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Fixed point for α_*-ψ-β_i-contractive set-valued mappings on Branciari S_b-metric space

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Abstract

In 1984, Khan et al. established some fixed point theorems in complete and compact metric spaces by altering distance functions. In 2020, Lotfy et al. introduced the α_*-ψ-common rational type mappings on generalized metric spaces applied to fractional integral equations. In 2022, Roy et al. described the notion of Branciari S_b-metric space and related fixed point theorems with an application. In this paper, we introduce the notion of fixed point theorems for α_*-ψ-β_i-contractive set-valued mappings on Branciari S_b-metric space with application to fractional integral equations.

Keywords: α_*-ψ-β_i-contractive, Branciari S_b-metric space, Fixed points, Fractional integral equations
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1 Introduction

We know that the fixed point theory has many applications and was extended by several authors from different views (see for example [1]-[22]). Samet et al [20] introduced the notion of α-ψ-contractive type mappings. Hassanzadeh Asl et al [11, 12] introduced the notion of common fixed point theorems for α_*-ψ-contractive multifunction. Farajzadeh et al [8] introduced the fixed point theorems for (ξ, α, η)-expansive mappings in complete metric spaces. Gungor et al established fixed point theorems on orthogonal metric spaces via altering distance functions. Lotfy et al [16] introduced the notion of α_*-ψ-common rational type mappings on generalized metric spaces with application to fractional integral equations. Roy et al [18] described the notion of Branciari S_b-metric space and related fixed point theorems with an application. This paper aims to introduce the notion of fixed point theorems for α_*-ψ-β_i-contractive set-valued mappings on Branciari S_b-metric space with application to fractional integral equations.

2 Preliminaries

In this section, we list some fundamental definitions that are useful tool in consequent analysis. Let 2^X denote the family of all nonempty subsets of X.

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Definition 2.1. ([15]) A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:
(ψ_1) $\psi(0) = 0$ and $\psi(t) > 0$ for all $t \in (0, +\infty)$;
(ψ_2) ψ is continuous and no-decreasing;
(ψ_3) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$;
(ψ_4) $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$;
for all $t_1, t_2 \in (0, +\infty)$.

These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) \leq t$ for all $t > 0$. We denote Ψ as the set of altering distance function ψ. The extended line is the ordered space $[-\infty; +\infty]$, considering of all points of the number line $\mathbb{R}$ and two points, denoted by $-\infty, +\infty$ with the usual order relation for points of $\mathbb{R}$.

Definition 2.2. ([4, 7]) Let X be a nonempty set and $\rho : X \times X \to [0, \infty]$ be a mapping. Then ρ is said to be a rectangular metric if it satisfies the following conditions, for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y:
$(GMS1)$ $\rho(x, y) = 0$ if and if $x = y$;
$(GMS2)$ $\rho(x, y) = \rho(y, x)$ for any points $x, y \in X$;
$(GMS3)$ $\rho(x, y) \leq \rho(x, u) + \rho(u, v) + \rho(v, y)$ for any points $x, y, u\&v \in X$ considering that if $d(x, u) = \infty$ or $\rho(u, v) = \infty$ or $d(v, y) = \infty$ then $\rho(x, u) + \rho(u, v) + d(v, y) = \infty$.

In this case the map ρ is called a generalized and abbreviated as GM. Here, the pair (X, ρ) is called a rectangular metric space and abbreviated as GMS. There are several rectangular metric spaces which are not usual metric spaces. Let us recall the following example.

In the above definition, if ρ satisfies only $GMS1$ and $GMS2$, then it is called a semi-metric.

Example 2.3. ([13]) Let $U = \{0, 2\}$, $V = \{\frac{1}{n} : n \geq 1$ and $X = U \cup V\}$.
Define $\rho : X^2 \to [0, \infty]$ by

$$\rho(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 1 & \text{if} \quad x \neq y \quad \text{and either} \quad x, y \in U \quad \text{or} \quad x, y \in V, \\ y & \text{if} \quad x \in U \quad \text{and} \quad y \in V, \\ x & \text{if} \quad x \in V \quad \text{and} \quad y \in U. \end{cases}$$

Then ρ is a rectangular metric on X but not an usual metric space.

$$\rho(0, 2) = 1 > \rho(0, \frac{1}{3}) + \rho(\frac{1}{3}, 2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Sedghi et al. [21] introduced a new type of metric structure consisting of three variables known as S-metric. Subsequently Souayah and Mlaiki [22] investigated the notion of S_b-metric spaces which generalized the concept of S-metric spaces.

Definition 2.4. ([19, 21]) Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, t \in X$:
(i) $S(x, y, z) = 0$ if and if $x = y = z$;
(ii) $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.
The pair (X, S) is called an S-metric space.

Example 2.5. ([21])
(1) Let $\mathbb{R}$ be the real line and $X = \mathbb{R}^n$ and $||.||$ a norm on X. Then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an S-metric on X.
(2) Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric on $\mathbb{R}$ is called the usual S-metric on $\mathbb{R}$.

Definition 2.6. ([17, 22]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $S_b : X^3 \to [0, \infty)$ is said to be S_b-metric if and if for all $x, y, z, t \in X$: the following conditions hold:

(i) $S_b(x,y,z)=0$ if and if $x=y=z$;
(ii) $S_b(x,y,z)\le s[S_b(x,x,t)+S_b(y,y,t)+S_b(z,z,t)]$.

The pair (X,S_b) is called an S_b-metric space.

Example 2.7. ([22]) Let X be a nonempty set and card$(X)\ge 5$. suppose $X=X_1\cup X_2$ a partition of X such that card$(X_1)\ge 4$. Let $s\ge 1$, then

$$S_b(x,y,z)=\begin{cases} 0 & \text{if} \quad x=y=z, \\ 5 & \text{if} \quad x=1=y \quad \text{and} \quad z=2, \\ \frac{1}{n+1} & \text{if} \quad x=1=y \quad \text{and} \quad z\ge 3, \\ \frac{1}{n+2} & \text{if} \quad x=2=y \quad \text{and} \quad z\ge 3, \\ 3 & \quad \text{otherwise.} \end{cases}$$

for all $x,y,z,t\in X$. Then S_b is an S_b-metric on X with coefficient s.

