq \wp \Longrightarrow \theta \preceq \wp^{-1} \preceq \jmath^{-1}$.

Definition 1.1. [2] Consider a nonempty set Π. A mapping $\mho : \Pi^2 \longrightarrow [0, \infty)$ is called a metric on Π, if:

1. $\mho(\zeta, \ell) = 0$ iff $\zeta = \ell$;
2. $\mho(\zeta, \ell) = \mho(\ell, \zeta)$ for each $\zeta, \ell \in \Pi$;
3. $\mho(\zeta, \ell) \leq \mho(\zeta, \eta) + \mho(\eta, \ell)$ for each $\zeta, \ell, \eta \in \Pi$.

Then $(\Pi, \mho)$ is called a metric space.

Definition 1.2. [20] Let the function $\mho : \Pi^2 \to \mathcal{A}$ (Π is a nonempty set) verifies the following for all $\zeta, \ell, \eta \in \Pi$:

(i) $\theta \preceq \mho(\zeta, \ell)$ and $\mho(\zeta, \ell) = \theta$ iff $\zeta = \ell$;

(ii) $\mho(\zeta, \ell) = \mho(\ell, \zeta)$;

(iii) $\mho(\zeta, \ell) \preceq \mho(\zeta, \eta) + \mho(\eta, \ell)$.

Then $(\Pi, \mathcal{A}, \mho)$ is called a C^*-algebra-valued metric space.

Definition 1.3. [14] Consider the self-mappings $\mathcal{L} : \Pi \to \Pi$ and $\mathcal{G} : \Pi \to \Pi$. If $\Im = \mathcal{L}\zeta = \mathcal{G}\zeta$ for some $\zeta \in \Pi$, then ζ is called a *coincidence point* of $\mathcal{L}$ and $\mathcal{G}$. $\Im$ is said to be a point of coincidence of $\mathcal{L}$ and $\mathcal{G}$.

Definition 1.4. [14] Consider the self-mappings $\mathcal{L} : \Pi \to \Pi$ and $\mathcal{G} : \Pi \to \Pi$. If $\mathcal{L}$ and $\mathcal{G}$ commute at their coincidence points, then they are called *w-compatible*.

Definition 1.5. [21] The function $\mathcal{W} : (0, +\infty) \times \Pi^2 \to \mathcal{A}$ is said to be a C^*-algebra-valued modular metric on nonempty set Π, if

1. $\mathcal{W}_\lambda(\zeta, \ell) = \theta$ iff $\zeta = \ell$ for all $\lambda > 0$;
2. $\mathcal{W}_\lambda(\zeta, \ell) = \mathcal{W}_\lambda(\ell, \zeta)$ for all $\lambda > 0$ and for all $\zeta, \ell \in \Pi$;
3. $\mathcal{W}_{\lambda+\mu}(\zeta, \ell) \preceq \mathcal{W}_\lambda(\zeta, \eta) + \mathcal{W}_\mu(\eta, \ell)$ for all $\zeta, \ell, \eta \in \Pi$ and all $\lambda, \mu > 0$.

Then the pair $(\Pi, \mathcal{W})$ is called a C^*-algebra-valued modular metric space.

Definition 1.6. [31] Let Π be a nonempty set. A function $\mathcal{P} : \Pi^2 \times (0, +\infty) \to [0, +\infty)$ is said to be a parametric metric on Π, if

1. $\mathcal{P}(\zeta, \ell, \iota) = 0$ iff $\zeta = \ell$;
2. $\mathcal{P}(\zeta, \ell, \iota) = \mathcal{P}(\ell, \zeta, \iota)$ for all $\iota > 0$;
3. $\mathcal{P}(\zeta, \ell, \iota) \leq \mathcal{P}(\zeta, \eta, \iota) + \mathcal{P}(\eta, \ell, \iota)$ for all $\zeta, \ell, \eta \in \Pi$ and all $\iota > 0$.

Then the pair $(\Pi, \mathcal{P})$ is called a parametric metric space.

Definition 1.7. [30] Let $\eta : \Pi^2 \to [0, \infty)$ (Π is a nonempty set). A mapping $T : \Pi \to \Pi$ is said to be an η-admissible mapping, if

$$\eta(\zeta, \ell) \geq 1 \Rightarrow \eta(T\zeta, T\ell) \geq 1, \textit{ for all } \zeta, \ell \in \Pi.$$

Definition 1.8. [1] The max function on C^*-algebra $\mathcal{A}$ with the partial order relation $\preceq$ is defined by:

$$\max\{\zeta, \ell\} = \ell \Leftrightarrow \zeta \preceq \ell \textit{ and } \| \zeta \| \leq \| \ell \|, \quad \textit{for all } \zeta, \ell \in \mathcal{A}^+.$$

2 Main results

2.1 η-admissible mappings

In this section, we present the concept of η-admissible mapping in a CAVMPMS to achieve a common fixed point for 4 maps. First, we introduce the following definitions:

Definition 2.1. Let $\mathcal{I}$ be a self-mapping on Π and let $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function. We say that $\mathcal{I}$ is an η-admissible mapping if

$$\eta_\lambda(\zeta, \ell, \iota) \succeq I_{\mathcal{A}} \Rightarrow \eta_\lambda(\mathcal{I}\zeta, \mathcal{I}\ell, \iota) \succeq I_{\mathcal{A}},$$

where $\zeta, \ell \in \Pi$ and $\lambda, \iota > 0$.

Definition 2.2. Let $\mathcal{L}$ and $\mathcal{G}$ be two self-mappings on a set Π and let $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function. A pair $(\mathcal{L}, \mathcal{G})$ is said to be,

(i) weakly η-admissible if $\eta_\lambda(\mathcal{L}\zeta, \mathcal{G}\mathcal{L}\zeta, \iota) \succeq I_{\mathcal{A}}$ and $\eta_\lambda(\mathcal{G}\zeta, \mathcal{L}\mathcal{G}\zeta, \iota) \succeq I_{\mathcal{A}}$ for all $\zeta \in \Pi$ and for all $\lambda, \iota > 0$,

(ii) partially weakly η-admissible if $\eta_\lambda(\mathcal{L}\zeta, \mathcal{G}\mathcal{L}\zeta, \iota) \succeq I_{\mathcal{A}}$ for all $\zeta \in \Pi$ and for all $\lambda, \iota > 0$.

Let Π be a nonempty set and $\mathcal{L} : \Pi \to \Pi$ be a given mapping. For every $\zeta \in \Pi$, let $\mathcal{L}^{-1}(\zeta) = \{\mathcal{U} \in X : \mathcal{L}\mathcal{U} = \mathcal{L}\zeta\}$.

Definition 2.3. Let Π be a set, $\mathcal{L}, \mathcal{G}, \mathcal{Q} : \Pi \to \Pi$ are mappings so that $\mathcal{L}\Pi \cup \mathcal{G}\Pi \subseteq \mathcal{Q}\Pi$ and let $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function. The ordered pair $(\mathcal{L}, \mathcal{G})$ is said to be:

(a) weakly η-admissible with respect to $\mathcal{Q}$ iff, $\eta_\lambda(\mathcal{L}\zeta, \mathcal{G}\ell, \iota) \succeq I_{\mathcal{A}}$ for all $\ell \in \mathcal{Q}^{-1}(\mathcal{L}\zeta)$ and $\eta_\lambda(\mathcal{G}\zeta, \mathcal{L}\ell, \iota) \succeq I_{\mathcal{A}}$ for all $\ell \in \mathcal{Q}^{-1}(\mathcal{G}\zeta)$,

(b) partially weakly η-admissible with respect to $\mathcal{Q}$ if $\eta_\lambda(\mathcal{L}\zeta, \mathcal{G}\ell, \iota) \succeq I_{\mathcal{A}}$ for all $\ell \in \mathcal{Q}^{-1}(\mathcal{L}\zeta)$, for all $\zeta \in \Pi$ and for all $\lambda, \iota > 0$.

Remark 2.4. In the above definition:

(i) if $\mathcal{G} = \mathcal{L}$, we say that $\mathcal{L}$ is weakly η-admissible (partially weakly η-admissible) with respect to $\mathcal{Q}$,

(ii) if $\mathcal{Q} = I_\Pi$ (the identity mapping on Π), then the above definition reduces to the concept of weakly η-admissible (partially weakly η-admissible) mapping.

From now on, we assume that $\lambda, \iota > 0$.

Definition 2.5. Let $\mathcal{L}$ and $\mathcal{G}$ be two self-maps on a set Π and let $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function. The weakly η-admissible (partially weakly η-admissible) pair $(\mathcal{L}, \mathcal{G})$ is said to be triangular weakly η-admissible (triangular partially weakly η-admissible) if $\eta_\lambda(\zeta, \ell, \iota) \succeq I_{\mathcal{A}}$ and $\eta_\lambda(\ell, \rho, \iota) \succeq I_{\mathcal{A}}$ imply $\eta_\lambda(\zeta, \rho, \iota) \succeq I_{\mathcal{A}}$ for all $\zeta, \ell, \rho \in \Pi$.

Definition 2.6. Let Π be a set, $\mathcal{L}, \mathcal{G}, \mathcal{Q} : \Pi \to \Pi$ are mappings such that $\mathcal{L}\Pi \cup \mathcal{G}\Pi \subseteq \mathcal{Q}\Pi$ and let $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function. The ordered pair $(\mathcal{L}, \mathcal{G})$ is said to be triangular weakly η-admissible (triangular partially weakly η-admissible) with respect to $\mathcal{Q}$ if it is weakly η-admissible (partially weakly η-admissible) with respect to $\mathcal{Q}$ and $\eta_\lambda(\zeta, \ell, \iota) \succeq I_{\mathcal{A}}$ and $\eta_\lambda(\ell, \rho, \iota) \succeq I_{\mathcal{A}}$ imply $\eta_\lambda(\zeta, \rho, \iota) \succeq I_{\mathcal{A}}$ for all $\zeta, \ell, \rho \in \Pi$.

