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Qualitative behaviour of local non-Lipschitz stochastic integrodifferential system with Rosenblatt process and infinite delay

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Abstract

The objective of this paper is to investigate the existence and uniqueness of mild solutions for stochastic integrodifferential evolution equations in Hilbert spaces with infinite delay and a Rosenblatt Process. The main results of this discussion are provided by Grimmer's resolvent operator theory and stochastic analysis. The theory is demonstrated with an example.

Keywords: C_0-semigroup, Grimmer resolvent operator, stochastic functional integrodifferential equations, Rosenblatt process
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1 Introduction

The theory of functional differential equations is intimately connected to the study of pure mathematics and the practical applications of mathematics in the real world. Different mathematical formulations of physical laws are described in functional differential equations. These equations include ordinary differential equations, partial differential equations, integral differential equations, integrodifferential equations, delay equations, and equations made by combining these different types of equations. The theory of differential equations and its applications have recently attracted great interest due to their successful modeling in many areas of science and engineering, including biomechanics, electrochemistry, financial markets, porous media, electromagnetic processes, and electrical circuits. Significant advancements have been made in the theory and applications of differential equations within this framework. For further information, refer to [1, 5, 9, 14, 22].

Noise, usually called random fluctuations, occurs frequently and predictably in both natural and artificial systems. Therefore, studying stochastic models rather than deterministic models is strongly advised. When a mathematical

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description of a particular occurrence incorporates a component of uncertainty, stochastic differential equations are used (SDEs).

In recent years, there has been a rise in interest in stochastic functional differential equations driven by fractional Brownian motion(hereafter, fBm), which has led to an increase in the amount of focus placed on these equations. We want to direct the reader to the extensive work presented in [2, 3, 12, 13, 17]. In addition, its distribution is typically Gaussian, and the calculus involved is significantly less complicated than those involved in other processes. However, if the data cannot support the Gaussianity assumption, the Rosenblatt process is a helpful tool at one's disposal and should be used whenever possible. As a direct result, the theory of the Rosenblatt process was developed to explain its remarkable qualities. Self-similarity, stationarity of the increments, long-range dependence, and other similar characteristics are some beautiful properties included here (see [10, 18, 23]). Tudor [24] gave more information, emphasizing the Rosenblatt process and the stochastic calculus that goes along with it. Maejima and Tudor [11] went on to establish some new properties within the Rosenblatt distribution after that. Recently, Shen et al. [19, 20] analyzed the stability and controllability of the stochastic functional differential equation driven by the Rosenblatt process. K. Dhanalakshmi and P. Balasubramaniam [6] explored higher-order fractional neutral stochastic differential system stability results with infinite delay driven by Poisson jumps and the Rosenblatt process.

In various fields of science, there is a growing interest in studying systems with memory or after-effects, such as the effect of infinite delay on state equations. We must discuss stochastic evolution systems with infinite delay. The theory of integrodifferential equations with resolvent operators has become an active area of research because they are used in many physical phenomena. There are few results on stochastic partial integrodifferential equations with the resolvent operator with an infinite delay. It is possible to view [4, 7, 15, 16] and the references inside them. Motivated by the previously mentioned problems, in this paper we will extend some of the results of mild solutions for the following stochastic integrodifferential equations driven by the Rosenblatt process and with infinite delay

$$\begin{cases} d\vartheta(\tau) = \left[\mathscr{A}\vartheta(\tau) + \int_0^\tau \mathscr{B}(\tau - s)\vartheta(s)ds + F(\tau, \vartheta_\tau)\right] d\tau + G(\tau, \vartheta_\tau)dW(\tau) + \sigma(\tau)dR^{\mathsf{H}}(\tau), \quad \tau \in [0, \mathsf{b}], \\ \vartheta_0(\cdot) = \varphi \in \mathscr{D}, \quad \tau \leq 0, \end{cases} \tag{1.1}$$

where the state $\vartheta(\cdot)$ takes values in a separable Hilbert space $\mathbb{V}$; $\mathscr{A}$ is the infinitesimal generator of a strongly continuous semigroup $(\mathsf{S}(\tau))_{\tau\geq 0}$ of bounded linear operators in a Hilbert space $\mathbb{V}$ with domain $D(\mathscr{A})$, $\mathscr{B}(\tau)$ is a closed linear operator on $\mathbb{V}$ with domain $D(\mathscr{B}) \supset D(\mathscr{A})$. The history $\vartheta_\tau : (-\infty, 0] \to \mathbb{V}$, $\vartheta_\tau(\theta) = \vartheta(\tau + \theta)$ for $\tau \geq 0$, belongs to the phase space $\mathscr{D}$, which will be defined in Section 2. Assume that the mappings $F : [0, \mathsf{b}] \times \mathscr{D} \to \mathbb{V}$, $G : [0, \mathsf{b}] \times \mathscr{D} \to \mathscr{L}_2^0(\mathbb{K}, \mathbb{V})$ and $\sigma : [0, \mathsf{b}] \to \mathscr{L}_2^0(\mathbb{K}, \mathbb{V})$ are appropriate functions to be specified later.The initial value φ is an $\mathscr{F}_0$-measurable $\mathscr{D}$-valued random variable independent of Rosenblatt process R^{H} and Wiener process W with the second finite moment.

The main contribution and advantage of this manuscript are listed as follows:

(i) Through the utilization of successive approximations in conjunction with the theory of resolvent operators for integrodifferential equations in the sense of Grimmer, the purpose of our paper is to investigate the solvability of (1.1) and to give findings on the existence of a mild solution to (1.1).

(ii) No study in the existing body of literature has documented stochastic integrodifferential equations with an infinite delay and a Rosenblatt process in the form of (1.1).

(iii) This work's objective is to examine such a subject to fill the gap that has been existing.

(iv) Our goal is to show that there are mild solutions to a group of stochastic integrodifferential equations with an infinite delay and a Rosenblatt process when certain local conditions are met. Additionally, we will ensure that these solutions are unique.

The rest of this paper is organized as follows: In Section 2, we briefly review the notations, concepts, and primary results concerning the Rosenblatt process and deterministic integrodifferential equations, which we utilize throughout this paper. Section 3 focuses on studying the existence and uniqueness of mild solutions for (1.1) along with their proofs, presenting the main results. An example is given in Section 4 to illustrate the results obtained.

2 Preliminaries

2.1 Rosenblatt process

In this segment, we recall the basic properties of the Rosenblatt process needed to establish our main results and the Wiener integral. For details of this section, we refer the reader to [3] and the references therein. Throughout this paper, $(\mathbb{V},\ \|\cdot\|_{\mathbb{V}}\ <\cdot,\cdot>)$ and $(\mathbb{W},\ \|\cdot\|_{\mathbb{W}}\ <\cdot,\cdot>)$ are two real separable Hilbert spaces. The notation $L^2(\Omega,\mathbb{V})$ stands for the space of all $\mathbb{V}$-valued random variables $\mathscr{G}$ such that $\mathbb{E}\parallel\mathscr{G}\parallel^2=\displaystyle\int_\Omega\parallel\mathscr{G}\parallel^2 d\mathbb{P}\ <\infty$. Let $\mathscr{L}(\mathbb{K},\mathbb{V})$ denotes the space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{V}$ and $Q\in\mathscr{L}(\mathbb{K},\mathbb{K})$ represents a non-negative self-adjoint operator. Let $\mathscr{L}_2^0=\mathscr{L}_2^0\big(\mathbb{K},\mathbb{V}\big)$ be the space of all functions $\Gamma\in(\mathbb{K},\mathbb{U})$ such that $\Gamma Q^{1/2}$ is a Hilbert-Schmidt operators. The norm is given by

$$\parallel\Gamma\parallel^2_{\mathscr{L}_Q^0}=\parallel\Gamma Q^{\frac{1}{2}}\parallel^2=Tr\big(\Gamma Q\Gamma^*\big)$$