Definition 2.8. ([18]) Let X be a nonempty set and $\lambda : X^3\to \mathbb{R}_0^+$ be a function. Then λ is said to be Branciari S_b-metric if it satisfies the following condition:
(i) $\lambda(x,y,z)=0$ if and if $x=y=z$;
(ii) for any $x,y,z\in X$ and for $a,b\in X\setminus\{x,y,z\}$ with $a\ne b$ we have

$$\lambda(x,y,z)\le k[\lambda(x,x,a)+\lambda(y,y,a)+\lambda(z,z,b)+\lambda(a,b,b)] \tag{2.1}$$

where $k\ge 1$. The pair (X,λ) is called Branciari S_b-metric space.

Definition 2.9. ([18]) A Branciari S_b-metric on a nonempty set X is said to be symmetric if $\lambda(x,x,y)=\sigma(y,y,x)$ for all $x,y\in X$.

Proposition 2.10. ([18]) (i) Let (X,S) be an S-metric spaces (see definition (2.4)). The X is also a Branciari S_b-metric space for $k=2$.
(ii) Let (X,S_b) be an S_b-metric space with coefficient $s\ge 1$ (see definition (2.8)). The X is also a Branciari S_b-metric space for $k=2s^2$.

Proposition 2.11. ([18]) Any S-metric space or S_b-metric space is also a Branciari S_b-metric space but there are several Branciari S_b-metric spaces which are neither S-metric spaces nor S_b-metric spaces.

Example 2.12. ([18]) Let $X=\mathbb{N}$ and $\lambda : X^3\to\mathbb{R}_0^+$ be defined by

$$\lambda(x,y,z)=\begin{cases} 0 & \text{if} \quad x=y=z, \\ 5 & \text{if} \quad x=1=y \quad \text{and} \quad z=2, \\ \frac{1}{n+1} & \text{if} \quad x=1=y \quad \text{and} \quad z\ge 3, \\ \frac{1}{n+2} & \text{if} \quad x=2=y \quad \text{and} \quad z\ge 3, \\ 3 & \quad \text{otherwise.} \end{cases}$$

for all $x,y,z,t\in X$. Also we take $\lambda(x,x,y)=\lambda(y,y,x)$ for all $x,y\in X$. Then λ is a symmetric S_b-metric space on X for $k=\frac{5}{3}$ but it is nether an S-metric nor an S_b-metric for any $k\ge 1$.

Definition 2.13. ([18]) Let (X,λ) be a Branciari S_b-metric space. Then
(i) A sequence $\{x_n\}$ in X is said to be Branciari convergent to some $z\in X$ if $\lambda(x_n,x_n,z)\to 0$ as $n\to\infty$.
(ii) A sequence $\{x_n\}$ in X is said to be Branciari cauchy if $\lambda(x_n,x_n,x_m)\to 0$ as $n,m\to\infty$.
(ii) X is said to be Branciari complete if every Branciari cauchy sequence in X is Branciari convergent to some element in X.

Definition 2.14. We say that (X,λ) has the property $\alpha-$regular Branciari S_b-metric space if, either

(i) $\{x_n\}$ is a monotone Branciari sequences in X such that $\alpha(x_n,x_n,x_{n+1})\ge 1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then there exists a Branciari subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x_{n_k},x)\ge 1$ for all k.

or

(ii) $\{x_n\}$ is a monotone Branciari sequences in X such that $\alpha(x_{n+1},x_{n+1},x_n)\ge 1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then there exists a Branciari subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x,x,x_{n_k})\ge 1$ for all k.

Definition 2.15. Let (X, λ) be a Branciari S_b-metric spaces. If $T : X \to 2^X$ is a set-valued mapping, then $x \in X$ is called fixed point for T if and only if $x \in F(x)$. The set

$$Fix(T) := \{x \in X \text{ such that } \ x \in Tx\}$$

is called the fixed point set of T.

Proposition 2.16. ([14, 7]) Suppose that $\{x_n\}$ is a Branciari Cauchy sequence in a (X, λ) be a Branciari S_b-metric space with $\lim_{n\to\infty} \lambda(x_n, x_n, u) = 0$ where $u \in X$. Then

$$\lim_{n\to\infty} \lambda(x_n, x_n, z) = \lambda(u, u, z)$$

for all $z \in X$. In particular, the Branciari sequence $\{x_n\}$ dose not Branciari converge to z if $z \neq u$.

Definition 2.17. Let (X, λ) be a Branciari S_b-metric space. A set-valued mapping $T : X \to 2^X$ is called Branciari order closed if for monotone Branciari sequences $x_n \in X$ and $y_n \in Tx_n$, with $x_n \to x$ and $y_n \to y$, implies $y \in Tx$.

Definition 2.18. Let (X, λ) be a Branciari S_b-metric space and $T : X \to 2^X$ with given set-valued, $\alpha : X \times X \times X \to [0, +\infty)$, $\alpha_* : 2^X \times 2^X \times 2^X \to [0, +\infty)$, $\alpha_*(A, A, B) = \inf\{\alpha(a, a, b) : a \in A, b \in B\}$, $\psi \in \Psi$, $\Lambda(s, s, Ts) = \inf\{\lambda(s, s, z)/z \in Ts\}$, H_λ is the Hausdorff metric

$$H_\lambda(Tx, Tx, Ty) = \max\{\sup_{a\in Tx} \Lambda(a, a, Ty), \sup_{b\in Ty} \Lambda(Tx, Tx, b)\}.$$

$\beta_i : \mathbb{R}^+ - \{0\} \to [0, 1)$ be four decreasing functions such that $\sum_{i=1}^{4} \beta_i(t) \leq 1$ for every $t > 0$. One says that T is α_*-ψ-β_i-contractive set-valued mappings whenever

$$\begin{aligned}\alpha_*(Tx, Tx, Ty)\psi(H_\lambda(Tx, Tx, Ty)) \leq & \beta_1(\lambda(x, x, y))\psi(\lambda(x, x, y)) + \beta_2(\lambda(x, x, y))\psi(\Lambda(x, x, Tx)) \\ & + \beta_3(\lambda(x, x, y))\psi(\Lambda(y, y, Ty)) + \beta_4(\lambda(x, x, y)) \min\{\psi(\Lambda(x, x, Ty), \psi(\Lambda(y, y, Tx))\}.\end{aligned} \tag{2.2}$$

One says that T are an α_* admissible if

$$\alpha(x, x, y) \geq 1 \Rightarrow \alpha_*(Tx, Tx, Ty) \geq 1 \tag{2.3}$$

for all $x, y \in X$.