Definition 2.7. The function $\mathcal{MP} : (0, \infty) \times \Pi^2 \times (0, \infty) \to [0, \infty]$ is called a modular parametric metric ($\mathcal{MP}$-metric) on nonempty set Π, if

(1) $\mathcal{MP}_\lambda(\zeta, \ell, \iota) = 0$ iff $\zeta = \ell$ for all $\zeta, \ell \in \Pi$ and for all $\lambda, \iota > 0$;

(2) $\mathcal{MP}_\lambda(\zeta, \ell, \iota) = \mathcal{MP}_\lambda(\ell, \zeta, \iota)$;

(3) $\mathcal{MP}_{\lambda+\mu}(\zeta, \ell, \iota) \leq \mathcal{MP}_\lambda(\zeta, \eta, \iota) + \mathcal{MP}_\mu(\eta, \ell, \iota)$ for all $\zeta, \ell, \eta \in \Pi$ and for all $\lambda, \iota > 0$.

Then the pair $(\Pi, \mathcal{MP})$ is called a $\mathcal{MP}$-metric space.

Definition 2.8. The function $\mathcal{P} : \Pi^2 \times (0, +\infty) \to \mathcal{A}^+$ is said to be a C^*-algebra-valued parametric metric on nonempty set Π if,

1. $\mathcal{P}(\zeta, \ell, \iota) = \theta$ iff $\zeta = \ell$ for all $\zeta, \ell \in \Pi$ and for all $\iota > 0$;
2. $\mathcal{P}(\zeta, \ell, \iota) = \mathcal{P}(\ell, \zeta, \iota)$;
3. $\mathcal{P}(\zeta, \ell, \iota) \preceq \mathcal{P}(\zeta, \eta, \iota) + \mathcal{P}(\eta, \ell, \iota)$ for all $\zeta, \ell, \eta \in \Pi$ and for all $\iota > 0$.

Then the pair $(\Pi, \mathcal{P})$ is called a C^*-algebra-valued parametric metric space.

Definition 2.9. The function $C : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ is called a C^*-algebra valued $\mathcal{MP}$-metric on nonempty set Π if:

(1) $\mathcal{C}_\lambda(\zeta, \ell, \iota) = 0$ iff $\zeta = \ell$ for all $\zeta, \ell \in \Pi$ and for all $\lambda, \iota > 0$;

(2) $\mathcal{C}_\lambda(\zeta, \ell, \iota) = \mathcal{C}_\lambda(\ell, \zeta, \iota)$;

(3) $\mathcal{C}_{\lambda+\mu}(\zeta, \ell, \iota) \preceq \mathcal{C}_\lambda(\zeta, \eta, \iota) + \mathcal{C}_\mu(\eta, \ell, \iota)$ for all $\zeta, \ell, \eta \in \Pi$ and for all $\lambda, \iota > 0$.

Then $(\Pi, \mathcal{A}, C)$ is called a CAVMPMS.

Definition 2.10. Let $(\Pi, \mathcal{A}, C)$ be a CAVMPMS and let $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function. We say that $(\Pi, \mathcal{A}, C)$ is η-regular if $\zeta_{\overline{\hbar}} \to \zeta$, where $\eta_\lambda(\zeta_{\overline{\hbar}}, \zeta_{\overline{\hbar}+1}, \iota) \succeq I_{\mathcal{A}}$ for all $\overline{\hbar} \in \mathbb{N}$, then $\eta_\lambda(\zeta_{\overline{\hbar}}, \zeta, \iota) \succeq I_{\mathcal{A}}$ for all $\overline{\hbar} \in \mathbb{N}$.

Theorem 2.11. Let $(\Pi, \mathcal{A}, C)$ be a complete CAVMPMS and $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{I}$ be self-mappings on Π, so that the following conditions are satisfies:

(i) $\mathcal{L}(\Pi) \subseteq \mathcal{I}(\Pi)$, $\mathcal{G}(\Pi) \subseteq \mathcal{Q}(\Pi)$ and $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function.

(ii) Suppose that for all $\zeta, \ell \in \Pi$ with $\eta_\lambda(\mathcal{Q}\zeta, \mathcal{I}\ell, \iota) \succeq I_{\mathcal{A}}$,

$$\mathcal{C}_\lambda(\mathcal{L}\zeta, \mathcal{G}\ell, \iota) \preceq \partial^*[\mathcal{C}_\lambda(\mathcal{Q}\zeta, \mathcal{I}\ell, \iota)]\partial \quad \textit{for all } \lambda, \iota > 0, \tag{2.1}$$

where $\| \partial \| < 1$.

Assume that $(\Pi, \mathcal{A}, C)$ is η-regular and the pairs $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are triangular partially weakly η-admissible with respect to $\mathcal{I}$ and $\mathcal{Q}$, respectively.
Then

(A) If one of $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi)$ and $\mathcal{Q}(\Pi) \cup \mathcal{I}(\Pi)$ be complete, then $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ have a coincidence point in Π. If $\eta_\lambda(\mathcal{QU}, \mathcal{IU}, \iota) \succeq I_{\mathcal{A}}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a coincidence point.

(B) if $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ be w-compatible, and if $\eta_\lambda(\mathcal{U}, \mathcal{IU}, \iota) \succeq I_{\mathcal{A}}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a common fixed point in Π. If $\eta_\lambda(\mathcal{QU}, \mathcal{IU}^*, \iota) \succeq I_{\mathcal{A}}$ for all common fixed points $\mathcal{U}$ and $\mathcal{U}^*$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a unique common fixed point in Π.

Proof . Let ζ_0 be an arbitrary point of Π. Choose $\zeta_1 \in \Pi$ such that $\mathcal{L}\zeta_0 = \mathcal{I}\zeta_1$ and $\zeta_2 \in \Pi$ such that $\mathcal{G}\zeta_1 = \mathcal{Q}\zeta_2$.

Continuing this way, construct a sequence $\{\ell_\hbar\}$ by:

$$\begin{cases} \mathcal{L}\zeta_{2\hbar} = \mathcal{I}\zeta_{2\hbar+1} = \ell_{2\hbar+1} \\ \mathcal{G}\zeta_{2\hbar+1} = \mathcal{Q}\zeta_{2\hbar+2} = \ell_{2\hbar+2}, \qquad \forall \hbar \geq 0. \end{cases}$$

As $\zeta_1 \in \mathcal{I}^{-1}(\mathcal{L}\zeta_0)$ and $\zeta_2 \in \mathcal{Q}^{-1}(\mathcal{G}\zeta_1)$ and the pairs $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are partially weakly η-admissible with respect to $\mathcal{I}$ and $\mathcal{Q}$, respectively, we have,

$$\eta_\lambda(\mathcal{I}\zeta_1 = \mathcal{L}\zeta_0, \mathcal{G}\zeta_1 = \mathcal{Q}\zeta_2, \iota) \succeq I_{\mathcal{A}}$$

and

$$\eta_\lambda(\mathcal{G}\zeta_1 = \mathcal{Q}\zeta_2, \mathcal{L}\zeta_2 = \mathcal{I}\zeta_3, \iota) \succeq I_{\mathcal{A}}.$$

Repeating this process, we obtain $\eta_\lambda(\mathcal{I}\zeta_{2\hbar+1}, \mathcal{Q}\zeta_{2\hbar+2}, \iota) = \eta_\lambda(\ell_{2\hbar+1}, \ell_{2\hbar+2}, \iota) \succeq I_{\mathcal{A}}$ for all $\hbar \geq 0$. So, we can apply contraction (2.1) which implies that

$$\begin{aligned}
\mathcal{C}_\lambda(\ell_{2\hbar+1}, \ell_{2\hbar+2}, \iota) &= \mathcal{C}_\lambda(\mathcal{L}\zeta_{2\hbar}, \mathcal{G}\zeta_{2\hbar+1}, \iota) \\
&\preceq \partial^*[\mathcal{C}_\lambda(\mathcal{Q}\zeta_{2\hbar}, \mathcal{I}\zeta_{2\hbar+1}, \iota)]\partial \\
&= \partial^*[\mathcal{C}_\lambda(\ell_{2\hbar}, \ell_{2\hbar+1}, \iota)]\partial.
\end{aligned}$$

So, by induction, we get

$$\mathcal{C}_\lambda(\ell_{2\hbar+1}, \ell_{2\hbar+2}, \iota) \preceq (\partial^*)^{2\hbar+1}[\mathcal{C}_\lambda(\ell_0, \ell_1, \iota)]\partial^{2\hbar+1}.$$

Similarly, it can be shown that

$$\mathcal{C}_\lambda(\ell_{2\hbar}, \ell_{2\hbar+1}, \iota) \preceq (\partial^*)^{2\hbar}[\mathcal{C}_\lambda(\ell_0, \ell_1, \iota)]\partial^{2\hbar}.$$

Now, for every $\hbar \in \mathbb{N}$ we can get

$$\begin{aligned}
\mathcal{C}_\lambda(\ell_\hbar, \ell_{\hbar+1}, \iota) &\preceq (\partial^*)^\hbar[\mathcal{C}_\lambda(\ell_0, \ell_1, \iota)]\partial^\hbar \\
&\preceq (\partial^*)^\hbar \varpi_0 \partial^\hbar,
\end{aligned}$$

where $\varpi_0 := \mathcal{C}_\lambda(\ell_0, \ell_1, \iota)$. Therefore,

$$\|\mathcal{C}_\lambda(\ell_\hbar, \ell_{\hbar+1}, \iota)\| \leq (\|\partial\|^2)^\hbar \|\varpi_0\|. \tag{2.2}$$