and Γ is called a Q-Hilbert-Schmidt operator from $\mathbb{K}$ to $\mathbb{V}$.
The Wiener-Ito multiple integral of order k with respect to the standard Brownian motion $\big(W_1(\vartheta_i)\big)_{\vartheta\in\mathbb{R}}$ is given by

$$R^k_{\mathsf{H}}(\tau)=q(\mathsf{H},k)\int_{\mathbb{R}^k}\int_0^\tau\left(\prod_{j=1}^k(s-\vartheta_j)_+^{\left(-\frac{1}{2}+\frac{1-\mathsf{H}}{k}\right)}\right)ds dW_1(\vartheta_1)\cdots dW_1(\vartheta_k), \tag{2.1}$$

where $\vartheta_+=\max(\vartheta,0)$ and the constant $c(\mathsf{H},k)$ is a positive normalization constant depending only on H and k that ensures $\mathbb{E}\big(R^k_{\mathsf{H}}(1)\big)^2=1$. The process $(R^k_{\mathsf{H}}(\tau))_{\tau\geq 0}$ is called as the Hermite process and it is H self-similar in the sense that for any $c>0$, $(R^k_{\mathsf{H}}(c\tau))\stackrel{d}{=}(c^{\mathsf{H}}R^k_{\mathsf{H}}(\tau))$ and it has stationary increments.

For $k=1$, the process given by (2.1) is the fBm with Hurst parameter $\mathsf{H}\in\big(\frac{1}{2},1\big)$, further the process is not Gaussian for $k=2$. Moreover, for $k=2$, the process given by (2.1) is called the Rosenblatt process.

Consider a time interval $[0,\mathsf{b}]$ with arbitrary fixed horizon b and $\{R^{\mathsf{H}}(\tau),\tau\in[0,\mathsf{b}]\}$ the one dimensional Rosenblatt process with parameter $\mathsf{H}\in(\dfrac{1}{2},1)$, R^{H} has the following integral representation [24]

$$R_{\mathsf{H}}(\tau)=q(\mathsf{H})\int_0^\tau\int_0^\tau\left[\int_{\vartheta_1\vee\vartheta_2}^\tau\frac{\partial K^{\mathsf{H}'}}{\partial u}(u,\vartheta_1)\frac{\partial K^{\mathsf{H}'}}{\partial u}(u,\vartheta_2)du\right]dW_1(\vartheta_1)dW_1(\vartheta_2), \tag{2.2}$$

where $K^{\mathsf{H}}(\tau,s)$ is given by

$$K^{\mathsf{H}}(\tau,s)=c_{\mathsf{H}}s^{\frac{1}{2}-\mathsf{H}}\int_s^\tau(u-s)^{\mathsf{H}-3/2}u^{\mathsf{H}-1/2}du \text{ for } \tau>s,$$

with

$$c_{\mathsf{H}}=\sqrt{\frac{\mathsf{H}(2\mathsf{H}-1)}{\beta(2-2\mathsf{H},\mathsf{H}-\frac{1}{2})}},$$

$\beta(\cdot,\cdot)$ denotes the Gamma function, $K^{\mathsf{H}}(\tau,s)=0$ when $\tau\leq s$, $\{W_1(\tau),\tau\in[0,\mathsf{b}]\}$ is a Brownian motion, $\mathsf{H}'=\frac{\mathsf{H}+1}{2}$ and $q(\mathsf{H})=\frac{1}{\mathsf{H}+1}\sqrt{\frac{\mathsf{H}}{2(2\mathsf{H}-1)}}$ is a normalizing constant. The covariance of the Rosenblatt process $\{R_{\mathsf{H}}(\tau),\tau\in[0,\mathsf{b}]\}$ satisfy

$$\mathbb{E}(R_{\mathsf{H}}(\tau)R_{\mathsf{H}}(s))=\frac{1}{2}\left(s^{2\mathsf{H}}+\tau^{2\mathsf{H}}-|s-\tau|^{2\mathsf{H}}\right)$$

and this structure of $\{R_{\mathsf{H}}(\gamma)\}_{\gamma\in[0,\mathsf{b}]}$ allows to represent it as a Wiener integral. Let $R^{\mathsf{H}}_Q(\tau)$ be a $\mathbb{K}$-valued Rosenblatt process with covariance Q as

$$R^{\mathsf{H}}_Q(\tau)=R_Q(\tau)=\sum_{n=1}^\infty\sqrt{\delta_n}\xi_n(\tau)e_n,\quad t\geq 0.$$

Let $\rho:[0,\mathsf{b}]\to L^2(Q^{1/2}\mathbb{K},\mathbb{V})$ such that

$$\sum_{n=1}^\infty\|K^*_{\mathsf{H}}(\rho Q^{1/2}e_n)\|_{L^2([0,\mathsf{b}];\mathbb{V})}<\infty. \tag{2.3}$$

Definition 2.1. (Tudor[24]). Let $\rho(l):\ [0,\mathsf{b}]:\to L^2(Q^{1/2}\mathbb{K},\mathbb{V})$ satisfy (2.3). In that case, the stochastic integral of ρ with respect to the Rosenblatt process $R_Q^{\mathsf{H}}(\tau)$ is defined for $\tau\geq 0$ as follows

$$\begin{aligned}\int_0^\tau \rho(l)dR_Q^{\mathsf{H}}(l) &:= \sum_{n=1}^{\infty}\int_0^\tau \rho(s)Q^{1/2}e_n dR_n(l)\\ &= \sum_{n=1}^{\infty}\int_0^\tau\int_0^\tau (K_{\mathsf{H}}^*(\rho Q^{1/2}e_n))(\vartheta_1,\vartheta_2)dW_1(\vartheta_1)dW_1(\vartheta_2).\end{aligned}$$

Lemma 2.2. ([20]) For any $\rho:[0,\mathsf{b}]\to L^2(Q^{1/2}\mathbb{K},\mathscr{H})$ such that $\sum_{n=1}^{\infty}\|\rho Q^{1/2}e_n\|_{L^{1/\mathsf{H}}([0,\mathsf{b}];\mathbb{V})}<\infty$ holds, and for any $\alpha,\beta\in[0,\mathsf{b}]$ with $\beta>\alpha$, we have

$$\mathbb{E}\left\|\int_\alpha^\beta \rho(\tau)dR_Q(\tau)\right\|^2\leq c_{\mathsf{H}}(\beta-\alpha)^{2\mathsf{H}-1}\sum_{n=1}^{\infty}\int_\alpha^\beta \|\rho(\tau)Q^{1/2}e_n\|^2 d\tau.$$

If, in addition,

$$\sum_{n=1}^{\infty}\|\rho(\tau)Q^{1/2}e_n\| \text{ is uniformly convergent for } \tau\in[0,\mathsf{b}],$$

then, it holds that

$$\mathbb{E}\left\|\int_\alpha^\beta \rho(\tau)dR_Q(\tau)\right\|^2\leq c_{\mathsf{H}}(\beta-\alpha)^{2\mathsf{H}-1}\int_\alpha^\beta \|\rho(\tau)\|^2_{L^2(Q^{1/2}\mathbb{K},\mathbb{V})}d\tau.$$

For further references [23, 24].