Definition 2.19. A subset $B \subseteq X$ is said to be an approximation if for each given $y \in X$, there exists $z \in B$ such that $\Lambda(B, B, y) = \lambda(z, z, y)$.

Definition 2.20. A set-valued mapping $T : X \longrightarrow 2^X$ is said to have an approximate values in X if Tx is an approximation for each $x \in X$.

3 Main result

Some fixed point theorems in symmetric Branciari S_b-metric space.

Theorem 3.1. Let (X, λ) be a complete symmetric Branciari S_b-metric space (not necessarily complete metric space), $T : X \to 2^X$ is α_*-ψ-β_i-Branciari contractive set-valued mappings satisfies the following conditions:

(i) T is α_*-admissible;

(ii) there exists $x_0 \in X$ such that

$$\alpha_*(\{x_0\}, \{x_0\}, T\{x_0\}) \geq 1, \alpha_*(\{x_0\}, \{x_0\}, T^2\{x_0\}) \geq 1;$$

(iii) (X, λ) has the property $\alpha-$regular Branciari S_b-metric space.

Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated Branciari sequences $\{x_n\}$ with $x_{n+1} \in Tx_n$ Branciari converges to the fixed point of T.

Proof . Let $x_0 \in X$ such that $\alpha_*(\{x_0\},\{x_0\},Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 > 1$, then $x^* = x_{n_0}$ are a fixed point for T. So, we can assume that $x_n \notin Tx_n$ for all $n \in \mathbb{N}_0$. Since T is α_*-admissible, we have

$$\alpha(x_0,x_0,x_1) \geq \alpha_*(\{x_0\},\{x_0\},Tx_0) \geq 1 \Rightarrow \alpha_*(Tx_0,Tx_0,Tx_1) \geq 1;$$

$$\alpha(x_1,x_1,x_2) \geq \alpha_*(Tx_0,Tx_0,Tx_1) \geq 1 \Rightarrow \alpha_*(Tx_1,Tx_1,Tx_2) \geq 1;$$

$$\alpha(x_2,x_2,x_3) \geq \alpha_*(Tx_1,Tx_1,Tx_2) \geq 1 \Rightarrow \alpha_*(Tx_2,Tx_2,Tx_3) \geq 1.$$

Inductively, we have

$$\alpha(x_n,x_n,x_{n+1}) \geq 1 \Rightarrow \alpha_*(Tx_n,Tx_n,Tx_{n+1}) \geq 1$$

for all $n \in \mathbb{N}_0$. Similarly, we have

$$\alpha(x_0,x_0,x_2) \geq \alpha_*(\{x_0\},\{x_0\},T^2x_0) \geq 1 \Rightarrow \alpha_*(Tx_0,Tx_0,Tx_2) \geq 1;$$

$$\alpha(x_1,x_1,x_3) \geq \alpha_*(Tx_0,Tx_0,Tx_2) \geq 1 \Rightarrow \alpha_*(Tx_1,Tx_1,Tx_3) \geq 1;$$

$$\alpha(x_2,x_2,x_4) \geq \alpha_*(Tx_1,Tx_1,Tx_3) \geq 1 \Rightarrow \alpha_*(Tx_2,Tx_2,Tx_4) \geq 1.$$

Inductively, we have

$$\alpha(x_n,x_n,x_{n+2}) \geq 1 \Rightarrow \alpha_*(Tx_n,Tx_n,Tx_{n+2}) \geq 1$$

for all $n \in \mathbb{N}_0$. Without loss of generality, we may assume that $T : X \to 2^X$ be a α_*-ψ-β_i-contractive set-valued mappings. Consider equation (2.2), with $x = x_{2n+1}$ and $y = x_{2n+2}$. Clearly, we have

$$\begin{aligned}
\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2})) \leq & \alpha_*(Tx_{2n},Tx_{2n},Tx_{2n+1})\psi(H_\lambda(Tx_{2n},Tx_{2n},Tx_{2n+1})) \\
\leq & \beta_1(\lambda(x_{2n},x_{2n},x_{2n+1}))\psi(\lambda(x_{2n},x_{2n},x_{2n+1})) + \beta_2(\Lambda(x_{2n},x_{2n},Tx_{2n}))\psi(\Lambda(x_{2n},x_{2n},Tx_{2n})) \\
& + \beta_3(\Lambda(x_{2n+1},x_{2n+1},Tx_{2n+1}))\psi(\Lambda(x_{2n+1},x_{2n+1},Tx_{2n+1})) \\
& \beta_4(H_\lambda(Tx_{2n},Tx_{2n},Tx_{2n+1}))\min\{\psi(\Lambda(x_{2n},x_{2n},Tx_{n+1}),\psi(\Lambda(x_{2n+1},x_{2n+1},Tx_{2n}))\} \\
\leq & \beta_1(\lambda(x_{2n},x_{2n},x_{2n+1}))\psi(\lambda(x_{2n},x_{2n},x_{2n+1})) + \beta_2(\lambda(x_{2n},x_{2n},x_{2n+1}))\psi(\lambda(x_{2n},x_{2n},x_{2n+1})) \\
& + \beta_3(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2})) \\
& \beta_4(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))\min\{\psi(\lambda(x_{2n},x_{2n},x_{2n+2}),\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+1}))\} \\
\leq & \beta_1(\lambda(x_{2n},x_{2n},x_{2n+1}))\psi(\lambda(x_{2n},x_{2n},x_{2n+1})) + \beta_2(\lambda(x_{2n},x_{2n},x_{2n+1}))\psi(\lambda(x_{2n},x_{2n},x_{2n+1})) \\
& + \beta_3(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))
\end{aligned} \tag{3.1}$$