Since $\| \partial \| < 1$, taking $\hbar \to \infty$, we have $\lim_{\hbar\to\infty} \|\mathcal{C}_\lambda(\ell_\hbar, \ell_{\hbar+1}, \iota)\| = \theta_{\mathcal{A}}$. We show that $\{\ell_\hbar\}$ is a Cauchy sequence in Π. Assume on contrary that, there exists $\varepsilon > 0$ for which we can find subsequences $\{\ell_{2m(k)}\}$ and $\{\ell_{2\overline{\hbar}(k)}\}$ of $\{\ell_{2\overline{\hbar}}\}$ such that $\overline{\hbar}(k) > m(k) \geq k$ and

$$\mathcal{C}_\lambda(\ell_{2m(k)}, \ell_{2\overline{\hbar}(k)}, \iota) \succeq \varepsilon \tag{2.3}$$

for some $\lambda, \iota > 0$ and $\overline{\hbar}(k)$ is the smallest number such that the above condition holds, *i.e.*,

$$\mathcal{C}_\lambda(\ell_{2m(k)}, \ell_{2\overline{\hbar}(k)-1}, \iota) \prec \varepsilon. \tag{2.4}$$

From triangle inequality and (2.3) and (2.4), we have

$$\varepsilon \preceq \mathcal{C}_\lambda(\ell_{2m(k)}, \ell_{2\overline{\hbar}(k)}, \iota) \preceq \mathcal{C}_{\frac{\lambda}{2}}(\ell_{2m(k)}, \ell_{2\overline{\hbar}(k)-1}, \iota) + \mathcal{C}_{\frac{\lambda}{2}}(\ell_{2\overline{\hbar}(k)-1}, \ell_{2\overline{\hbar}(k)}, \iota). \tag{2.5}$$

Taking the limit as $k \to \infty$ in (2.5), from (2.2) we obtain that,

$$\limsup_{k\to\infty} \mathcal{C}_\lambda(\ell_{2m(k)}, \ell_{2\overline{\hbar}(k)}, \iota) = \varepsilon. \tag{2.6}$$

Using triangle inequality again, we have

$$\mathcal{C}_\lambda(\ell_{2m(k)}, \ell_{2\overline{\hbar}(k)}, \iota) \preceq \mathcal{C}_{\frac{\lambda}{2}}(\ell_{2m(k)}, \ell_{2m(k)+1}, \iota) + \mathcal{C}_{\frac{\lambda}{2}}(\ell_{2m(k)+1}, \ell_{2\overline{\hbar}(k)}, \iota).$$

Making $k \to \infty$ in the above inequality, we have

$$\varepsilon \preceq \limsup_{k\to\infty} \mathcal{C}_\lambda(\ell_{2m(k)+1}, \ell_{2\overline{\hbar}(k)}, \iota). \tag{2.7}$$

We know that $2\overline{\hbar}(k) - 1 \geq 2m(k)$ and $\eta_\lambda(\mathcal{Q}\zeta_{2\overline{\hbar}+2}, \mathcal{I}\zeta_{\overline{\hbar}+1}, \iota) = \eta_\lambda(\mathcal{G}\zeta_{2\overline{\hbar}+1}, \mathcal{L}\zeta_{2\overline{\hbar}}, \iota) \succeq I_{\mathcal{A}}$ for all $\overline{\hbar} \in \mathbb{N}$. On the other hand, the pairs $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are triangular partially weakly η-admissible with respect to $\mathcal{I}$ and $\mathcal{Q}$, respectively. So, $\eta_\lambda(\mathcal{I}\zeta_{2\overline{\hbar}(k)-1}, \mathcal{Q}\zeta_{2\overline{\hbar}(k)-2}, \iota) \succeq I_{\mathcal{A}}$ and $\eta_\lambda(\mathcal{Q}\zeta_{2\overline{\hbar}(k)-2}, \mathcal{I}\zeta_{2\overline{\hbar}(k)-3}, \iota) \succeq I_{\mathcal{A}}$ implies $\eta_\lambda(\mathcal{I}\zeta_{2\overline{\hbar}(k)-1}, \mathcal{I}\zeta_{2\overline{\hbar}(k)-3}, \iota) \succeq I_{\mathcal{A}}$.

Also, $\eta_\lambda(\mathcal{I}\zeta_{2\bar{\hbar}(k)-1}, \mathcal{I}\zeta_{2\bar{\hbar}(k)-3}, \iota) \succeq I_\mathcal{A}$ and $\eta_\lambda(\mathcal{I}\zeta_{2\bar{\hbar}(k)-3}, \mathcal{Q}\zeta_{2\bar{\hbar}(k)-4}, \iota) \succeq I_\mathcal{A}$ implies that $\eta_\lambda(\mathcal{I}\zeta_{2\bar{\hbar}(k)-1}, \mathcal{Q}\zeta_{2\bar{\hbar}(k)-4}, \iota) \succeq I_\mathcal{A}$. By continuing this process, we get $\eta_\lambda(\mathcal{I}\zeta_{2\bar{\hbar}(k)-1}, \mathcal{Q}\zeta_{2m(k)}, \iota) \succeq I_\mathcal{A}$. Now by applying (2.1) we have

$$\begin{aligned} \mathcal{C}_\lambda(\ell_{2m(k)+1}, \ell_{2\bar{\hbar}(k)}, \iota) &\preceq \mathcal{C}_{\frac{\lambda}{2}}(\ell_{2m(k)+1}, \ell_{2\bar{\hbar}(k)}, \iota) \\ &= \mathcal{C}_{\frac{\lambda}{2}}(\mathcal{L}\zeta_{2m(k)}, \mathcal{G}\zeta_{2\bar{\hbar}(k)-1}, \iota) \\ &\preceq \partial^*[\mathcal{C}_{\frac{\lambda}{2}}(\mathcal{Q}\zeta_{2m(k)}, \mathcal{I}\zeta_{2\bar{\hbar}(k)-1}, \iota)]\partial \\ &= \partial^*[\mathcal{C}_{\frac{\lambda}{2}}(\ell_{2m(k)}, \ell_{2\bar{\hbar}(k)-1}, \iota)]\partial. \end{aligned} \tag{2.8}$$

Taking the limit as $k \to \infty$ in (2.8), we have

$$\varepsilon \preceq \partial^*[\varepsilon]\partial \tag{2.9}$$

which after getting the norm leads to a contradiction. Hence, $\{\ell_\hbar\}$ is a Cauchy sequence in Π. Assume that $\mathcal{Q}(\Pi)\cup\mathcal{I}(\Pi)$ be complete. In this case, there is $\mathcal{U} \in \mathcal{Q}(\Pi) \cup \mathcal{I}(\Pi)$ so that $\ell_\hbar \to \mathcal{U}$ as $\hbar \to \infty$. Further, the subsequences $\{\mathcal{Q}\zeta_{2\hbar+2}\} = \{\mathcal{G}\zeta_{2\hbar+1}\} = \{\ell_{2\hbar+2}\}$ and $\{\mathcal{I}\zeta_{2\hbar+1}\} = \{\mathcal{L}\zeta_{2\hbar}\} = \{\ell_{2\hbar+1}\}$ of $\{\ell_\hbar\}$ also converge to the point $\mathcal{U}$. Since $\mathcal{U} \in \mathcal{Q}(\Pi) \cup \mathcal{I}(\Pi)$, we have $\mathcal{U} \in \mathcal{Q}(\Pi)$ or $\mathcal{U} \in \mathcal{I}(\Pi)$. If $\mathcal{U} \in \mathcal{Q}(\Pi)$, then we can find $\mathcal{V} \in \Pi$ so that $\mathcal{Q}\mathcal{V} = \mathcal{U}$. We claim that $\mathcal{L}\mathcal{V} = \mathcal{U}$. η-regularity of Π implies that $\eta_\lambda(\mathcal{Q}\mathcal{V}, \mathcal{I}\zeta_{2\hbar+1}, \iota) \succeq I_\mathcal{A}$. So, we see that

$$\begin{aligned} C_{2\lambda}(\mathcal{L}\mathcal{V}, \mathcal{U}, \iota) &\preceq \mathcal{C}_\lambda(\mathcal{L}\mathcal{V}, \mathcal{G}\zeta_{2\hbar+1}, \iota) + \mathcal{C}_\lambda(\mathcal{G}\zeta_{2\hbar+1}, \mathcal{U}, \iota) \\ &\preceq \partial^*[\mathcal{C}_\lambda(\mathcal{Q}\mathcal{V}, \mathcal{I}\zeta_{2\hbar+1}, \iota)]\partial + \mathcal{C}_\lambda(\mathcal{G}\zeta_{2\hbar+1}, \mathcal{U}, \iota). \end{aligned}$$

So, we get

$$\| C_{2\lambda}(\mathcal{L}\mathcal{V}, \mathcal{U}, \iota) \| \leq \| \partial \|^2 \| \mathcal{C}_\lambda(\mathcal{U}, \mathcal{I}\zeta_{2\hbar+\mathbf{1}}, \iota) \| + \| \mathcal{C}_\lambda(\mathcal{G}\zeta_{2\hbar+1}, \mathcal{U}, \iota) \|.$$

Since $\| \partial \| < 1$, making $\hbar \to \infty$, we have $\mathcal{L}\mathcal{V} = \mathcal{Q}\mathcal{V} = \mathcal{U}$. Since $\mathcal{U} \in \mathcal{L}(\Pi) \subset \mathcal{I}(\Pi)$, there is $\Im \in \Pi$ such that $\mathcal{I}\Im = \mathcal{U}$. Now, we show that $\mathcal{G}\Im = \mathcal{U}$. In fact, as $\eta_\lambda(\mathcal{I}\Im, \mathcal{Q}\zeta_{2\hbar}, \iota) \succeq I_\mathcal{A}$, we have

$$\begin{aligned} C_{2\lambda}(\mathcal{G}\Im, \mathcal{U}, \iota) &\preceq \mathcal{C}_\lambda(\mathcal{G}\Im, \mathcal{L}\zeta_{2\hbar}, \iota) + \mathcal{C}_\lambda(\mathcal{L}\zeta_{2\hbar}, \mathcal{U}, \iota) \\ &\preceq \partial^*[\mathcal{C}_\lambda(\mathcal{I}\Im, \mathcal{Q}\zeta_{2\hbar}, \iota)]\partial + \mathcal{C}_\lambda(\mathcal{L}\zeta_{2\hbar}, \mathcal{U}, \iota), \end{aligned}$$

so, we get

$$\| C_{2\lambda}(\mathcal{G}\Im, \mathcal{U}, \iota) \| \leq \| \partial \|^2 \| \mathcal{C}_\lambda(\mathcal{U}, \mathcal{Q}\zeta_{2\hbar}, \iota) \| + \| \mathcal{C}_\lambda(\mathcal{L}\zeta_{2\hbar}, \mathcal{U}, \iota) \|.$$