2.2 Partial integrodifferential equation in Banach space

In this part, we recall some basic results about the resolvent operators for the following integro-differential equation

$$\begin{cases}\vartheta'(\tau)=\mathscr{A}\vartheta(\tau)+\displaystyle\int_0^\tau \mathscr{B}(\tau-s)\vartheta(s)ds \text{ for } \tau\geq 0\\ \vartheta(0)=\vartheta_0\in\mathbb{Y},\end{cases}\tag{2.4}$$

where $\mathscr{A}$ and $\mathscr{B}(\tau)$ are closed linear operators on $\mathbb{Y}$. Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces. $\mathscr{L}(\mathbb{X},\mathbb{Y})$ denotes the space of all bounded linear operator from $\mathbb{X}$ to $\mathbb{Y}$. To simplify, we write $\mathscr{L}(\mathbb{X})$ when $\mathbb{X}=\mathbb{Y}$. Let $\mathbb{X}$ be the Banach space $\mathscr{D}(\mathscr{A})$ equipped with the graph norm given by

$$\|\vartheta\|_{\mathbb{X}}=\|\mathscr{A}\vartheta\|+\|\vartheta\| \text{ for } \vartheta\in\mathbb{X}.$$

The notation $\mathscr{C}(\mathbb{R}^+,\mathbb{X})$ stands for the space of all continuous functions from $\mathbb{R}^+$ into $\mathbb{X}$.

Definition 2.3. [8] A bounded linear operator valued function $\mathscr{R}(\tau)\in\mathscr{L}(\mathbb{Y})$, $\tau\geq 0$ is called the resolvent operator for system (2.4) if it satisfies the following conditions:

(i) $\mathscr{R}(0)=\mathsf{I}$ and $\|\mathscr{R}(\tau)\|_{\mathscr{L}(\mathbb{Y})}\leq\tilde{M}e^{\gamma\tau}$ for some constants $\tilde{M}$ and γ;

(ii) For all $\vartheta\in\mathbb{Y}$, $\mathscr{R}(\tau)$ is strongly continuous for $\tau\geq 0$;

(iii) For $\vartheta\in\mathbb{X}$, $\mathscr{R}(\cdot)\vartheta\in\mathscr{C}^1(\mathbb{R}_+,\mathbb{Y})\cap\mathscr{C}(\mathbb{R}_+,\mathbb{X})$ and

$$\begin{aligned}\mathscr{R}'(\tau)\vartheta &= \mathscr{A}\mathscr{R}(\tau)\vartheta+\int_0^\tau \mathscr{B}(\tau-s)\mathscr{R}(s)\vartheta ds,\\ &= \mathscr{R}(\tau)\mathscr{A}\vartheta+\int_0^\tau \mathscr{R}(\tau-s)\mathscr{B}(s)\vartheta ds,\ \tau\geq 0.\end{aligned}$$

In what follows, we suppose the following assumptions.

(H1) $\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup $\{\mathsf{S}(\tau)\}_{\tau\geq 0}$.

(H2) For all $\tau \geq 0$, $\mathscr{B}(\tau)$ is a continuous linear operator from $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ into $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$. Moreover, there exists an integrable function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $\vartheta \in \mathbb{X}$, $\tau \mapsto \mathscr{B}(\tau)\vartheta$ belongs to $W^{1,1}(\mathbb{R}_+, \mathbb{Y})$ and

$$\left\| \frac{d}{d\tau}\mathscr{B}(\tau)\vartheta \right\|_{\mathbb{Y}} \leq \mu(\tau)\|\vartheta\|_{\mathbb{X}} \text{ for } \vartheta \in \mathbb{X} \text{ and } \tau \geq 0.$$

Theorem 2.4. [8] Assume that **(H1)** and **(H2)** are satisfied. Then equation (2.4) has a unique resolvent operator $(\mathscr{R}(\tau))_{\tau\geq 0}$.

Now, we present some results on the existence of solutions for the following integrodifferential equation:

$$\begin{cases} \vartheta'(\tau) = \mathscr{A}\vartheta(\tau) + \displaystyle\int_0^\tau \mathscr{B}(\tau - s)\vartheta(s)ds + \Xi(\tau) \text{ for } \tau \geq 0, \\ \vartheta(0) = \vartheta_0 \in \mathbb{Y}, \end{cases} \tag{2.5}$$

where $\Xi : \mathbb{R}_+ \to \mathbb{Y}$ is a continuous function.

Definition 2.5. A continuous function $\vartheta : [0, \infty[\to \mathbb{Y}$ is said to be a strict solution for equation (2.5) if

1. $\vartheta \in \mathscr{C}^1(\mathbb{R}_+, \mathbb{Y}) \cap \mathscr{C}(\mathbb{R}_+, \mathbb{X})$,
2. ϑ satisfies equation (2.5) for $\tau \geq 0$.

Remark 2.6. From this definition, we deduce that $\vartheta(\tau) \in \mathscr{D}(\mathscr{A})$, and the function $s \mapsto \mathscr{B}(\tau - s)\vartheta(s)$ is integrable, for all $\tau > 0$ and $s \geq 0$.

Theorem 2.7. [8] Suppose that hypotheses **(H1)** and **(H2)** hold. If ϑ is a strict solution of (2.5), then the following variation of the constants formula holds.

$$\vartheta(\tau) = \mathscr{R}(\tau)\vartheta_0 + \int_0^\tau \mathscr{R}(\tau - s)\Xi(s)ds, \text{ for } \tau \geq 0. \tag{2.6}$$

Consequently, we can establish the following definition.

Definition 2.8. [8] A function $\vartheta : \mathbb{R}_+ \to \mathbb{Y}$ is called a mild solution of (2.5) for $\vartheta_0 \in \mathbb{Y}$, if ϑ sastisfies the variation of constants formula (2.6).

Theorem 2.9. [8] Let $\Xi \in \mathscr{C}^1([0, +\infty[; \mathbb{Y})$ and ϑ be defined by (2.6). If $\vartheta_0 \in \mathscr{D}(\mathscr{A})$, then ϑ is a strict solution for equation (2.4).

In this paper, $\mathscr{D}((-\infty, 0]; L^2(\Omega, \mathbb{V}))$ (denoted by $\mathscr{D}$ simply) denotes the family of all $\mathscr{F}_0$-measurable, bounded continuous functions $\varphi : (-\infty, 0] \to L^2(\Omega, \mathbb{V})$ endowed with the norm $\|\varphi\|^2 = \sup_{\theta\in(-\infty,0]} \mathbb{E}\,\|\varphi(\theta)\|^2_{\mathbb{V}}$. Let $\mathscr{D}_{\mathscr{F}_0}((-\infty, 0]; \mathbb{V})$ denote the family of almost surely bounded, $\mathscr{F}_0$-measurable, $\mathscr{D}$-valued random variables.

Moreover, let $\mathscr{D}_{\mathsf{b}}$ denote the Banach space of all $\mathscr{F}_\tau$ adpted processes $\varphi(\tau, \omega)$ which are almost surely continuous in τ for fixed $\omega \in \Omega$ with norm $\|\varphi\|_{\mathscr{D}_{\mathsf{b}}} < \infty$, where

$$\|\varphi\|_{\mathscr{D}_{\mathsf{b}}} = \Big(\sup_{0\leq\tau\leq\mathsf{b}} \|\varphi\|^2_\tau \Big)^{\frac{1}{2}}, \qquad \text{and} \qquad \|\varphi\|_\tau = \sup_{-\infty\leq s\leq\tau} \mathbb{E}\|\varphi(s)\|.$$

3 Existence and uniqueness

This section discusses the existence and uniqueness of a mild solution for the stochastic functional equation (1.1). For this equation, we assume that the following conditions hold.

(H3) **(a)** There exists a function $\mathscr{K}: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mathscr{K}(\tau, r)$ is locally integrable in $\tau \geq 0$ for any fixed $r \geq 0$ and is continuous, monotone nondecreasing and concave in r for any fixed $\tau \in [0, \mathsf{b}]$. Moreover, for any fixed $\tau \in [0, \mathsf{b}]$ and $\varphi_1 \in \mathscr{D}$, the following inequality is satisfied:

$$\|F(\tau, \varphi_1)\|^2 + \|G(\tau, \varphi_1)\|^2_{L_2^0} \leq \mathscr{K}(\tau, \|\varphi_1\|^2_\tau).$$

(b) For any constant $\mathscr{M} > 0$, the following differential equation

$$\begin{cases} \dfrac{d\vartheta}{d\tau} = \mathscr{M}\mathscr{K}(\tau, \vartheta) \text{ for } \tau \geq 0 \\ \vartheta(0) = \vartheta_0, \end{cases}$$

has a global solution on $\mathbb{R}_+$ for any initial value $\vartheta_0 > 0$.