Then

$$\begin{aligned}
&(1-\beta_3(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2})) \leq \\
&\beta_1(\lambda(x_{2n},x_{2n},x_{2n+1})) + \beta_2(\lambda(x_{2n},x_{2n},x_{2n+1}))\psi(\lambda(x_{2n},x_{2n},x_{2n+1}))
\end{aligned} \tag{3.2}$$

and

$$\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2})) \leq \frac{(\beta_1(\lambda(x_{2n},x_{2n},x_{2n+1})) + \beta_2(\lambda(x_{2n},x_{2n},x_{2n+1})))}{(1-\beta_3(\lambda(x_{2n+1},x_{2n+1},x_{2n+2}))}\psi(\lambda(x_{2n},x_{2n},x_{2n+1})) \tag{3.3}$$

Thus

$$\psi(\lambda(x_{2n+1},x_{2n+1},x_{2n+2})) \leq \psi(\lambda(x_{2n},x_{2n},x_{2n+1})). \tag{3.4}$$

for all $n \in \mathbb{N}_0$. Similarly,

$$\psi(\lambda(x_{2n},x_{2n},x_{2n+1})) \leq \psi(\lambda(x_{2n-1},x_{2n-1},x_{2n})). \tag{3.5}$$

for all $n \in \mathbb{N}_0$. We have

$$\psi(\lambda(x_{n+1},x_{n+1},x_{n+2})) \leq \psi(\lambda(x_n,x_n,x_{n+1})) \leq \ldots \leq \psi^n(\lambda(x_0,x_0,x_1)), \tag{3.6}$$

for all $n \in \mathbb{N}$. From the property of ψ, we conclude that

$$\lambda(x_n,x_n,x_{n+1}) \leq \lambda(x_{n-1},x_{n-1},x_n), \tag{3.7}$$

for all $n \in \mathbb{N}$, it is clear that

$$\lim_{n\to\infty} \lambda(x_{n+1},x_{n+1},x_{n+2}) = 0. \tag{3.8}$$

Consider equation (2.2), with $x = x_{2n-1}$ and $y = x_{2n+1}$. Clearly, we have

$$\begin{aligned}\psi(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})) \leq&\alpha_*(Tx_{2n-2}, Tx_{2n-2}, Tx_{2n})\psi(H_\lambda(Tx_{2n-2}, Tx_{2n-2}, Tx_{2n}))\\ \leq&\beta_1(\lambda(x_{2n-2}, x_{2n-2}, x_{2n}))\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n}))\\ &+ \beta_2(\Lambda(x_{2n-2}, x_{2n-2}, Tx_{2n-2}))\psi(\Lambda(x_{2n-2}, x_{2n-2}, Tx_{2n-2}))\\ &+ \beta_3(\Lambda(x_{2n}, x_{2n}, Tx_{2n}))\psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n}))\\ &+ \beta_4(H_\lambda(Tx_{2n-2}, Tx_{2n-2}, Tx_{2n}))\min\{\psi(\Lambda(x_{2n-2}, x_{2n-2}, Tx_{2n}), \psi(\Lambda(x_{2n}, x_{2n}, Tx_{2n-2}))\}\\ \leq&\beta_1(\lambda(x_{2n-2}, x_{2n-2}, x_{2n}))\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n}))\\ &+ \beta_2(\lambda(x_{2n-2}, x_{2n-2}, x_{2n-1}))\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n-1}))\\ &+ \beta_3(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\psi(\lambda(x_{2n}, x_{2n}, x_{2n+1}))\\ &+ \beta_4(\lambda(x_{2n-1}, x_{2n-1}, x_{2n+1}))\min\{\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n+1}), \psi(\lambda(x_{2n}, x_{2n}, x_{2n-1}))\}\end{aligned} \tag{3.9}$$

Define $a_{2n} = \lambda(x_{2n-1}, x_{2n-1}, x_{2n+1})$ and $b_{2n} = \lambda(x_{2n}, x_{2n}, x_{2n+1})$. Then

$$\psi(a_{2n}) \leq \beta_1(a_{2n-1})\psi(a_{2n-1}) + \beta_2(b_{2n-1})\psi(b_{2n-1}) + \beta_3(b_{2n})\psi(b_{2n}) + \beta_4(a_{2n})\min\{\psi(\lambda(x_{2n-2}, x_{2n-2}, x_{2n+1}), \psi(b_{2n-1})\}. \tag{3.10}$$

From the (3.8) $\lim_{n\to\infty} b_{2n} = \lim_{n\to\infty} \lambda(x_{2n}, x_{2n}, x_{2n+1}) = 0$. We get

$$\psi(a_{2n}) \leq \beta_1(a_{2n-1})\psi(a_{2n-1}) \leq \psi(a_{2n-1}) \tag{3.11}$$

and hence,

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} d(x_{2n-1}, x_{2n+1}) = 0 \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \lambda(x_{n-1}, x_{n-1}, x_{n+1}) = 0.$$