If $\hbar \to \infty$, $\| C_{2\lambda}(\mathcal{G}\Im, \mathcal{U}, \iota) \| = \theta_\mathcal{A}$, since $\| \partial \| < 1$. So, $\mathcal{G}\Im = \mathcal{I}\Im = \mathcal{U}$. Thus, the pairs $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ have a coincidence point in Π. If $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ be w-compatible, $\mathcal{L}\mathcal{U} = \mathcal{L}\mathcal{Q}\mathcal{V} = \mathcal{Q}\mathcal{L}\mathcal{V} = \mathcal{Q}\mathcal{U} := \Im_1$ and $\mathcal{G}\mathcal{U} = \mathcal{G}\mathcal{I}\Im = \mathcal{I}\mathcal{G}\Im = \mathcal{I}\mathcal{U} := \Im_2$. Now, as $\eta_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) = \eta_\lambda(\Im_1, \Im_2, \iota) \succeq I_\mathcal{A}$,

$$\begin{aligned} \mathcal{C}_\lambda(\Im_1, \Im_2, \iota) &= \mathcal{C}_\lambda(\mathcal{L}\mathcal{U}, \mathcal{G}\mathcal{U}, \iota) \\ &\preceq \partial^*[\mathcal{C}_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{I}\mathcal{U}, \iota)]\partial \\ &= \partial^*[\mathcal{C}_\lambda(\Im_1, \Im_2, \iota)]\partial. \end{aligned}$$

This implies that

$$\|C_{\mathcal{A}_\lambda}(\Im_1, \Im_2, \iota)\| \leq \|\partial\|^2 \|\mathcal{C}_\lambda(\Im_1, \Im_2, \iota)\|.$$

Since $\| \partial \| < 1$, this fact implies that $\Im_1 = \Im_2$ and hence $\mathcal{L}\mathcal{U} = \mathcal{G}\mathcal{U} = \mathcal{Q}\mathcal{U} = \mathcal{I}\mathcal{U}$, that is, the point $\mathcal{U}$ is a coincidence point of the pairs $\{\mathcal{L}, \mathcal{Q}\}$ and $\{\mathcal{G}, \mathcal{I}\}$. Now, we show that $\mathcal{U} = \mathcal{G}\mathcal{U}$. In fact, as $\eta_\lambda(\mathcal{U} = \mathcal{Q}\mathcal{V}, \mathcal{I}\mathcal{U}, \iota) \succeq I_\mathcal{A}$, we have

$$\begin{aligned} \mathcal{C}_\lambda(\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) = \mathcal{C}_\lambda(\mathcal{U}, \mathcal{G}\mathcal{U}, \iota) &= \mathcal{C}_\lambda(\mathcal{L}\mathcal{V}, \mathcal{G}\mathcal{U}, \iota) \\ &\preceq \partial^*[\mathcal{C}_\lambda(\mathcal{Q}\mathcal{V}, \mathcal{I}\mathcal{U}, \iota)]\partial = \partial^*[\mathcal{C}_\lambda(\mathcal{U}, \mathcal{I}\mathcal{U}, \iota)]\partial, \end{aligned}$$

which conclude that

$$\| C_{\mathcal{A}_\lambda}(\mathcal{U}, \mathcal{G}\mathcal{U}, \iota) \| \leq \theta_\mathcal{A}.$$

Hence, $\mathcal{GU} = \mathcal{U}$ and therefore $\mathcal{U}$ is a common fixed point of $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$.

Finally, to show the uniqueness of point $\mathcal{U}$, suppose that $\mathcal{U}^*$ be another common fixed point of $\mathcal{L}, \mathcal{G}, \mathcal{Q}$ and $\mathcal{I}$. From (2.1), as $\eta_\lambda(\mathcal{QU}, \mathcal{IU}^*, \iota) \succeq I_\mathcal{A}$ it follows that

$$\begin{aligned} \mathcal{C}_\lambda(\mathcal{U}, \mathcal{U}^*, \iota) &= \mathcal{C}_\lambda(\mathcal{LU}, \mathcal{GU}^*, \iota) \\ &\preceq \partial^*[\mathcal{C}_\lambda(\mathcal{QU}, \mathcal{IU}^*, \iota)]\partial. \end{aligned}$$

This implies that

$$\|\mathcal{C}_\lambda(\mathcal{U}, \mathcal{U}^*, \iota)\| \leq \|\partial\|^2 \|\mathcal{C}_\lambda(\mathcal{U}, \mathcal{U}^*, \iota)\|,$$

since $\| \partial \| < 1$, we have $\mathcal{U} = \mathcal{U}^*$. Suppose that $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi)$ be complete and $\mathcal{U} \in \mathcal{I}(\Pi)$. In this case, the proof is similar to the completeness of $\mathcal{Q}(\Pi) \cup \mathcal{I}(\Pi)$ and $\mathcal{U} \in \mathcal{I}(\Pi)$. □

Corollary 2.12. Let $(\Pi, \mathcal{A}, \mathcal{C}_\mathcal{A})$ be a complete CAVMPMS and $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{I}$ be self-mappings on Π satisfying $\mathcal{L}(\Pi) \subseteq \mathcal{I}(\Pi)$, $\mathcal{G}(\Pi) \subseteq \mathcal{Q}(\Pi)$ and $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function, so that for all $\zeta, \ell \in \Pi$ with $\eta_\lambda(\mathcal{Q}^\Re \zeta, \mathcal{I}^\hbar \ell, \iota) \succeq I_\mathcal{A}$ and for some $\Re, \hbar \geq 1$

$$\mathcal{C}_\lambda(\mathcal{L}^\Re \zeta, \mathcal{G}^\hbar \ell, \iota) \preceq \partial^*[\mathcal{C}_\lambda(\mathcal{Q}^\Re \zeta, \mathcal{I}^\hbar \ell, \iota)]\partial, \quad \textit{for all } \lambda, \iota > 0, \tag{2.10}$$

where $\| \partial \| < 1$.

Assume that $(\Pi, \mathcal{A}, C)$ is η-regular and the pairs $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are triangular partially weakly η-admissible with respect to $\mathcal{I}$ and $\mathcal{Q}$, respectively.
Then

(A) If one of $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi)$ and $\mathcal{Q}(\Pi) \cup \mathcal{I}(\Pi)$ be complete, then $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ have a coincidence point in Π. If $\eta_\lambda(\mathcal{QU}, \mathcal{IU}, \iota) \succeq I_\mathcal{A}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a coincidence point.

(B) if $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ be w-compatible, and if $\eta_\lambda(\mathcal{U}, \mathcal{IU}, \iota) \succeq I_\mathcal{A}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a common fixed point in Π. If $\eta_\lambda(\mathcal{QU}, \mathcal{IU}^*, \iota) \succeq I_\mathcal{A}$ for all common fixed points $\mathcal{U}$ and $\mathcal{U}^*$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a unique common fixed point in Π.

Proof . By using Theorem 2.11, it follows that $(\mathcal{L}^\Re, \mathcal{Q}^\Re)$ and $(\mathcal{G}^\hbar, \mathcal{I}^\hbar)$ have a unique common fixed point $\Im \in \Pi$. Now, we have

$$\begin{aligned} \mathcal{L}(\Im) &= \mathcal{L}(\mathcal{L}^\Re(\Im)) = \mathcal{L}^{\Re+1}(\Im) = \mathcal{L}^\Re(\mathcal{L}(\Im)), \\ \mathcal{Q}(\Im) &= \mathcal{Q}(\mathcal{Q}^\Re(\Im)) = \mathcal{Q}^{\Re+1}(\Im) = \mathcal{Q}^\Re(\mathcal{Q}(\Im)) \end{aligned}$$

and therefore $\mathcal{L}(\Im)$ and $\mathcal{Q}(\Im)$ are also fixed points for mappings $\mathcal{L}^\Re$ and $\mathcal{Q}^\Re$. Hence, $\mathcal{L}(\Im) = \mathcal{Q}(\Im) = \Im$. Using the same reasoning in the proof of Theorem 2.11, we get $\mathcal{G}(\Im) = \mathcal{I}(\Im) = \Im$. Therefore, the proof is complete. □

Corollary 2.13. Let $\mathcal{L}$, $\mathcal{G}$ and $\mathcal{I}$ be self-mappings on complete CAVMPMS $(\Pi, \mathcal{A}, C)$, satisfying $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi) \subset \mathcal{I}(\Pi)$, and $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function, so that for all $\zeta, \ell \in \Pi$ with $\eta_\lambda(\mathcal{I}\zeta, \mathcal{I}\ell, \iota) \succeq I_\mathcal{A}$,

$$\mathcal{C}_\lambda(\mathcal{L}\zeta, \mathcal{G}\ell, \iota) \preceq \partial^*[\mathcal{C}_\lambda(\mathcal{I}\zeta, \mathcal{I}\ell, \iota)]\partial, \quad \textit{for all } \lambda, \iota > 0, \tag{2.11}$$

where $\| \partial \| < 1$.