(H4) (global conditions)
(a) There exists a function $\mathscr{Z}: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mathscr{Z}(\tau, r)$ is locally integrable in τ for any fixed $r \geq 0$ and is continuous, monotone nondecreasing and concave in r for any fixed $\tau \in [0, \mathsf{b}]$, $\mathscr{Z}(\tau, 0) = 0$ for any $\tau \in [0, \mathsf{b}]$. Moreover, for any fixed $\tau \in [0, \mathsf{b}]$, $\varphi_1, \varphi_2 \in \mathscr{B}$, the following inequality is satisfied:

$$\|F(\tau, \varphi_2) - F(\tau, \varphi_1)\|^2_{\mathbb{V}} + \|G(\tau, \varphi_2) - G(\tau, \varphi_1)\|^2_{\mathscr{L}_2^0} \leq \mathscr{Z}(\tau, \|\varphi_2 - \varphi_1\|^2_\tau).$$

(**b**) For any constant $C > 0$, if a nonnegative, continuous function $v(\tau)$ satisfies

$$v(\tau) \leq C \int_0^\tau \mathscr{Z}(s, v(s)) ds \ \text{ for } \ \tau \in [0, \mathsf{b}],$$

then $v(\tau) \equiv 0$ for all $\tau \in [0, \mathsf{b}]$.

(H5) There exists a constant $p > 1$ such that the function $\sigma : [0, +\infty) \to \mathscr{L}_2^0$ satisfies the following

$$\int_0^{\mathsf{b}} \|\sigma(s)\|^{2p}_{\mathscr{L}_2^0} ds < \infty.$$

(H6) (**a**) (the local condition) For any integer $\eta > 0$, there exists a function $\mathscr{Z}_\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mathscr{Z}_\eta(\tau, r)$ is locally integrable in τ for any fixed $r \geq 0$ and is continuous, monotone nondecreasing and concave in r for any fixed $\tau \in [0, \mathsf{b}]$ with $\mathscr{Z}_\eta(\tau, 0) = 0$. Futhermore, the following inequality is satisfied: for any $\varphi_1, \varphi_2 \in \mathscr{D}$ with $\|\varphi_1\|_\tau \leq \eta, \|\varphi_2\|_\tau \leq \eta$, the following inequality holds:

$$\|F(\tau, \varphi_2) - F(\tau, \varphi_1)\|^2_{\mathbb{V}} + \|G(\tau, \varphi_2) - G(\tau, \varphi_1)\|^2_{\mathscr{L}_2^0} \leq \mathscr{Z}_\eta(\tau, \|\varphi_2 - \varphi_1\|^2_\tau).$$

(**b**) For any constant $C > 0$, if a nonnegative function v(t) satisfies that

$$v(\tau) \leq C \int_0^\tau \mathscr{Z}_\eta(s, v(s)) ds,$$

for all $\tau \in J$, then $v(\tau) \equiv 0$ holds for any $\tau \in [0, \mathsf{b}]$.

Remark 3.1. Let $\mathscr{Z}(\tau, \vartheta) = \beta(\tau)\bar{\mathscr{Z}}(\vartheta), \ \tau \in [0, \mathsf{b}]$, where $\beta(\tau) \geq 0$ is locally integrable and $\bar{\mathscr{Z}}(\vartheta)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\bar{\mathscr{Z}}(0) = 0$, $\bar{\mathscr{Z}}(\vartheta) > 0$ for $\vartheta > 0$ and $\displaystyle\int_{0^+} \frac{1}{\bar{\mathscr{Z}}(\vartheta)} d\vartheta = \infty$. Then, by the comparison theorem of differential equations, we know that assumption **(H4-b)** holds.
We propose now some concrete examples of the function $\bar{\mathscr{Z}}(\cdot)$. Let $\xi > 0$ and let $\delta \in (0, 1)$ be sufficient small. Define $\bar{\mathscr{Z}}_1(\vartheta) = \xi\vartheta, \quad \vartheta \geq 0$

$$\bar{\mathscr{Z}}_2(\vartheta) = \begin{cases} \vartheta \log(\vartheta^{-1}), & 0 \leq \vartheta \leq \delta, \\ \delta \log(\delta^{-1}) + \bar{\mathscr{Z}}_2'(\delta-)(\vartheta - \delta), & \vartheta > \delta, \end{cases}$$

where $\bar{\mathscr{Z}}_2'$ denotes the derivative of function $\bar{\mathscr{Z}}_2$. They are all concave nondecreasing functions satisfying $\displaystyle\int_{0^+} \frac{1}{\bar{\mathscr{Z}}_i(\vartheta)} d\vartheta = \infty \quad (i = 1, 2)$.

Now, we introduce the definition of mild solutions for problem (1.1).

Definition 3.2. An $\mathscr{F}_\tau$-adapted $\mathbb{V}$-valued stochastic process $\vartheta(\tau)$ defined on $-\infty < \tau \leq \mathsf{b}$ is called the mild solution for Eq. (1.1) if

(1) $\vartheta(\tau)$ is continuous and $\{\vartheta_\tau : \tau \in [0, \mathsf{b}]\}$ is a $\mathscr{D}$-valued stochastic process;

(2) for arbitrary $\tau \in [0, \mathsf{b}]$, $\vartheta(\tau)$ satisfies the following integral equation:

$$\begin{cases} \vartheta(\tau) = \mathscr{R}(\tau)\varphi(0) + \int_0^\tau \mathscr{R}(\tau - s)F(s, \vartheta_s)ds + \int_0^\tau \mathscr{R}(\tau - s)G(s, \vartheta_s)dW(s) + \int_0^\tau \mathscr{R}(\tau - s)\sigma(s)dR_Q^{\mathsf{H}}(s), \\ \vartheta_0(\cdot) = \varphi \in \mathscr{D}. \end{cases} \tag{3.1}$$

Next, we prove the existence and uniqueness of mild solution for (1.1).

Theorem 3.3. Assume that (**H1**) – (**H5**) are satisfied. Then the system (1.1) has a unique mild solution $\vartheta(\tau) \in \mathscr{D}_{\mathsf{b}}$.

Proof . To prove this theorem, let us introduce the following iteration procedure. Define for each integer $n = 1, 2, 3, \cdots$

$$\vartheta^n(\tau) = \mathscr{R}(\tau)\varphi(0) + \int_0^\tau \mathscr{R}(\tau - s)F(s, \vartheta_s^{n-1})ds + \int_0^\tau \mathscr{R}(\tau - s)G(s, \vartheta_s^{n-1})dW(s) + \int_0^\tau \mathscr{R}(\tau - s)\sigma(s)dR_Q^{\mathsf{H}}(s)ds, \tag{3.2}$$

and for $n = 0$, $\vartheta^0(\tau) = \mathscr{R}(\tau)\varphi(0)$, $\tau \in [0, \mathsf{b}]$, while for $n = 1, 2, 3, \cdots$, $\vartheta^n(\tau) = \varphi(\tau)$, $-\infty < \tau \leq 0$. To ensure the existence of mild solutions, we split the proof into several steps.