Now, we shall prove that $x_n \neq x_m$ for all $n \neq m$. Assume on the contrary that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $n \neq m$. Since $d(x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1, m = 2k$ and $n = 2l$ for $k, l \in \mathbb{N}$. Substitute again $x = x_{2l} = x_{2k}$ and $y = x_{2l+1} = x_{2k+1}$ in (2.2), (3.7) which yields

$$\begin{aligned}\psi(\lambda(x_{2l}, x_{2l}, x_{2l+1})) =&\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\\ \leq&\alpha_*(H_\lambda(Tx_{2k-1}, Tx_{2k-1}, Tx_{2k}))\psi(H_\lambda(Tx_{2k-1}, Tx_{2k-1}, Tx_{2k}))\\ \leq&\beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\\ &+ \beta_2(\Lambda(x_{2k-1}, x_{2k-1}, Tx_{2k-1}))\psi(\Lambda(x_{2k-1}, x_{2k-1}, Tx_{2k-1}))\\ &+ \beta_3(\Lambda(x_{2k}, x_{2k}, Tx_{2k}))\psi(\Lambda(x_{2k}, x_{2k}, Tx_{2k}))\\ &\beta_4(H_\lambda(Tx_{2k}, Tx_{2k}, Tx_{2k-1}))\min\{\psi(\Lambda(x_{2k}, x_{2k}, Tx_{2k-1}), \psi(\Lambda(x_{2k-1}, x_{2k-1}, Tx_{2k}))\}\\ \leq&\beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\\ &+ \beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\\ &+ \beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\\ &\beta_4(\lambda(x_{2k+1}, x_{2k+1}, x_{2k}))\min\{\psi(\lambda(x_{2k}, x_{2k}, x_{2k}), \psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k+1}))\}\\ \leq&\beta_1(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\\ &+ \beta_2(\lambda(x_{2k-1}, x_{2k-1}, x_{2k}))\psi(\lambda(x_{2k-1}, x_{2k-1}, x_{2k})) + \beta_3(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\psi(\lambda(x_{2k}, x_{2k}, x_{2k+1}))\end{aligned} \tag{3.12}$$

which is impossible. From this it follows that $x_n \neq x_m$ for all $n, m \quad (n \neq m) \in \mathbb{N}$.

Case I: Suppose that $S_n = \lambda(x_n, x_n, x_{n+1})$, $\psi(S_n) = \alpha_n S_n$ and $\alpha \in (0, \frac{1}{\sqrt{k}})$. Then

$$\begin{aligned}&S_n = \lambda(x_n, x_n, x_{n+1}) \leq \psi(\lambda(x_{n-1}, x_{n-1}, x_n)) = \alpha_{n-1}\lambda(x_{n-1}, x_{n-1}, x_n)\\ &\leq \alpha_{n-1}\psi(\lambda(x_{n-2}, x_{n-2}, x_{n-1})) \leq \cdots \leq \alpha_{n-1}\alpha_{n-2}\cdots\alpha_1\alpha_0\lambda(x_0, x_0, x_1) = \alpha^n S_0\end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned}&S_n^* = \lambda(x_n, x_n, x_{n+2}) \leq \psi(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) = \alpha_{n-1}\lambda(x_{n-1}, x_{n-1}, x_{n+1})\\ &\leq \alpha_{n-1}\psi(\lambda(x_{n-2}, x_{n-2}, x_n)) \leq \cdots \leq \alpha_{n-1}\alpha_{n-2}\cdots\alpha_1\alpha_0\lambda(x_0, x_0, x_1) = \alpha^n S_0^*\end{aligned} \tag{3.14}$$

for all $n \geq 1$ and $\alpha = \max_{0 \leq i \leq n-1}\{\alpha_i\}$. Now, we shall prove that $\{x_n\}$ is a Branciari Cauchy sequence, that is,

$$\lim_{n\to\infty} \lambda(x_n, x_n, x_{n+l}) = 0,$$

for all $l \in \mathbb{N}$. We have already proved the cases for $l = 1$ and $l = 2$ in (3.7) and (3.10), respectively. Now for $l = 2m+1$, where $m \geq 1$. Using the inequality (2.1), we have

$$\begin{aligned}
&\lambda(x_n, x_n, x_{n+l}) \leq k[\lambda(x_n, x_n, x_{n+1}) + \lambda(x_n, x_n, x_{n+1}) + \lambda(x_{n+l}, x_{n+l}, x_{n+2})\\
&+\lambda(x_{n+1}, x_{n+1}, x_{n+2})]\\
&= 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+l}, x_{n+l}, x_{n+2}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2})]\\
&\overset{Symmetric}{=} 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + k\lambda(x_{n+2}, x_{n+2}, x_{n+l})\\
&\leq 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + k(k[\lambda(x_{n+2}, x_{n+2}, x_{n+3})\\
&+\lambda(x_{n+2}, x_{n+2}, x_{n+3}) + \lambda(x_{n+l}, x_{n+l}, x_{n+4}) + \lambda(x_{n+3}, x_{n+3}, x_{n+4})])\\
&\overset{Symmetric}{=} 2k\lambda(x_n, x_n, x_{n+1}) + k\lambda(x_{n+1}, x_{n+1}, x_{n+2}) + 2k^2\lambda(x_{n+2}, x_{n+2}, x_{n+3})\\
&+k^2\lambda(x_{n+3}, x_{n+3}, x_{n+4}) + k^2\lambda(x_{n+4}, x_{n+4}, x_{n+2m+1})\\
&\leq \cdots\\
&\vdots\\
&\leq 2k[\lambda(x_n, x_n, x_{n+1}) + \lambda(x_{n+1}, x_{n+1}, x_{n+2})] + 2k^2[\lambda(x_{n+2}, x_{n+2}, x_{n+3})\\
&+\lambda(x_{n+3}, x_{n+3}, x_{n+4})]\\
&+ \cdots + 2k^m[\lambda(x_{n+2m-2}, x_{n+2m-2}, x_{n+2m-1}) + \lambda(x_{n+2m-1}, x_{n+2m-1}, x_{n+2m})]\\
&+k^m\lambda(x_{n+2m}, x_{n+2m}, x_{n+2m+1})\\
&\leq 2[\{k(\alpha_0^n + \alpha_0^{n+1}) + k^2(\alpha_0^{n+2} + \alpha_0^{n+3}) + \cdots + k^m(\alpha_0^{n+2m-2} + \alpha_0^{n+2m-1})\}\\
&+k^m\alpha_0^{n+2m}]S_0 = 2k(1+\alpha_0)\alpha_0^n[1 + k\alpha_0^2 + \cdots + k^m\alpha_0^{2m}]S_0 = \tfrac{2k(1+\alpha_0)}{1+k\alpha_0^2}\alpha_0^n S_0
\end{aligned} \tag{3.15}$$

for all $n \geq 1$. Also for $l = 2m$ we get

$$\lambda(x_n, x_n, x_{n+2m}) \leq \cdots \leq \tfrac{2k(1+\alpha_0)}{1+k\alpha_0^2}\alpha_0^n S_0 + \alpha_0^n(k\alpha^2)^{m-1}S_0^* \tag{3.16}$$

for all $n \geq 1$. Thus we proved that $\{x_n\}$ is a Branciari Cauchy sequence in the complete metric space (X, λ), there exists $x^* \in X$ such that

$$\lim_{n\to\infty} \lambda(x_n, x_n, x^*) = 0$$

by (X, λ) has the property $\alpha-$regular Branciari S_b-metric space. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha_*(\{x_{2n_k+1}\}, \{x_{2n_k+1}\}, \{x^*\}) \geq \alpha_*(Tx_{2n_k}, Tx_{2n_k}, Tx^*) \geq 1 \text{ for all } k. \tag{3.17}$$