Assume that $(\Pi, \mathcal{A}, C)$ is η-regular and the pairs $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are triangular partially weakly η-admissible with respect to $\mathcal{I}$, respectively.
Then

(A) If one of $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi)$ and $\mathcal{I}(\Pi)$ be complete, then $(\mathcal{L}, \mathcal{I})$ and $(\mathcal{G}, \mathcal{I})$ have a coincidence point in Π. If $\eta_\lambda(\mathcal{IU}, \mathcal{IU}, \iota) \succeq I_\mathcal{A}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{I}$ have a coincidence point.

(B) if $(\mathcal{L}, \mathcal{I})$ and $(\mathcal{G}, \mathcal{I})$ be w-compatible, and if $\eta_\lambda(\mathcal{U}, \mathcal{IU}, \iota) \succeq I_\mathcal{A}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{I}$ have a common fixed point in Π. If $\eta_\lambda(\mathcal{IU}, \mathcal{IU}^*, \iota) \succeq I_\mathcal{A}$ for all common fixed points $\mathcal{U}$ and $\mathcal{U}^*$, then $\mathcal{L}$, $\mathcal{G}$ and $\mathcal{I}$ have a unique common fixed point in Π.

Corollary 2.14. Let $\mathcal{L}$ and $\mathcal{I}$ be self-mappings on complete CAVMPMS $(\Pi, \mathcal{A}, C)$, satisfying $\mathcal{L}(\Pi) \subset \mathcal{I}(\Pi)$, and $\eta : (0,\infty) \times \Pi^2 \times (0,\infty) \to \mathcal{A}^+$ be a function, so that for all $\zeta, \ell \in \Pi$, with $\eta_\lambda(\mathcal{I}\zeta, \mathcal{I}\ell, \iota) \succeq I_{\mathcal{A}}$,

$$\mathcal{C}_\lambda(\mathcal{L}\zeta, \mathcal{L}\ell, \iota) \preceq \partial^*[\mathcal{C}_\lambda(\mathcal{I}\zeta, \mathcal{I}\ell, \iota)]\partial, \quad \textit{for all } \lambda, \iota > 0, \tag{2.12}$$

where $\parallel \partial \parallel < 1$.

Assume that $(\Pi, \mathcal{A}, C)$ is η-regular and $\mathcal{L}$ is triangular partially weakly η-admissible with respect to $\mathcal{I}$.
Then

(A) If one of $\mathcal{L}(\Pi)$ and $\mathcal{I}(\Pi)$ be complete, then $(\mathcal{L}, \mathcal{I})$ have a coincidence point in Π. If $\eta_\lambda(\mathcal{I}\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_{\mathcal{A}}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$ and $\mathcal{I}$ have a coincidence point.

(B) if $(\mathcal{L}, \mathcal{I})$ be w-compatible, and if $\eta_\lambda(\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_{\mathcal{A}}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, and $\mathcal{I}$ have a common fixed point in Π. If $\eta_\lambda(\mathcal{I}\mathcal{U}, \mathcal{I}\mathcal{U}^*, \iota) \succeq I_{\mathcal{A}}$ for all common fixed points $\mathcal{U}$ and $\mathcal{U}^*$, then $\mathcal{L}$ and $\mathcal{I}$ have a unique common fixed point in Π.

2.2 η-admissible mapping and Kannan-Ćirić C^*-contractions

Now, we generalize the Kannan-Ćirić contraction condition [22] as follows.

Theorem 2.15. Let $(\Pi, \mathcal{A}, C)$ be a complete CAVMPMS and $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{I}$ be self-mappings on Π, so that the following conditions are satisfies:

(i) $\mathcal{L}(\Pi) \subseteq \mathcal{I}(\Pi)$, $\mathcal{G}(\Pi) \subseteq \mathcal{Q}(\Pi)$ and $\eta : (0,\infty) \times \Pi^2 \times (0,\infty) \to \mathcal{A}^+$ be a function.

(ii) Suppose that for all $\zeta, \ell \in \Pi$ with $\eta_\lambda(\mathcal{Q}\zeta, \mathcal{I}\ell, \iota) \succeq I_{\mathcal{A}}$,

$$\mathcal{C}_\lambda(\mathcal{L}\zeta, \mathcal{G}\ell, \iota) \preceq \partial^*[\mathbf{P}(\zeta, \ell, \iota)]\partial, \; \textit{for all } \zeta, \ell \in \Pi, \; \lambda, \iota > 0, \tag{2.13}$$

where $\parallel \partial \parallel < 1$ and

$$\mathbf{P}(\zeta, \ell, \iota) = \max\{\mathcal{C}_\lambda(\mathcal{Q}\zeta, \mathcal{L}\zeta, \iota), \mathcal{C}_\lambda(\mathcal{I}\ell, \mathcal{G}\ell, \iota)\}.$$

Assume that $(\Pi, \mathcal{A}, C)$ is η-regular and the pairs $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are triangular partially weakly η-admissible with respect to $\mathcal{I}$ and $\mathcal{Q}$, respectively.
Then

(A) If one of $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi)$ and $\mathcal{Q}(\Pi) \cup \mathcal{I}(\Pi)$ be complete, then $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ have a coincidence point in Π. If $\eta_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_{\mathcal{A}}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a coincidence point.

(B) if $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ be w-compatible, and if $\eta_\lambda(\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_{\mathcal{A}}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a common fixed point in Π. If $\eta_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{I}\mathcal{U}^*, \iota) \succeq I_{\mathcal{A}}$ for all common fixed points $\mathcal{U}$ and $\mathcal{U}^*$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a unique common fixed point in Π.

Proof . Let ζ_0 be an arbitrary point of Π. Choose $\zeta_1 \in \Pi$ such that $\mathcal{L}\zeta_0 = \mathcal{I}\zeta_1$ and $\zeta_2 \in \Pi$ such that $\mathcal{G}\zeta_1 = \mathcal{Q}\zeta_2$.

Continuing this way, construct a sequence $\{\ell_\hbar\}$ defined by:

$$\begin{cases} \mathcal{L}\zeta_{2\hbar} = \mathcal{I}\zeta_{2\hbar+1} = \ell_{2\hbar+1} \\ \mathcal{G}\zeta_{2\hbar+1} = \mathcal{Q}\zeta_{2\hbar+2} = \ell_{2\hbar+2}, \qquad \forall \hbar \geq 0. \end{cases}$$

As $\zeta_1 \in \mathcal{I}^{-1}(\mathcal{L}\zeta_0)$ and $\zeta_2 \in \mathcal{Q}^{-1}(\mathcal{G}\zeta_1)$ and the pairs $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are partially weakly η-admissible with respect to $\mathcal{I}$ and $\mathcal{Q}$, respectively, we have

$$\eta_\lambda(\mathcal{I}\zeta_1 = \mathcal{L}\zeta_0, \mathcal{G}\zeta_1 = \mathcal{Q}\zeta_2, \iota) \succeq I_{\mathcal{A}}$$

and

$$\eta_\lambda(\mathcal{G}\zeta_1 = \mathcal{Q}\zeta_2, \mathcal{L}\zeta_2 = \mathcal{I}\zeta_3, \iota) \succeq I_{\mathcal{A}}.$$

Repeating this process, we obtain $\eta_\lambda(\mathcal{I}\zeta_{2\hbar+1}, \mathcal{Q}\zeta_{2\hbar+2}, \iota) = \eta_\lambda(\ell_{2\hbar+1}, \ell_{2\hbar+2}, \iota) \succeq I_{\mathcal{A}}$ for all $\hbar \geq 0$. According to (2.13)

$$\begin{aligned} \mathcal{C}_\lambda(\ell_{2\hbar+1}, \ell_{2\hbar+2}, \iota) &= \mathcal{C}_\lambda(\mathcal{L}\zeta_{2\hbar}, \mathcal{G}\zeta_{2\hbar+1}, \iota) \\ &\preceq \partial^*[\mathbf{P}(\zeta_{2\hbar}, \zeta_{2\hbar+1}, \iota)]\partial, \end{aligned}$$

where

$$\begin{aligned}\mathbf{P}(\zeta_{2\hbar},\zeta_{2\hbar+1},\iota) &= \max\{\mathcal{C}_\lambda(\mathcal{Q}\zeta_{2\hbar},\mathcal{L}\zeta_{2\hbar},\iota),\mathcal{C}_\lambda(\mathcal{I}\zeta_{2\hbar+1},\mathcal{G}\zeta_{2\hbar+1},\iota)\\ &= \max\{\mathcal{C}_\lambda(\ell_{2\hbar},\ell_{2\hbar+1},\iota),\mathcal{C}_\lambda(\ell_{2\hbar+1},\ell_{2\hbar+2},\iota)\}.\end{aligned}$$

If $\mathbf{P}(\zeta_{2\hbar},\zeta_{2\hbar+1},\iota)=\mathcal{C}_\lambda(\ell_{2\hbar+1},\ell_{2\hbar+2},\iota)$, then we simply see that it is impossible. So, $\mathbf{P}(\zeta_{2\hbar},\zeta_{2\hbar+1},\iota)=\mathcal{C}_\lambda(\ell_{2\hbar},\ell_{2\hbar+1},\iota)$ for all $\hbar\in\mathbb{N}$, and

$$\begin{aligned}\mathcal{C}_\lambda(\ell_{2\hbar+1},\ell_{2\hbar+2},\iota) &\preceq \partial^*[\mathbf{P}(\zeta_{2\hbar},\zeta_{2\hbar+1},\iota)]\partial\\ &= \partial^*[\mathcal{C}_\lambda(\ell_{2\hbar},\ell_{2\hbar+1},\iota)]\partial.\end{aligned}$$

By induction, we get

$$\mathcal{C}_\lambda(\ell_{2\hbar+1},\ell_{2\hbar+2},\iota)\preceq(\partial^*)^{2\hbar+1}[\mathcal{C}_\lambda(\ell_0,\ell_1,\iota)]\partial^{2\hbar+1}.$$

Similarly, it can be shown that

$$\mathcal{C}_\lambda(\ell_{2\hbar},\ell_{2\hbar+1},\iota)\preceq(\partial^*)^{2\hbar}[\mathcal{C}_\lambda(\ell_0,\ell_1,\iota)]\partial^{2\hbar}.$$

Now, for every $\hbar\in\mathcal{N}$, we can get

$$\begin{aligned}\mathcal{C}_\lambda(\ell_\hbar,\ell_{\hbar+1},\iota) &\preceq (\partial^*)^\hbar[\mathcal{C}_\lambda(\ell_0,\ell_1,\iota)]\partial^\hbar\\ &\preceq (\partial^*)^\hbar\varpi_0\partial^\hbar,\end{aligned}$$

where $\varpi_0:=\mathcal{C}_\lambda(\ell_0,\ell_1,\iota)$.