Step 1. For all $n \in \mathbb{N}$, $s \in (-\infty, \tau]$, $\vartheta^n(\cdot) \in \mathscr{D}_{\mathsf{b}}, n \geq 0$ is bounded. It is obvious that $\vartheta^0(\tau) \in \mathscr{D}_{\mathsf{b}}$. By elementary inequality to (3.2), for $\tau \in [0, \mathsf{b}]$, it is seen

$$\begin{aligned} \mathbb{E}\left\|\vartheta^n(s)\right\|^2 &\leq 4\mathbb{E}\left\|\mathscr{R}(s)\varphi(0)\right\|^2 + 4\mathbb{E}\left\|\int_0^s \mathscr{R}(s - r)F(r, \vartheta_r^{n-1})dr\right\|^2 + 4\mathbb{E}\left\|\int_0^s \mathscr{R}(s - r)G(r, \vartheta_r^{n-1})dW(r)\right\|^2 \\ &\quad + 4\mathbb{E}\left\|\int_0^s \mathscr{R}(s - r)\sigma(r)dR_Q^{\mathsf{H}}(r)\right\|^2 \\ &=: 4\sum_{i=1}^4 I_i. \end{aligned} \tag{3.3}$$

Thus, by (**H3**), one has

$$I_1 \leq M^2\mathbb{E}\left\|\varphi(0)\right\|^2. \tag{3.4}$$

From (**H3**) and Hölder's inequality, the following relation holds:

$$\begin{aligned} I_2 &\leq M^2\mathbb{E}\Big(\int_0^s F(r, \vartheta_r^{n-1})dr\Big)^2 \\ &\leq M^2\Big(\int_0^s dr\Big)\mathbb{E}\int_0^s \mathscr{K}(r, \|\vartheta^{n-1}\|_r^2)dr \\ &\leq M^2\mathsf{b}\mathbb{E}\int_0^s \mathscr{K}(r, \|\vartheta^{n-1}\|_r^2)dr. \end{aligned} \tag{3.5}$$

By (**H3**), Hölder's inequality, Doob's martingale inequality and hypothesis (**H3**), we know that there exists a positive constant $\mathscr{C}_1$ such that

$$I_3 \leq M^2\mathsf{b}\mathscr{C}_1\mathbb{E}\int_0^s \mathscr{K}(r, \|\vartheta^{n-1}\|_r^2)dr. \tag{3.6}$$

For I_4, by Lemma 2.2 and assumption **(H5)**, we obtain for $p>1$,

$$\begin{aligned} I_4 &\leq c_{\mathsf{H}}\mathsf{H}(2\mathsf{H}-1)\mathsf{b}^{2\mathsf{H}-1}M^2\int_0^s \|\sigma(r)\|^2_{\mathscr{L}_2^0}dr \\ &\leq M^2 c_{\mathsf{H}}\mathsf{H}(2\mathsf{H}-1)\mathsf{b}^{2\mathsf{H}-1+1-\frac{1}{p}}\left(\int_0^\tau \|\sigma(r)\|^{2p}_{\mathscr{L}_2^0}dr\right)^{\frac{1}{p}} \\ &\leq M^2 c_{\mathsf{H}}\mathsf{H}(2\mathsf{H}-1)\mathsf{b}^{2\mathsf{H}-\frac{1}{p}}\left(\int_0^{\mathsf{b}} \|\sigma(r)\|^{2p}_{\mathscr{L}_2^0}dr\right)^{\frac{1}{p}} \\ &< \infty. \end{aligned} \tag{3.7}$$

Hence, substituting (3.4) – (3.7) into (3.3) yields

$$\mathbb{E}\left\|\vartheta^n(s)\right\|^2 \leq \Delta_0+\Delta_1\mathbb{E}\int_0^s \mathscr{K}(r,\|\vartheta^{n-1}\|_r^2)dr,$$

where we have used the notation

$$\Delta_0 = M^2\mathbb{E}\left\|\varphi(0)\right\|^2 + M^2 c_{\mathsf{H}}\mathsf{H}(2\mathsf{H}-1)\mathsf{b}^{2\mathsf{H}+1-\frac{1}{p}}\left(\int_0^{\mathsf{b}} \|\sigma(r)\|^{2p}_{\mathscr{L}_2^0}dr\right)^{\frac{1}{p}}$$

and $\Delta_1=\mathsf{b}M^2(1+\mathscr{C}_1)$. Then

$$\begin{aligned} \mathbb{E}\|\vartheta^n\|_\tau^2 &= \sup_{-\infty<s\leq\tau}\mathbb{E}\|\vartheta^n(s)\|^2 \\ &\leq \mathbb{E}\sup_{-\infty<\theta\leq 0}\|\varphi(\theta)\|^2+\Delta_0+\Delta_1\mathbb{E}\int_0^s\mathscr{K}(r,\|\vartheta^{n-1}\|_r^2)dr \\ &\leq \Delta_2+\Delta_1\mathbb{E}\int_0^\tau\mathscr{K}(s,\|\vartheta_s^{n-1}\|_s^2)ds, \end{aligned}$$

where $\Delta_2=\mathbb{E}\sup_{-\infty<\theta\leq 0}\|\varphi(\theta)\|^2+\Delta_0$. Using Jensen's inequality, we obtain

$$\mathbb{E}\|\vartheta^n\|_\tau^2\leq \Delta_2+\Delta_1\int_0^\tau\mathscr{K}(s,\mathbb{E}\|\vartheta^{n-1}\|_s^2)ds.$$

From assumption (**H3-b**) there is a solution z_τ which satisfies

$$z_\tau = 2\Delta_2+2\Delta_1\int_0^\tau\mathscr{K}(r,z_r)dr.$$

Since $\mathbb{E}\|\vartheta^0(\tau)\|\leq M^2\|\varphi(0)\|<\infty$, it reads $\mathbb{E}\|\vartheta^n\|_\tau^2\leq z_\tau\leq z_{\mathsf{b}}<\infty$. Therefore $\{\vartheta^n(\tau),n\geq 0\}$is uniformly bounded, and Step 1 is then fulfilled.

Step 2. We claim that $\{\vartheta^n(\tau),n\geq 0\}$ is a Cauchy sequence. For all $n,m\geq 0$ and $\tau\in J$, from (3.3), we have

$$\mathbb{E}\left\|\vartheta^{n+1}(s)-\vartheta^{m+1}(s)\right\|^2\leq 2\mathbb{E}\left\|\int_0^s\mathscr{R}(s-r)\Big(F(r,\vartheta_r^n)-F(r,\vartheta_r^m)\Big)dr\right\|^2+2\mathbb{E}\left\|\int_0^s\mathscr{R}(s-r)\Big(G(r,\vartheta_r^n)-G(r,\vartheta_r^m)\Big)dW(r)\right\|^2$$

By **(H4)** and Burkhölder-Davis-Gundy's inequality, there exists a positive constant $\mathscr{C}_2$ such that

$$\mathbb{E}\left\|\vartheta^{n+1}(s)-\vartheta^{m+1}(s)\right\|^2\leq 2M^2\mathsf{b}(1+\mathscr{C}_2)\mathbb{E}\int_0^s\mathscr{Z}\big(r,\|\vartheta_r^n-\vartheta_r^m\|_r^2\big)dr.$$

Then, by Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}\left\|\vartheta^{n+1}-\vartheta^{m+1}\right\|_\tau^2 &= \sup_{-\infty<s\leq\tau}\mathbb{E}\left\|\vartheta^{n+1}(s)-\vartheta^{m+1}(s)\right\|^2 \\ &\leq \Delta_3\int_0^\tau\mathscr{Z}\big(s,\mathbb{E}\|\vartheta^{n+1}-\vartheta^{m+1}\|_s^2\big)ds, \end{aligned} \tag{3.8}$$

where $\Delta_3 = 2M^2\mathsf{b}(1+\mathscr{C}_2)$. By (5.1) and Fatou's Lemma, we have

$$\lim_{n,m\to\infty}\left(\sup_{0\le s\le \mathsf{b}} \mathbb{E}\left\|\vartheta^{n+1}(s)-\vartheta^{m+1}(s)\right\|^2\right) \le \Delta_3\int_0^\tau \mathscr{Z}\left(s, \lim_{n,m\to\infty}\left(\sup_{0\le\theta\le s}\mathbb{E}\left\|\vartheta^{n+1}(\theta)-\vartheta^{m+1}(\theta)\right\|^2\right)\right)ds.$$

By assumption (**H4-b**) we obtain

$$\lim_{n,m\to\infty}\sup_{0\le s\le \mathsf{b}}\mathbb{E}\left\|\vartheta^{n+1}(s)-\vartheta^{m+1}(s)\right\|^2 = 0.$$

This implies that $\{\vartheta^n, n\ge 0\}$ is Cauchy in $\mathscr{D}_{\mathsf{b}}$.