Thus

$$\begin{aligned}
\psi(\Lambda(x^*, x^*, Tx^*)) \leq&\psi(\lambda(x^*, x^*, x_{2n_k+1})) + \psi(\Lambda(x_{2n_k+1}, x_{2n_k+1}, Tx^*))\\
\leq&\psi(\lambda(x^*, x^*, x_{2n_k+1})) + \alpha_*(Tx_{2n_k}, Tx_{2n_k}, Tx^*)\psi(H_\lambda(Tx_{2n_k}, Tx_{2n_k}, Tx^*))\\
\leq&\psi(\lambda(x^*, x^*, x_{2n_k+1})) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\lambda(x_{2n_k}, x_{2n_k}, x^*))\\
&+ \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx_{2n_k})) + \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*))\\
&\beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*))\min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*), \psi(\Lambda(x^*, x^*, Tx_{2n_k}))\}\\
\leq&\psi(\lambda(x^*, x^*, x_{2n_k+1})) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\lambda(x_{2n_k}, x_{2n_k}, x^*))\\
&+ \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1}))\psi(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1})) + \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*))\\
&\beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*))\min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*), \psi(\lambda(x^*, x^*, x_{2n_k+1}))\}\\
\leq&\psi(0) + \beta_1(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(0) + \beta_2(\lambda(x_{2n_k}, x_{2n_k}, x_{2n_k+1}))\psi(0)\\
&+ \beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*))\beta_4(\lambda(x_{2n_k}, x_{2n_k}, x^*))\min\{\psi(\Lambda(x_{2n_k}, x_{2n_k}, Tx^*), \psi(0)\}\\
\leq&\beta_3(\lambda(x_{2n_k}, x_{2n_k}, x^*))\psi(\Lambda(x^*, x^*, Tx^*))\\
\leq&\psi(\Lambda(x^*, x^*, Tx^*))
\end{aligned} \tag{3.18}$$

for all k, which is impossible. Hence, $\Lambda(x^*, x^*, Tx^*) = \Lambda(Tx^*, Tx^*, x^*) = 0$ and so $x^* \in Tx^*$.

Case-(II): $\alpha \in [\frac{1}{\sqrt{k}}, 1)$. Then there exists $N \in \mathbb{N}$ such that $\alpha^N \in (\frac{1}{\sqrt{k}}, 1)$. Now due to the contractive condition (2.2) we see that also satisfies the contractive condition (2.2) for the Lipschitz constant therefore by Case-(I) T^N has a fixed point in X and thus in this case also T has a fixed point. □

Example 3.2. ([18]) Let $X = \mathbb{N}$ and $\lambda : X^3 \to \mathbb{R}_0^+$ be defined $\lambda(x,x,x) = 0$ and $\lambda(x,x,y) = \lambda(y,y,x)$ for all $x,y \in X$ with

$$\lambda(x,y,z) = \begin{cases} 10 & \text{if} \quad x = 1 = y \quad \text{and} \quad z = 2, \\ \frac{1}{2(n+1)} & \text{if} \quad x = 1 = y \quad \text{and} \quad z \geq 3, \\ \frac{1}{n+2} & \text{if} \quad x = 2 = y \quad \text{and} \quad z \geq 3, \\ 5 & \text{otherwise.} \end{cases}$$

Then λ is a complete symmetric Branciari S_b-metric space on X for $k = 4$ but it is nether an S-metric nor an S_b-metric for any $k \geq 1$. Let $T : X \to 2^X$ be

$$Tx = \begin{cases} \{3,4\} & \text{if} \quad x \in \{1,2\}, \\ \{5,6\} & \text{otherwise.} \end{cases}$$

Then T^2 satisfies the contractive condition (2.2) for any $\psi(x) = \frac{x^2}{1+x^2}$ and thus T^2 has a fixed point in X. Therefore T has a fixed point $x = 5$ in X.

Corollary 3.3. ([18]) (Analogue to Banach Contraction Theorem) Let (X, λ) be a complete symmetric Branciari S_b-metric space and $T : X \to X$ satisfies

$$\lambda(Tx,Tx,Ty) \leq \alpha(\lambda(x,x,y))$$

for all $x, y \in X$, where $\alpha \in (0,1)$. Then T has a unique fixed point in X.

Example 3.4. Let $X = \mathbb{Z}$ and $Y \subseteq X$ be a finite set defined as $Y = \{1,2,4,8\}$. Define $\lambda : Y \times Y \times Y \to [0,\infty)$ as:
$\lambda(1,1,1) = \lambda(2,2,2) = \lambda(4,4,4) = \lambda(8,8,8) = 0$,
$\lambda(1,1,2) = \lambda(2,2,1) = 3$,
$\lambda(2,2,8) = \lambda(8,8,2) = \lambda(1,1,8) = \lambda(8,8,1) = 1$ and
$\lambda(1,1,4) = \lambda(4,4,1) = \lambda(2,2,4) = \lambda(4,4,2) = \lambda(8,8,4) = \lambda(4,4,8) = \frac{1}{2}$.
The function λ is not a metric on Y. Indeed, note

$$3 = \lambda(1,1,2) \geq \lambda(1,1,8) + \lambda(8,8,2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However, λ is a Branciari S_b-metric on Y and moreover (Y, λ) is a complete Branciari S_b-metric space. Define $T : Y \to 2^Y$ as: $T1 = T2 = T8 = \{2,4\}, T4 = \{1,8\}$ and $T1 = T2 = T4 = \{2,8\}, T8 = \{1,2\}$, $\alpha : Y \times Y \times Y \to [0,+\infty), \alpha_* = \inf \alpha$ as $\alpha(x,x,y) = \alpha(y,y,x)) = 1$ $\psi(t) = \frac{2}{3}t$. Clearly, T satisfies the conditions of Theorem (3.1) and has a fixed point $x = 2$.