Step II. Following similar lines in the proof of Theorem 2.11 we can show that $\{\ell_\hbar\}$ is a Cauchy sequence in Π.

Suppose that $\mathcal{Q}(\Pi)\cup\mathcal{I}(\Pi)$ be complete. In this case there is $\mathcal{U}\in\mathcal{Q}(\Pi)\cup\mathcal{I}(\Pi)$ so that $\ell_\hbar\to\mathcal{U}$ as $\hbar\to\infty$. Furthermore, the subsequences $\{\mathcal{Q}\zeta_{2\hbar+2}\}=\{\mathcal{G}\zeta_{2\hbar+1}\}=\{\ell_{2\hbar+2}\}$ and $\{\mathcal{I}\zeta_{2\hbar+1}\}=\{\mathcal{L}\zeta_{2\hbar}\}=\{\ell_{2\hbar+1}\}$ of $\{\ell_\hbar\}$ also converge to the point $\mathcal{U}$. Since $\mathcal{U}\in\mathcal{Q}(\Pi)\cup\mathcal{I}(\Pi)$, we have $\mathcal{U}\in\mathcal{Q}(\Pi)$ or $\mathcal{U}\in\mathcal{I}(\Pi)$. If $\mathcal{U}\in\mathcal{Q}(\Pi)$, then we can find $\mathcal{V}\in\Pi$ so that $\mathcal{Q}\mathcal{V}=\mathcal{U}$. Now, we claim that $\mathcal{L}\mathcal{V}=\mathcal{U}$. For this, as $\eta_\lambda(\mathcal{Q}\mathcal{V},(\mathcal{I}\zeta_{2\hbar+1},\iota)\succeq I_{\mathcal{A}}$ we see that

$$\begin{aligned}C_{2\lambda}(\mathcal{L}\mathcal{V},\mathcal{U},\iota) &\preceq \mathcal{C}_\lambda(\mathcal{L}\mathcal{V},\mathcal{G}\zeta_{2\hbar+1},\iota)+\mathcal{C}_\lambda(\mathcal{G}\zeta_{2\hbar+1},\mathcal{U},\iota)\\ &\preceq \partial^*[\mathbf{P}(\mathcal{V},\zeta_{2\hbar+1},\iota)]\partial+\mathcal{C}_\lambda(\mathcal{G}\zeta_{2\hbar+1},\mathcal{U},\iota),\end{aligned}$$

where

$$\mathbf{P}(\mathcal{V},\zeta_{2\hbar+1},\iota)=\max\{\mathcal{C}_\lambda(\mathcal{Q}\mathcal{V},\mathcal{L}\mathcal{V},\iota),\mathcal{C}_\lambda(\mathcal{I}\zeta_{2\hbar+1},\mathcal{G}\zeta_{2\hbar+1},\iota)\}.$$

Therefore, we get

$$\| C_{\mathcal{A}_{2\lambda}}(\mathcal{L}\mathcal{V},\mathcal{U},\iota)\|\le\|\partial\|^2\|\mathbf{P}(\mathcal{V},\zeta_{2\hbar+1},\iota)\|+\|\mathcal{C}_\lambda(\mathcal{G}\zeta_{2\hbar+1},\mathcal{U},\iota)\|.$$

Since $\|\partial\|<1$, making $\hbar\to\infty$, we have a contradiction, so $\mathcal{C}_{2\lambda}(\mathcal{L}\mathcal{V},\mathcal{U},\iota)=0$. Consequently, we have $\mathcal{L}\mathcal{V}=\mathcal{Q}\mathcal{V}=\mathcal{U}$ and since $\mathcal{U}\in\mathcal{L}(\Pi)\subset\mathcal{I}(\Pi)$, there is $\Im\in\Pi$ so that $\mathcal{I}\Im=\mathcal{U}$.

Now, we show that $\mathcal{G}\Im=\mathcal{U}$. So, we have

$$\begin{aligned}C_{2\lambda}(\mathcal{G}\Im,\mathcal{U},\iota) &\preceq \mathcal{C}_\lambda(\mathcal{G}\Im,\mathcal{L}\zeta_{2\hbar},\iota)+\mathcal{C}_\lambda(\mathcal{L}\zeta_{2\hbar},\mathcal{U},\iota)\\ &\preceq \partial^*[\mathbf{P}(\Im,\zeta_{2\hbar},\iota)]\partial+\mathcal{C}_\lambda(\mathcal{L}\zeta_{2\hbar},\mathcal{U},\iota),\end{aligned}$$

because $\eta_\lambda(\mathcal{Q}\Im,\mathcal{I}\zeta_{2\hbar},\iota)\succeq I_{\mathcal{A}}$ where

$$\mathbf{P}(\Im,\zeta_{2\hbar},\iota)=\max\{\mathcal{C}_\lambda(\mathcal{Q}\Im,\mathcal{L}\Im,\iota),\mathcal{C}_\lambda(\mathcal{I}\zeta_{2\hbar},\mathcal{G}\zeta_{2\hbar},\iota)\}.$$

We get

$$\| C_{2\lambda}(\mathcal{G}\Im,\mathcal{U},\iota)\|\le\|\partial\|^2\|\mathcal{U}(\Im,\zeta_{2\hbar},\iota)\|+\|\mathcal{C}_\lambda(\mathcal{L}\zeta_{2\hbar},\mathcal{U},\iota)\|.$$

If $\hbar \to \infty$, since $\| \partial \| < 1$, $C_{2\lambda}(\mathcal{G}\Im, \mathcal{U}, \iota) \preceq \theta_{\mathcal{A}}$. So, $\mathcal{G}\Im = \mathcal{I}\Im = \mathcal{U}$. Hence, $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ have a coincidence point in Π. Now, if $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ be w-compatible, $\mathcal{L}\mathcal{U} = \mathcal{L}\mathcal{Q}\mathcal{V} = \mathcal{Q}\mathcal{L}\mathcal{V} = \mathcal{Q}\mathcal{U} := \Im_1$ and $\mathcal{G}\mathcal{U} = \mathcal{G}\mathcal{I}\Im = \mathcal{I}\mathcal{G}\Im = \mathcal{I}\mathcal{U} := \Im_2$. Now

$$\begin{aligned}\mathcal{C}_\lambda(\Im_1, \Im_2, \iota) &= \mathcal{C}_\lambda(\mathcal{L}\mathcal{U}, \mathcal{G}\mathcal{U}, \iota)\\ &\preceq \partial^*[\mathbf{P}(\mathcal{U}, \mathcal{U}, \iota)]\partial,\end{aligned}$$

because we have $\eta_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) = \eta_\lambda(\Im_1, \Im_2, \iota) \succeq I_{\mathcal{A}}$, where

$$\mathbf{P}(\mathcal{U}, \mathcal{U}, \iota) = \max\{\mathcal{C}_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{L}\mathcal{U}, \iota), \mathcal{C}_\lambda(\mathcal{I}\mathcal{U}, \mathcal{G}\mathcal{U}, \iota)\}.$$

This implies that

$$C_\lambda(\Im_1, \Im_2, \iota) \preceq \theta_{\mathcal{A}}.$$

So $\Im_1 = \Im_2$ and hence $\mathcal{L}\mathcal{U} = \mathcal{G}\mathcal{U} = \mathcal{Q}\mathcal{U} = \mathcal{I}\mathcal{U}$, i.e., the point $\mathcal{U}$ is a coincidence point of $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$. Now, we show that $\mathcal{U} = \mathcal{G}\mathcal{U}$. We have

$$\begin{aligned}\mathcal{C}_\lambda(\mathcal{U}, \mathcal{G}\mathcal{U}, \iota) &= \mathcal{C}_\lambda(\mathcal{L}\mathcal{V}, \mathcal{G}\mathcal{U}, \iota)\\ &\preceq \partial^*[\mathbf{P}(\mathcal{V}, \mathcal{U}, \iota)]\partial,\end{aligned}$$

because $\eta_\lambda(\mathcal{Q}\mathcal{V}, \mathcal{I}\mathcal{U}, \iota) \succeq I_{\mathcal{A}}$ where

$$\begin{aligned}\mathbf{P}(\mathcal{V}, \mathcal{U}, \iota) &= \max\{\mathcal{C}_\lambda(\mathcal{Q}\mathcal{V}, \mathcal{L}\mathcal{V}, \iota), \mathcal{C}_\lambda(\mathcal{I}\mathcal{U}, \mathcal{G}\mathcal{U}, \iota)\}\\ &= C_{\mathcal{A}_\lambda}(\mathcal{U}, \mathcal{G}\mathcal{U}, \iota).\end{aligned}$$

So, we get

$$\| C_{\mathcal{A}_\lambda}(\mathcal{U}, \mathcal{G}\mathcal{U}, \iota) \| \leq \| \partial \|^2 \|C_{\mathcal{A}_\lambda}(\mathcal{U}, \mathcal{G}\mathcal{U}, \iota)\|.$$