Step 3. The completeness of $\mathscr{B}_{\mathsf{b}}$ guarantees the existence of a process $\vartheta \in \mathscr{B}_{\mathsf{b}}$, such that

$$\lim_{n\to\infty}\sup_{0\le s\le \mathsf{b}}\mathbb{E}\left\|\vartheta^{n+1}(s)-\vartheta(s)\right\|^2 = 0.$$

Hence, letting $n\to\infty$ and taking limits on both sides of (3.3), we obtain that $\vartheta(\tau)$ is a solution to (1.1). The proof existence is complete. Appendix A shows uniqueness. Hence, the Theorem 3.3 is completed. □

Theorem 3.4. Assume that **(H1)**, **(H2)**, **(H3)** and **(H6)** hold. Then the system (1.1) has a unique mild solution $\vartheta(\tau)\in\mathscr{D}_{\mathsf{b}}$.

Proof . Let η be a natural integer and let $\mathsf{b}_0\in(0,\mathsf{b})$. We define the sequence of functions $\{F_\eta(\tau,v)\}$ and $\{G_\eta(\tau,v)\}$ for $(\tau,v)\in[0,\mathsf{b}_0]\times\mathbb{V}$ as follows:

$$F_\eta(\tau,v)=\begin{cases} F(\tau,v) & \text{if } \|v\|\le\eta,\\ F(\tau,\dfrac{\eta v}{\|v\|}) & \text{if } \|v\|>\eta,\end{cases} \qquad \text{and} \qquad G_\eta(\tau,v)=\begin{cases} G(\tau,v) & \text{if } \|v\|\le\eta,\\ G(\tau,\dfrac{\eta v}{\|v\|}) & \text{if } \|v\|>\eta.\end{cases}$$

Then, the functions F_η and G_η satisfy assumption **(H3)**, and the following inequality hold:

$$\|F_\eta(\tau,\vartheta)-F_\eta(\tau,v)\|^2+\|G_\eta(\tau,\vartheta)-G_\eta(\tau,v)\|^2_{L_2^0}\le \mathscr{Z}_\eta(\tau,\|\vartheta-v\|^2_\tau),$$

for any $\vartheta, v\in\mathscr{B}$, $\tau\in[0,\mathsf{b}_0]$. So, by Theorem 3.3, there exist the unique mild solutions $\vartheta_\eta(\tau)$ and $\vartheta_{\eta+1}(\tau)$, respectively to the following integral equations:

$$\begin{cases}\vartheta_\eta(\tau)=\mathscr{R}(\tau)\vartheta_0+\displaystyle\int_0^\tau\mathscr{R}(\tau-s)F_\eta(s,\vartheta_\eta(s))ds+\int_0^\tau\mathscr{R}(\tau-s)G_\eta(s,\vartheta_\eta(s))dW(s)\\ \qquad\qquad +\displaystyle\int_0^\tau\mathscr{R}(\tau-s)\sigma(s)dR_Q^{\mathsf{H}}(s)ds,\quad \tau\in[0,\mathsf{b}],\\ \vartheta^\eta(\tau)=\varphi(\tau),\quad \tau\le 0,\end{cases} \tag{3.9}$$

$$\begin{cases}\vartheta^{\eta+1}(\tau)=\mathscr{R}(\tau)\vartheta_0+\displaystyle\int_0^\tau\mathscr{R}(\tau-s)F_{\eta+1}(s,\vartheta_{\eta+1}(s))ds+\int_0^\tau\mathscr{R}(\tau-s)G_{\eta+1}(s,\vartheta_{\eta+1}(s))dW(s)\\ \qquad\qquad +\displaystyle\int_0^\tau\mathscr{R}(\tau-s)\sigma(s)dR_Q^{\mathsf{H}}(s)ds\quad \tau\in[0,\mathsf{b}],\\ \vartheta^{\eta+1}(\tau)=\varphi(\tau),\quad \tau\le 0.\end{cases} \tag{3.10}$$

Define the stopping times

$$\begin{aligned}\sigma_\eta &:= \mathsf{b}_0\wedge\inf\{\tau\in[0,\mathsf{b}]:\|\vartheta_\eta(\tau)\|\ge\eta\},\\ \sigma_{\eta+1} &:= \mathsf{b}_0\wedge\inf\{\tau\in[0,\mathsf{b}]:\|\vartheta_{\eta+1}(\tau)\|\ge\eta+1\},\\ \kappa_\eta &:= \sigma_\eta\wedge\sigma_{\eta+1}.\end{aligned}$$

In view of (3.9) and (3.10), we obtain

$$\begin{aligned}\mathbb{E}\|\vartheta_{\eta+1}(s)-\vartheta_\eta(s)\|^2 &\leq 2\mathbb{E}\left\|\int_0^s \mathscr{R}(s-r)[F_{\eta+1}(r,\vartheta_{\eta+1}(s))-F_\eta(r,\vartheta_\eta(r))]dr\right\|^2\\ &\quad +2\mathbb{E}\left\|\int_0^s \mathscr{R}(s-r)[G_{\eta+1}(r,\vartheta_{\eta+1}(r))-G_\eta(r,\vartheta_\eta(r))]dW(r)\right\|^2\\ &= 2\sum_{i=1}^2 I_i,\end{aligned}$$

which we have used the fact that for $0 \leq r \leq \tau_\eta$,

$$F_{\eta+1}(r,\vartheta_\eta(r)) = F_\eta(r,\vartheta_\eta(r)),\ \ G_{\eta+1}(r,\vartheta_\eta(r)) = G_\eta(r,\vartheta_\eta(r)).$$

Employing assumption (**H6**) and Hölder's inequality, it follows that

$$\begin{aligned}\sup_{0\leq s\leq \tau\wedge\kappa_\eta} I_1 &\leq M^2\mathsf{b}\mathbb{E}\sup_{0\leq s\leq \tau\wedge\kappa_\eta}\int_0^s \|F_{\eta+1}(r,\vartheta_{\eta+1}(r))-F_\eta(r,\vartheta_\eta(r))\|^2\,dr\\ &\leq M^2\mathsf{b}\sup_{0\leq s\leq \tau\wedge\kappa_\eta}\mathbb{E}\int_0^s \mathscr{Z}_{\eta+1}\left(r,\|\vartheta_{\eta+1}-\vartheta_\eta\|_r^2\right)dr.\end{aligned}$$

Combining Burkhölder-Davis-Gundy's inequality, there exist a positive constant K_1 such that

$$\begin{aligned}\sup_{0\leq s\leq \tau\wedge\kappa_\eta} I_2 &\leq K_1M^2\sup_{0\leq s\leq \tau\wedge\kappa_\eta}\mathbb{E}\int_0^s \left\|\left[G_{\eta+1}(r,\vartheta_{\eta+1}(r))-G_\eta(r,\vartheta_\eta(r))\right]\right\|_{\mathscr{L}_2^0}^2 dr\\ &\leq K_1M^2\sup_{0\leq s\leq \tau\wedge\kappa_\eta}\mathbb{E}\left(\int_0^s \mathscr{Z}_{\eta+1}\left(r,\|\vartheta_{\eta+1}-\vartheta_\eta\|_r^2\right)dr\right).\end{aligned}\tag{3.11}$$