3.1 Analogue to Kannan fixed point theorem

Theorem 3.5. (Analogue to Kannan fixed point theorem) Let (X, λ) be complete symmetric Branciari S_b-metric space and $T : X \to 2^X$ satisfies

$$\alpha_*(Tx,Tx,Ty)H_\lambda(Tx,Tx,Ty) \leq \beta_1(\lambda(x,x,y))\psi_1(\Lambda(x,x,Tx)) + \beta_2(\lambda(y,y,x))\psi_2(\Lambda(y,y,Ty)) \tag{3.19}$$

for all $x, y \in X$ where $\psi_i \in \Psi$ and $\sum_{i=1}^{2} \beta_i(\lambda(x,x,y)) \in (0, \frac{1}{2})$. Then T has a fixed point in X.

Proof . Let $x_0 \in X$ be taken as arbitrary and let us construct the sequence $\{x_n\}$ in X by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 > 1$, then $x^* = x_{n_0}$ are a fixed point for T. So, we can assume that $x_n \notin Tx_n$ for all $n \in \mathbb{N}_0$. Here we show that $\{x_n\}$ is Cauchy sequence in X.

Case$-I$: $\sum_{i=1}^{2} \beta_i(\lambda(x,x,y)) \in (0, \frac{1}{k+1})$. From the contraction condition (3.19), we get

$$\begin{aligned} \lambda(x_n,x_n,x_{n+1}) \leq & \alpha_*(Tx_{n-1},Tx_{n-1},Tx_n)H_\lambda(Tx_{n-1},Tx_{n-1},Tx_n) \\ \leq & \beta_1(\lambda(x_{n-1},x_{n-1},x_n))\psi_1(\Lambda(x_{n-1},x_{n-1},Tx_{n-1})) + \beta_2(\lambda(x_n,x_n,x_{n-1}))\psi_2(\Lambda(x_n,x_n,Tx_n)) \\ \leq & \beta_1(\lambda(x_{n-1},x_{n-1},x_n))\psi_1(\lambda(x_{n-1},x_{n-1},x_n)) + \beta_2(\lambda(x_n,x_n,x_{n-1}))\psi_2(\lambda(x_n,x_n,x_{n+1})) \\ \leq & \beta_1(\lambda(x_{n-1},x_{n-1},x_n))\lambda(x_{n-1},x_{n-1},x_n) + \beta_2(\lambda(x_n,x_n,x_{n-1}))\lambda(x_n,x_n,x_{n+1}) \end{aligned}$$

for all $n \geq 1$. From which we get

$$S_n = \lambda(x_n, x_n, x_{n+1}) \leq \frac{\beta_1}{1-\beta_2}\lambda(x_{n-1}, x_{n-1}, x_n) = \gamma\lambda(x_{n-1}, x_{n-1}, x_n) = \gamma S_{n-1} \leq \cdots \leq \gamma^n S_0$$

for all $n \in \mathbb{N}$, where $\gamma = \frac{\beta_1}{1-\beta_2} < \frac{1}{k}$. Also we have,

$$\begin{aligned}
S_n^* = \lambda(x_n, x_n, x_{n+2}) \leq & \alpha_*(Tx_{n-1}, Tx_{n-1}, Tx_{n+1})H_\lambda(Tx_{n-1}, Tx_{n-1}, Tx_{n+1}) \\
\leq & \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1}))\psi_1(\Lambda(x_{n-1}, x_{n-1}, Tx_{n-1})) \\
& + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1}))\psi_2(\Lambda(x_{n+1}, x_{n+1}, Tx_{n+1})) \\
\leq & \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1}))\psi_1(\lambda(x_{n-1}, x_{n-1}, x_n)) \\
& + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1}))\psi_2(\lambda(x_{n+1}, x_{n+1}, x_{n+2})) \\
\leq & \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1}))\lambda(x_{n-1}, x_{n-1}, x_n) + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1}))\lambda(x_{n+1}, x_{n+1}, x_{n+2}) \\
\leq & \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1}))S_{n-1} + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1}))S_{n+1} \\
\leq & \beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1}))\gamma^{n-1}S_0 + \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1}))\gamma^{n+1}S_0 \\
\leq & \beta(\lambda(x_{n-1}, x_{n-1}, x_{n+1}))\gamma^{n-1}S_0 + \beta(\lambda(x_{n+1}, x_{n+1}, x_{n-1}))\gamma^{n+1}S_0 \\
\leq & \beta(\lambda(x_{n-1}, x_{n-1}, x_{n+1}))[\gamma^{n-1} + \gamma^{n+1}]S_0 \\
= & \beta(\lambda(x_{n-1}, x_{n-1}, x_{n+1}))[1 + \gamma^2]\gamma^{n-1}S_0
\end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\beta(\lambda(x_{n-1}, x_{n-1}, x_{n+1})) = \max\{\beta_1(\lambda(x_{n-1}, x_{n-1}, x_{n+1})), \beta_2(\lambda(x_{n+1}, x_{n+1}, x_{n-1}))\}.$$