Since $\| \partial \| < 1$, $\mathcal{G}\mathcal{U} = \mathcal{U}$ and hence $\mathcal{U}$ is a common fixed point of $\mathcal{L}, \mathcal{G}, \mathcal{Q}$ and $\mathcal{I}$. Finally, to show the uniqueness of point $\mathcal{U}$, suppose that $\mathcal{U}^*$ be another common fixed point of $\mathcal{L}, \mathcal{G}, \mathcal{Q}$ and $\mathcal{I}$. From (2.13), it follows that

$$\begin{aligned}\mathcal{C}_\lambda(\mathcal{U}, \mathcal{U}^*, \iota) &= \mathcal{C}_\lambda(\mathcal{L}\mathcal{U}, \mathcal{G}\mathcal{U}^*, \iota)\\ &\preceq \partial^*[\mathbf{P}(\mathcal{U}, \mathcal{U}^*, \iota)]\partial,\end{aligned}$$

because $\eta_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{I}\mathcal{U}^*, \iota) = \eta_\lambda(\mathcal{U}, \mathcal{U}^*, \iota) \succeq I_{\mathcal{A}}$ in which

$$\mathbf{P}(\mathcal{U}, \mathcal{U}^*, \iota) = \max\{\mathcal{C}_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{L}\mathcal{U}, \iota), \mathcal{C}_\lambda(\mathcal{I}\mathcal{U}^*, \mathcal{G}\mathcal{U}^*, \iota)\}.$$

This implies that

$$\mathcal{C}_\lambda(\mathcal{U}, \mathcal{U}^*, \iota) \preceq \theta_{\mathcal{A}}.$$

Then $\mathcal{U} = \mathcal{U}^*$. Suppose that $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi)$ be complete and $\mathcal{U} \in \mathcal{I}(\Pi)$. In this case, the proof is similar to the completeness of $\mathcal{Q}(\Pi) \cup \mathcal{I}(\Pi)$ and $\mathcal{U} \in \mathcal{I}(\Pi)$. □

Corollary 2.16. Let $(\Pi, \mathcal{A}, \mathcal{C}_{\mathcal{A}})$ be a complete CAVMPMS and $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$, $\mathcal{I}$ be self-mappings on Π satisfying $\mathcal{L}(\Pi) \subset \mathcal{I}(\Pi)$, $\mathcal{G}(\Pi) \subset \mathcal{Q}(\Pi)$, and $\eta : (0, \infty) \times \Pi^2 \times (0, \infty) \to \mathcal{A}^+$ be a function, so that for all $\zeta, \ell \in \Pi$ with $\eta_\lambda(\mathcal{Q}^{\Re}\zeta, \mathcal{I}^{\hbar}\ell, \iota) \succeq I_{\mathcal{A}}$ and for some $\Re, \hbar \geq 1$

$$\mathcal{C}_\lambda(\mathcal{L}^{\Re}\zeta, \mathcal{G}^{\hbar}\ell, \iota) \preceq \partial^*[\mathbf{P}(\zeta, \ell, \iota)]\partial, \quad \textit{for all } \lambda, \iota > 0, \tag{2.14}$$

where $\| \partial \| < 1$ and

$$\mathbf{P}(\zeta, \ell, \iota) = \max\{\mathcal{C}_\lambda(\mathcal{Q}^{\Re}\zeta, \mathcal{L}^{\Re}\zeta, \iota), \mathcal{C}_\lambda(\mathcal{I}^{\hbar}\ell, \mathcal{G}^{\hbar}\ell, \iota)\}.$$

Assume that $(\Pi, \mathcal{A}, C)$ is η-regular and the pairs $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are triangular partially weakly η-admissible with respect to $\mathcal{I}$ and $\mathcal{Q}$, respectively. Then

(A) If one of $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi)$ and $\mathcal{Q}(\Pi) \cup \mathcal{I}(\Pi)$ be complete, then $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ have a coincidence point in Π. If $\eta_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_{\mathcal{A}}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a coincidence point.

(B) if $(\mathcal{L}, \mathcal{Q})$ and $(\mathcal{G}, \mathcal{I})$ be w-compatible, and if $\eta_\lambda(\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_{\mathcal{A}}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a common fixed point in Π. If $\eta_\lambda(\mathcal{Q}\mathcal{U}, \mathcal{I}\mathcal{U}^*, \iota) \succeq I_{\mathcal{A}}$ for all common fixed points $\mathcal{U}$ and $\mathcal{U}^*$, then $\mathcal{L}$, $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{I}$ have a unique common fixed point in Π.

Corollary 2.17. Let $\mathcal{L}$, $\mathcal{G}$ and $\mathcal{I}$ be self-mappings on complete CAVMPMS $(\Pi, \mathcal{A}, C)$, satisfying $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi) \subset \mathcal{I}(\Pi)$, and $\eta : (0,\infty) \times \Pi^2 \times (0,\infty) \to \mathcal{A}^+$ be a function, so that for all $\zeta, \ell \in \Pi$ with $\eta_\lambda(\mathcal{I}\zeta, \mathcal{I}\ell, \iota) \succeq I_\mathcal{A}$,

$$\mathcal{C}_\lambda(\mathcal{L}\zeta, \mathcal{G}\ell, \iota) \preceq \partial^*[\mathbf{P}(\zeta, \ell, \iota)]\partial, \quad \textit{for all } \lambda, \iota > 0, \tag{2.15}$$

where $\| \partial \| < 1$ and

$$\mathbf{P}(\zeta, \ell, \iota) = \max\{\mathcal{C}_\lambda(\mathcal{I}\zeta, \mathcal{L}\zeta, \iota), \mathcal{C}_\lambda(\mathcal{I}\ell, \mathcal{G}\ell, \iota)\}.$$

Assume that $(\Pi, \mathcal{A}, C)$ is η-regular and the pairs $(\mathcal{L}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{L})$ are triangular partially weakly η-admissible with respect to $\mathcal{I}$.
Then

(A) If one of $\mathcal{L}(\Pi) \cup \mathcal{G}(\Pi)$ and $\mathcal{I}(\Pi)$ be complete, then $(\mathcal{L}, \mathcal{I})$ and $(\mathcal{G}, \mathcal{I})$ have a coincidence point in Π. If $\eta_\lambda(\mathcal{I}\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_\mathcal{A}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{I}$ have a coincidence point.

(B) if $(\mathcal{L}, \mathcal{I})$ and $(\mathcal{G}, \mathcal{I})$ be w-compatible, and if $\eta_\lambda(\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_\mathcal{A}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$, $\mathcal{G}$, and $\mathcal{I}$ have a common fixed point in Π. If $\eta_\lambda(\mathcal{I}\mathcal{U}, \mathcal{I}\mathcal{U}^*, \iota) \succeq I_\mathcal{A}$ for all common fixed points $\mathcal{U}$ and $\mathcal{U}^*$, then $\mathcal{L}$, $\mathcal{G}$ and $\mathcal{I}$ have a unique common fixed point in Π.

Corollary 2.18. Let $\mathcal{L}$ and $\mathcal{I}$ be self-mappings on complete CAVMPMS $(\Pi, \mathcal{A}, C)$, satisfying $\mathcal{L}(\Pi) \subset \mathcal{I}(\Pi)$, and $\eta : (0,\infty) \times \Pi^2 \times (0,\infty) \to \mathcal{A}^+$ be a function, so that for all $\zeta, \ell \in \Pi$, with $\eta_\lambda(\mathcal{I}\zeta, \mathcal{I}\ell, \iota) \succeq I_\mathcal{A}$,

$$\mathcal{C}_\lambda(\mathcal{L}\zeta, \mathcal{L}\ell, \iota) \preceq \partial^*[\mathbf{P}(\zeta, \ell, \iota)]\partial, \quad \textit{for all } \lambda, \iota > 0, \tag{2.16}$$

where $\| \partial \| < 1$ and

$$\mathbf{P}(\zeta, \ell, \iota) = \max\{\mathcal{C}_\lambda(\mathcal{I}\zeta, \mathcal{L}\zeta, \iota), \mathcal{C}_\lambda(\mathcal{I}\ell, \mathcal{L}\ell, \iota)\}.$$

Assume that $(\Pi, \mathcal{A}, C)$ is η-regular and $\mathcal{L}$ is triangular partially weakly η-admissible with respect to $\mathcal{I}$. Then

(A) If one of $\mathcal{L}(\Pi)$ and $\mathcal{I}(\Pi)$ be complete, then $(\mathcal{L}, \mathcal{I})$ have a coincidence point in Π. If $\eta_\lambda(\mathcal{I}\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_\mathcal{A}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$ and $\mathcal{I}$ have a coincidence point.

(B) if $(\mathcal{L}, \mathcal{I})$ be w-compatible, and if $\eta_\lambda(\mathcal{U}, \mathcal{I}\mathcal{U}, \iota) \succeq I_\mathcal{A}$ for all coincidence point $\mathcal{U}$, then $\mathcal{L}$ and $\mathcal{I}$ have a common fixed point in Π. If $\eta_\lambda(\mathcal{I}\mathcal{U}, \mathcal{I}\mathcal{U}^*, \iota) \succeq I_\mathcal{A}$ for all common fixed points $\mathcal{U}$ and $\mathcal{U}^*$, then $\mathcal{L}$ and $\mathcal{I}$ have a unique common fixed point in Π.

Example 2.19. Let $\mathfrak{H}$ be a Hilbert space and let $\mathfrak{L}(\mathfrak{H})$ be the set of all linear bounded operators on $\mathfrak{H}$. Let $\{\mathfrak{D}_i\} \subseteq \mathfrak{L}(\mathfrak{H})$, with $\sum_{\hbar=1}^{\infty} \| \mathfrak{D}_\hbar \|^2 < 1$, $\Pi \in \mathfrak{L}(\mathfrak{H})$ and $\mathfrak{P} \in \mathfrak{L}(\mathfrak{H})_+$. Then the operator equation

$$\mathfrak{k} - \sum_{\hbar=1}^{\infty} \mathfrak{D}_\hbar^* \mathfrak{k} \mathfrak{D}_\hbar = \mathfrak{P},$$

has a unique solution in $\mathfrak{L}(\mathfrak{H})$.