Therefore, we have

$$\sup_{0\leq s\leq \tau\wedge\kappa_\eta}\mathbb{E}\left\|\vartheta_{\eta+1}(s)-\vartheta_\eta(s)\right\|^2 \leq \Delta_4\sup_{0\leq s\leq \tau\wedge\kappa_\eta}\mathbb{E}\left(\int_0^s \mathscr{Z}_{\eta+1}\left(r,\|\vartheta_{\eta+1}-\vartheta_\eta\|_r^2\right)dr\right)$$

where $\Delta_4 = M^2(\mathsf{b}+K_1)$. Then for all $\tau \in [0,\mathsf{b}_0]$, by Jensen's inequality, we have

$$\sup_{-\infty<s\leq \tau\wedge\kappa_\eta}\mathbb{E}\left\|\vartheta_{\eta+1}(s)-\vartheta_\eta(s)\right\|^2 \leq C_4\int_0^s \mathscr{Z}_{\eta+1}(r,\sup_{-\infty<r\leq s\wedge\kappa_\eta}\mathbb{E}\|\vartheta_{\eta+1}-\vartheta_\eta\|_r^2)ds.$$

The assumption **(H6)** indicates that

$$\sup_{-\infty<s\leq \tau\wedge\kappa_\eta}\mathbb{E}\|\vartheta_{\eta+1}(s)-\vartheta_\eta(s)\|^2 = 0.$$

Thus, for a.e. ω,

$$\vartheta_{\eta+1}(\tau) = \vartheta_\eta(\tau),\quad \text{for }\ 0\leq \tau\leq \mathsf{b}_0\wedge\kappa_\eta.$$

Note that for each $\omega \in \Omega$, there exists an $\eta_0(\omega) > 0$ such that $0 < \mathsf{b}_0 \leq \kappa_{\eta_0}$. Define $\vartheta(\tau)$ by $\vartheta(\tau) = \vartheta_0\eta_0(\tau)$ for $\tau \in [0,\mathsf{b}_0]$. Since $\vartheta(\tau\wedge\kappa_\eta) = \vartheta_\eta(\tau\wedge\kappa_\eta)$, it holds that

$$\begin{aligned}\vartheta(\tau\wedge\kappa_\eta) &= \mathscr{R}(\tau\wedge\kappa_\eta)\varphi(0) + \int_0^{\tau\wedge\kappa_\eta}\mathscr{R}(\tau\wedge\kappa_\eta-s)F_\eta(s,\vartheta_\eta(s))ds + \int_0^{\tau\wedge\kappa_\eta}\mathscr{R}(\tau-s)G_\eta(s,\vartheta_\eta(s))dW(s)\\ &\quad + \int_0^{\tau\wedge\kappa_\eta}\mathscr{R}(\tau\wedge\kappa_\eta-s)\sigma(s)dR_Q^{\mathsf{H}}(s)ds\end{aligned}$$

$$
\begin{aligned}
&= \mathscr{R}(\tau\wedge\kappa_\eta)\varphi(0)+\int_0^{\tau\wedge\kappa_\eta}\mathscr{R}(\tau\wedge\kappa_\eta-s)F(s,\vartheta(s))ds+\int_0^{\tau\wedge\kappa_\eta}\mathscr{R}(\tau-s)G(s,\vartheta(s))dW(s)\\
&\quad+\int_0^{\tau\wedge\kappa_\eta}\mathscr{R}(\tau\wedge\kappa_\eta-s)\sigma(s)dR_Q^{\mathsf{H}}(s)ds. \qquad (3.12)
\end{aligned}
$$

Taking $\eta\to\infty$, we have

$$
\vartheta(\tau) = \mathscr{R}(\tau)\varphi(0)+\int_0^{\tau}\mathscr{R}(\tau-s)F(s,\vartheta(s))ds+\int_0^{\tau}\mathscr{R}(\tau-s)G(s,\vartheta(s))dW(s)+\int_0^{\tau}\mathscr{R}(\tau-s)\sigma(s)dR_Q^{\mathsf{H}}(s)ds,
$$

which completes the proof. □

4 Example

This section presents an example for illustrating Theorem 3.4. For that, we consider the following stochastic partial functional integrodifferential equation:

$$
\begin{cases}
\dfrac{\partial}{\partial\tau}\beta(\tau,\xi)=\dfrac{\partial^2}{\partial\xi^2}\beta(\tau,\xi)+\displaystyle\int_0^{\tau}\varLambda(\tau-s)\dfrac{\partial^2}{\partial\xi^2}\beta(s,\xi)ds+\lambda(\tau)F_1\left(\beta(\tau-r,\xi)\right)\beta(\tau-r,\xi)d\tau+\gamma_1\beta(\tau-r,\xi)dW(t)\\
\qquad\qquad+\sigma(\tau)dR^{\mathsf{H}}(\tau), r>0, \xi\in[0,\pi],\\
\beta(\tau,0)=\beta(\tau,\pi)=0, \quad \tau\in[0,\mathsf{b}],\\
\beta(0,\xi)=\beta_0(\theta,\xi), \quad \theta\in(-\infty,0], \xi\in[0,\pi],
\end{cases}
\tag{4.1}
$$

where, $\varLambda : \mathbb{R}^+\to\mathbb{R}$ is a continuous function, $\gamma_1>0$ $W(\tau)$ denotes a $\mathbb{R}$-valued Wiener process, R^{H} a Rosenblatt process, $\lambda^2(\tau)>0$ is a locally integrable function, $F_1:\mathbb{R}^+\to\mathbb{R}$ is a bounded continuous functions, $\varLambda\in\mathscr{C}^1(\mathbb{R}^+,\mathbb{R})$ and $\beta_0:[0,\pi]\to\mathbb{R}$ is a given function such that $\beta_0(\cdot)\in L^2(0,\pi)$, is $\mathscr{F}_0$-measurable and satisfies $\mathbb{E}\left\|\beta_0\right\|^2<\infty$.

Let $\mathbb{V}=L^2(0,\pi)$ with the norm $\|\cdot\|$ and $e_n:=\sqrt{\frac{2}{\pi}}\sin(nx)$, $(n=1,2,3,\ldots)$ denote a complete orthonormal basis in $\mathbb{V}$.We assume that there exists the product $\vartheta_1\vartheta_2\in\mathbb{V}$ for $\vartheta_1,\vartheta_2\in\mathbb{V}$. We note that there exists an $M\geq 1$ such that $|\vartheta_1\vartheta_2|\leq M\|\vartheta_1\|\|\vartheta_2\|$. Let $W(\tau):=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\delta_n(\tau)e_n$ $(\lambda_n>0)$, where $\delta_n(\tau)$ are one-dimensional standard Brownian motion mutually independent of an usual complete probability space $(\Omega,\mathscr{F},\{\mathscr{F}_\tau\}_{\tau\geq 0},\mathbb{P})$.