Proof . Set $\varrho = \sum_{\hbar=1}^{\infty} \| \mathfrak{D}_\hbar \|^2$. Clearly, if $\varrho = 0$, then the $\mathfrak{D}_\hbar = \theta_\mathcal{A}$ $(\hbar \in \mathbb{N})$, and the equation has a unique solution in $\mathfrak{L}(\mathfrak{H})$. We suppose that $\varrho > 0$. Choose a positive operator $\mathcal{T} \in \mathfrak{L}(\mathfrak{H})$. For $\mathfrak{k}, \mathfrak{Y} \in \mathfrak{L}(\mathfrak{H})$, set

$$\mathcal{C}_\lambda(\mathfrak{k}, \mathfrak{Y}, \iota) = \| \iota \frac{\mathfrak{k} - \mathfrak{Y}}{\lambda} \| \mathcal{T}.$$

It is clear that $(\mathfrak{L}(\mathfrak{H}), \mathcal{A}, C)$ is a CAVMPMS which is complete since $\mathfrak{L}(\mathfrak{H})$ is a Banach space. We defined $\eta_\lambda(\zeta, \ell, \iota) = I_\mathcal{A}$ and the mapping $\mathcal{F} : \mathfrak{L}(\mathfrak{H}) \to \mathfrak{L}(\mathfrak{H})$ by

$$\mathcal{F}(\mathfrak{k}) = \sum_{\hbar=1}^{\infty} \mathfrak{D}_\hbar^* \mathfrak{k} \mathfrak{D}_\hbar + \mathfrak{P}.$$

Then

$$
\begin{aligned}
\mathcal{C}_\lambda(\mathcal{F}\mathfrak{k},\mathcal{F}\mathfrak{Y},\iota) &= \parallel \iota\frac{\mathcal{F}\mathfrak{k}-\mathcal{F}\mathfrak{Y}}{\lambda} \parallel \mathcal{T}\\
&= \parallel \iota\frac{\sum_{\hbar=1}^{\infty}\mathfrak{D}_\hbar^*(\mathfrak{k}-\mathfrak{Y})\mathfrak{D}_\hbar}{\lambda} \parallel \mathcal{T}\\
&\preceq \sum_{\hbar=1}^{\infty} \parallel \mathfrak{D}_\hbar \parallel^2 \parallel \iota\frac{(\mathfrak{k}-\mathfrak{Y})}{\lambda} \parallel \mathcal{T}\\
&= \varrho \parallel \iota\frac{(\mathfrak{k}-\mathfrak{Y})}{\lambda} \parallel \mathcal{T}\\
&= (\varrho^{\frac{1}{2}} I_{\mathcal{A}})^*[\parallel \iota\frac{(\mathfrak{k}-\mathfrak{Y})}{\lambda} \parallel \mathcal{T}](\varrho^{\frac{1}{2}} I_{\mathcal{A}})\\
&= (\varrho^{\frac{1}{2}} I_{\mathcal{A}})^*[\mathcal{C}_\lambda(\mathfrak{k},\mathfrak{Y},\iota)](\varrho^{\frac{1}{2}} I_{\mathcal{A}}).
\end{aligned}
$$

Using Theorem 2.11 for mapping $\mathcal{F}$, there is a unique fixed point $\mathfrak{X} \in \mathfrak{L}(\mathfrak{H})$. Moreover, since $\sum_{\hbar=1}^{\infty}\mathfrak{D}_\hbar^*\mathfrak{X}\mathfrak{D}_\hbar + \mathfrak{P}$ is a positive operator, the solution is a Hermitian operator. □

3 Application

Let $\mathfrak{X} = \mathfrak{L}^\infty(\mathfrak{S})$ be the set of essentially bounded measurable functions on $\mathfrak{S}$. Consider the Hilbert space $\mathfrak{H} = \mathfrak{L}^2(\mathfrak{S})$ and $\mathfrak{L}(\mathfrak{H}) = \mathcal{A}$, where $\mathfrak{S}$ is a Lebesgue measurable set and $m(\mathfrak{S}) < \infty$. We consider a Fredholm integral equation as follows:

$$
\zeta(\iota) = \int_{\mathfrak{S}} \Upsilon(\iota,\varsigma,\zeta(\varsigma))d\varsigma + \mathfrak{h}(\iota), \ \textit{for all } \varsigma,\iota \in \mathfrak{S}, \tag{3.1}
$$

where $\Upsilon : \mathfrak{S}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathfrak{h} \in \mathfrak{L}^\infty(\mathfrak{S})$. Define $C : (0,\infty) \times [\mathfrak{L}^\infty(\mathfrak{S})]^2 \times (0,\infty) \to \mathfrak{L}(\mathfrak{H})$ by $\mathcal{C}_\lambda(\zeta,\ell,\iota) = M_{|\frac{\iota|\zeta-\ell|}{\lambda}|}$, for all $\zeta,\ell \in \mathfrak{X}$ and for all $\lambda,\iota > 0$, where M_τ is the multiplication operator on $\mathfrak{L}^2(\mathfrak{S})$ which is given by $M_\tau(\varpi) = \tau.\varpi$, for all $\varpi \in \mathfrak{L}^2(\mathfrak{S})$. Then $(\mathfrak{X},\mathcal{A},C)$ is a complete CAVMPMS. Now we consider the following assumption:

There is $\kappa \in (0,\frac{1}{2})$ such that for all $\zeta,\ell \in \mathfrak{X}$:

$$
|\Upsilon(\varsigma,\iota,\zeta(\varsigma)) - \Upsilon(\varsigma,\iota,\ell(\varsigma))| \le \kappa\big(|\zeta(\varsigma)-\ell(\varsigma)|\big).
$$

Theorem 3.1. Under the above assumption, the integral equation (3.1) has a unique solution in $\mathfrak{X}$.

Proof . We define $\mathcal{L} : \mathfrak{X} \to \mathfrak{X}$ by

$$
\mathcal{L}(\zeta)(\iota) = \int_{\mathfrak{S}} \Upsilon(\iota,\varsigma,\zeta(\varsigma))d\varsigma + \mathfrak{h}(\iota), \quad \forall \varsigma,\iota \in \mathfrak{S},
$$

and $\eta : (0,\infty) \times \mathfrak{X}^2 \times (0,\infty) \to \mathcal{A}^+$ by $\eta_\lambda(\zeta,\ell,\iota) = I_{\mathcal{A}}$. Set $\varrho = \kappa I_{\mathcal{A}}$, then $\varrho \in \mathfrak{L}(\mathfrak{H})_+$ and $\parallel \varrho \parallel = \kappa < 1$. For every $\varphi \in \mathfrak{H}$, we have

$$
\begin{aligned}
\parallel \mathcal{C}_\lambda(\mathcal{L}\zeta,\mathcal{L}\ell,\iota) \parallel &= \parallel M_{|\frac{\iota|\mathcal{L}\zeta-\mathcal{L}\ell|}{\lambda}|} \parallel\\
&= \sup_{\|\varphi\|=1} (M_{|\frac{\iota|\mathcal{L}\zeta-\mathcal{L}\ell|}{\lambda}|}\varphi,\varphi)\\
&\le \sup_{\|\varphi\|=1} \int_{\mathfrak{S}} \Big[\frac{\iota}{\lambda}\Big|\int_{\mathfrak{S}} \Upsilon(\iota,\varsigma,\zeta(\varsigma))d\varsigma - \int_{\mathfrak{S}} \Upsilon(\iota,\varsigma,\ell(\varsigma))d\varsigma\Big|\Big]\varphi(r)\overline{\varphi(r)}dr\\
&\le \sup_{\|\varphi\|=1} \int_{\mathfrak{S}} \Big[\frac{\iota}{\lambda}\int_{\mathfrak{S}} |\Upsilon(\iota,\varsigma,\zeta(\varsigma)) - \Upsilon(\iota,\varsigma,\ell(\varsigma))|d\varsigma\Big]|\varphi(r)|^2dr\\
&\le \sup_{\|\varphi\|=1} \int_{\mathfrak{S}} |\varphi(r)|^2dr \ \iota\big(\kappa \parallel \frac{\zeta-\ell}{\lambda} \parallel_\infty \big)\\
&\le \kappa\big(\parallel \iota\frac{\zeta-\ell}{\lambda} \parallel_\infty \big)\\
&= \kappa\big(\parallel M_{|\frac{\iota|\zeta-\ell|}{\lambda}|} \parallel\\
&= \parallel \varrho \parallel \parallel \mathcal{C}_\lambda(\zeta,\ell,\iota) \parallel .
\end{aligned}
$$

This implies that

$$\| \mathcal{C}_\lambda(\mathcal{L}\zeta, \mathcal{L}\ell, \iota) \| \leq \| \varrho \| \| \mathcal{C}_\lambda(\zeta, \ell, \iota) \|,$$

for $\lambda, \iota > 0$. Since $\| \varrho \| < 1$, $\mathcal{L}$ is a contractive mapping and Theorem 2.11 hold for a mapping $\mathcal{L}$. Therefore, the equation (3.1) has a unique solution, that is, $\mathcal{L}$ has a unique fixed point. □

4 Conclusion

In this article, we reviewed and revised the chapter published in CRC Press [12]. To improve the previous version, definitions and assumptions needed to prove the main results have been added. In this way, we expressed the concept of η-admissiblity in C^*-algebra-valued modular parametric metric spaces for C^*-contractions and also Kannan-Ćirić C^*-contractions. In fact, we have combined the concepts of modular metric, parametric metric and C^*-algebra-valued metric spaces. Using this new space, we presented a new development of the Banach contraction principle. To confirm the new results, we provide an example and an application about the solvability of operator equations and integral equations. Our results extend and generalize the relevant results in [16, 18, 20, 22, 25, 30].

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