Define $\mathscr{A}:D(\mathscr{A})\subset\mathbb{V}\to\mathbb{V}$ by $\mathscr{A}=\frac{\partial^2}{\partial z^2}$, with domain $D(\mathscr{A})=H^2(0,\pi)\cap H_0^1(0,\pi)$. Then

$$
\mathscr{A}\hat{h}=-\sum_{n=1}^{\infty}n^2<\hat{h},e_n>e_n, \quad \hat{h}\in D(\mathscr{A}),
$$

where e_n, $n=1,2,3,\cdots$, is also the orthonormal set of eigenvectors of $\mathscr{A}$. It is well-known that $\mathscr{A}$ is the infinitesimal generator of a strongly continuous semigroup $\mathsf{S}(\tau)$ on $\mathbb{V}$, given by

$$
\mathsf{S}(\tau)\hat{h}=\sum_{n=1}^{\infty}e^{-n^2\tau}<\hat{h},e_n>e_n \text{ for } \hat{h}\in\mathbb{V},
$$

which is compact. Let $\mathscr{B}:D(\mathscr{A})\subset\mathbb{V}\to\mathbb{V}$ be the operator defined by

$$
\mathscr{B}(\tau)(\tilde{z})=\varLambda(\tau)\mathscr{A}\tilde{z} \text{ for } \tau\geq 0 \text{ and } \tilde{z}\in D(\mathscr{A}).
$$

Let ϕ be as follows :

$$
\phi(x)=\begin{cases}0, x=0,\\ cx\left(\log\frac{1}{x}\right), 0<x\leq\delta,\\ c\delta\left(\log\frac{1}{\delta}\right), x>\delta,\end{cases}
$$

or

$$
\phi(x)=\begin{cases}0, x=0,\\ cx\left(\log\frac{1}{x}\right)^{\frac{1}{3}}\log\log\frac{1}{x}, 0<x\leq\delta,\\ c\delta\left(\log\frac{1}{\delta}\right)^{\frac{1}{3}}\log\log\frac{1}{\delta}, x>\delta,\end{cases}
$$

with $c > 0$ and $0 < \delta < 1$ is sufficiently small. In order to rewrite Eq. (4.1) in an abstract form in $\mathbb{V}$, we introduce the following notation

$$\begin{cases} X(\tau) &= \beta(\tau,\xi) \quad \text{for} \quad \tau \geq 0 \quad \text{and} \quad \xi \in [0,\pi], \\ \varphi(\theta)(\xi) &= \beta_0(\theta,\xi), \quad \theta \in (-\infty,0], \xi \in [0,\pi]. \end{cases}$$

Assume that:

1. $|F_1(z_2) - F_1(z_1)|^2 \leq \phi\left(|z_2 - z_1|^2\right), z_1, z_2 \in \mathbb{R}$.
2. There exists a constant $p > 1$ such that the function $\sigma : [0,+\infty) \to \mathscr{L}_2^0$ satisfies the following

$$\int_0^{\mathsf{b}} \|\sigma(s)\|_{\mathscr{L}_2^0}^{2p} ds < \infty.$$

For $t \geq 0$, $\xi \in [0,\pi]$ and ψ a $\mathscr{D}$-valued function, define the operators $F, G : [0,\mathsf{b}] \times \mathscr{D} \to \mathbb{V}$ by

$$F(\tau,\psi)(\xi) = \lambda(\tau)F_1(\psi(-r))(\xi)\psi(-r)(\xi), \tag{4.2}$$

$$G(\tau,\psi)(\xi) = \lambda_1\psi(-r)(\xi). \tag{4.3}$$

Then, for any $\varphi \in \mathscr{D}$, equation (4.1) takes the following abstract form

$$\begin{cases} dX(\tau) = [\mathscr{A}X(\tau) + \int_0^\tau \mathscr{B}(\tau - s)X(s)ds]d\tau + F(\tau,X_\tau)d\tau + G(\tau,X_\tau)dW(\tau) + \sigma(\tau)dR^{\mathsf{H}}(\tau) \\ X(0) = x_0 \in \mathbb{H}. \end{cases} \tag{4.4}$$

Moreover, if Λ is bounded and $\mathscr{C}^1$-function such that Λ' is bounded and uniformly continuous, then (**H2**) is satisfied and hence, by Theorem 2.4, Eq. (2.4) has a resolvent operator $(\mathscr{R}(\tau))_{\tau\geq 0}$ on $\mathbb{V}$. Since $F_1(\vartheta)$ is a bounded function, there exists $B_1 > 0$ such that $|F(\vartheta)| \leq B_1$ for any $\vartheta \in \mathbb{V}$. For any nonnegative real number $r \in \mathbb{R}^+$ we set

$$\Phi_\eta(r) := 2M^2\eta^2\phi(r), \tag{4.5}$$

$$\Psi(r) := \left(2M^2B_1^2 + \gamma_1^2 trace(Q)\right) r. \tag{4.6}$$

We define $\rho(\tau)$ by

$$\rho(\tau) := \begin{cases} 1 \text{ if } \lambda(\tau) \leq 1, \\ \lambda^2(\tau) \text{ if } \lambda(\tau) > 1. \end{cases}$$

Then $\rho(\tau)$ is a locally integrable function. And we have that

$$|\lambda(\tau)F_1(\vartheta)\vartheta - \lambda(\tau)F_1(u)u|^2 + |\gamma_1\vartheta - \gamma_1 u|_{L_2^0}^2 \leq \rho(\tau)\bigg(\Phi_\eta\left[|\vartheta - u|^2\right] + \Psi\left[|\vartheta - u|^2\right]\bigg), \ \vartheta, u \in \mathbb{V}.$$

On the other hand, set

$$a := \left(2M^2B_1^2 + \gamma_1^2 trace(Q)\right).$$

Since ϕ is a concave function, it follows that $\phi(r) \geq \phi(1)r$ for $0 \leq r < 1$. Thus it holds that

$$\begin{aligned} \int_{0^+}^{+\infty} \frac{1}{\Phi_\eta(r) + \Psi(r)} dr &= \int_{0^+}^{+\infty} \frac{1}{2M^2\eta^2\phi(r) + ar} dr \\ &\geq \frac{\phi(1)}{(a + 2M^2\eta^2\phi(1))} \int_{0^+}^{+\infty} \frac{1}{\phi(r)} dr \\ &= \infty. \end{aligned}$$

Thus by the example of Lemma 3 [[21], p.157] we have that **(H5)** holds. Therefore, the proof of the example is complete by Theorem 3.4.

5 Conclusion

Our article focuses on whether or not mild solutions to local non-Lipschitz stochastic integrodifferential equations with Rosenblatt processes and infinite delays exist and whether or not these solutions are unique. The primary findings of our study are deduced from various theoretical frameworks, including resolvent operator theory in the sense of Grimmer and stochastic analysis. The approximate controllability of these equations will be the topic of our next paper.

Appendix A

Proof . (Proof of uniqueness)
Let $\vartheta_1,\ \vartheta_2$ be two solutions of Equation (1.1). Then the uniqueness is obvious on the interval $]-\infty,0]$, and for $\tau\in[0,\mathsf{b}]$ by similar analysis of Equation(3.3), elementary inequality, Burkhölder-Davis-Gundy's inequality and hypothesis **(H4)**, it is easy to obtain

$$\mathbb{E}\,\|\vartheta_2(s)-\vartheta_1(s)\|^2\leq 2M^2\mathsf{b}(1+\mathscr{C}_2)\mathbb{E}\int_0^s\mathscr{Z}\big(r,\|\vartheta_2-\vartheta_1\|_r^2\big)dr.$$

Then, by Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}\,\|\vartheta_2(s)-\vartheta_1(s)\|_\tau^2 &= \sup_{-\infty<s\leq\tau}\mathbb{E}\,\|\vartheta_2(s)-\vartheta_1(s)\|^2\\ &\leq \Delta_3\int_0^\tau\mathscr{Z}\big(s,\mathbb{E}\|\vartheta_2-\vartheta_1\|_s^2\big)ds,\end{aligned}\tag{5.1}$$

which, with the aid of **(H4)** $-\,b$, gives

$$\mathbb{E}\,\|\vartheta_2(s)-\vartheta_1(s)\|_\tau^2=0,\ \ 0\leq\tau\leq\mathsf{b}.$$

Therefore $\vartheta_2(s)=\vartheta_1(s)$ for all $0\leq\tau\leq\mathsf{b}$. Hence the uniqueness is proved. □

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