Show that x_n is Cauchy sequence in X and therefore due to the completeness of X there exist a $u \in X$ such that $x_n \to u$ as $n \to \infty$. Now,

$$\begin{aligned}
\Lambda(x_{n+1}, x_{n+1}, Tu) \leq & \alpha_*(Tx_n, Tx_n, Tu)H_\lambda(Tx_n, Tx_n, Tu) \\
\leq & \beta_1(\lambda(x_n, x_n, u))\psi_1(\Lambda(x_n, x_n, Tx_n)) + \beta_2(\lambda(u, u, x_n))\psi_2(\Lambda(u, u, Tu)) \\
= & \beta_1(\lambda(x_n, x_n, u))\psi_1(\lambda(x_n, x_n, x_{n+1})) + \beta_2(\lambda(u, u, x_n))\psi_2(\Lambda(u, u, Tu)) \\
\leq & \beta_1(\lambda(x_n, x_n, u))\lambda(x_n, x_n, x_{n+1}) + \beta_2(\lambda(u, u, x_n))\Lambda(u, u, Tu) \\
\leq & \beta_1(\lambda(x_n, x_n, u))\lambda(x_n, x_n, x_{n+1}) + \beta_2(\lambda(u, u, x_n))k[2\lambda(u, u, x_n) + \lambda(Tu, Tu, x_{n+1}) + \lambda(x_n, x_n, x_{n+1})] \\
\leq & \beta(\lambda(x_n, x_n, u))\lambda(x_n, x_n, x_{n+1}) + \beta(\lambda(u, u, x_n))k[2\lambda(u, u, x_n) + \lambda(Tu, Tu, x_{n+1}) + \lambda(x_n, x_n, x_{n+1})],
\end{aligned}$$

for all $n \geq 1$. Therefore

$$\Lambda(x_{n+1}, x_{n+1}, Tu) \leq \frac{\beta(\lambda(x_n, x_n, u))(1+k)\lambda(x_n, x_n, x_{n+1}) + 2k\beta(\lambda(x_n, x_n, u))\lambda(x_n, x_n, u)}{1 - k\beta(\lambda(x_n, x_n, u))} \to 0$$

as $n \to \infty$ and $\beta(\lambda(x_n, x_n, u)) = \max\{\beta_1(\lambda(x_n, x_n, u)), \beta_2(\lambda(x_n, x_n, u))\}$. Hence $u \in Tu$ and u is a fixed point of T.

Case$-II$: $\beta = \max\{\beta_1, \beta_2\} \in [\frac{1}{k+1}, \frac{1}{2})$. Then there exists $N \in \mathbb{N}$ such that $\beta\gamma^{N-1} \in (0, \frac{1}{k+1})$. $\square$

Example 3.6. Let $X = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ and $\lambda : X^3 \to [0, \infty)$ be defined by $\lambda(x, x, x) = 0$ and $\lambda(x, x, y) = \lambda(y, y, x)$ for all $x, y \in X$ with

$$\lambda(x, y, z) = \begin{cases} |n-m| & \text{if} \quad x = \frac{1}{n} = y \quad, z = \frac{1}{m} \quad \text{and} \quad |n-m| > 1, \\ \frac{1}{3} & \text{if} \quad x = \frac{1}{n} = y \quad, z = \frac{1}{m} \quad \text{and} \quad |n-m| = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then λ is a complete symmetric Branciari S_b-metric space for $k = 3$ but not an S-metric, since

$$\lambda(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}) = 2 > 2\lambda(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}) + \lambda(\frac{1}{4}, \frac{1}{4}, \frac{1}{3}) = 1.$$

Let $T : X \to 2^X$ be given by

$$Tx = \begin{cases} \{\frac{1}{3}, \frac{1}{4}\} & \text{if} \quad x = \frac{1}{2}, \\ \{\frac{1}{5}, \frac{1}{6}\} & \text{if} \quad x \leq \frac{1}{3}. \end{cases}$$

Then T satisfies the contractive condition (3.19) for $\sum_{i=1}^{2} \beta_i = \frac{1}{6}$ and thus T has a fixed point $x = \frac{1}{5}$ in X.

In this section we give some consequences of the main results presented above. Specifically, we apply our results to complete symmetric Branciari S_b-metric space endowed with a partial order.

3.2 Fixed point theorems for weakly increasing on X has the property α-regular Branciari S_b-metric space

In the following we provide set-valued versions of the preceding theorem. The results are related to those in ([9]). Let X be a topological space and $\preceq$ be a partial order on X.

Definition 3.7. ([3]). Let A, B be two nonempty subsets of X, the relations between A and B are definers follows:
(r_1) If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
(r_2) If for every $b \in B$ there exists $a \in A$, such that $a \preceq b$, then $A \prec_2 B$.
(r_3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Definition 3.8. ([5], [6]). Let $(X, \preceq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ hold for all $x \in X$.

Definition 3.9. ([2]) Let $(X, \preceq)$ be a partially ordered set. Two mapping $F, G : X \to 2^X$ are said to be weakly increasing with respect to $\prec_1$ if for any $x \in X$ we have $Fx \prec_1 Gy$ for all $y \in Fx$ and $Gx \prec_1 Fy$ for all $y \in Gx$. Similarly two maps $F, G : X \to 2^X$ are said to be weakly increasing with respect to $\prec_2$ if for any $x \in X$ we have $Gy \prec_2 Fx$ for all $y \in Fx$ and $Fy \prec_2 Gx$ for all $y \in Gx$.

Now we give some examples.

Example 3.10. ([2]) Let $X = [1, \infty)$ and $\leq$ be usual order on X. Consider two mappings $F, G : X \to 2^X$ defined by $Fx = [1, x^2]$ and $Gx = [1, 2x]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_2$ but not $\prec_1$. Indeed, since

$$Gy = [1, 2y] \prec_2 [1, x^2] = Fx \text{ for all } y \in Fx$$

and

$$Fy = [1, y^2] \prec_2 [1, 2x] = Gx \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to $\prec_2$ but $F2 = [1, 4] \not\prec_1 [1, 2] = G1$ for $1 \in F2$, so F and G are not weakly increasing with respect to $\prec_1$.

Example 3.11. ([2]) Let $X = [1, \infty)$ and $\leq$ be usual order on X. Consider two mappings $F, G : X \to 2^X$ defined by $Fx = [0, 1]$ and $Gx = [x, 1]$ for all $x \in X$. Then the pair of mappings F and G are weakly increasing with respect to $\prec_1$ but not $\prec_2$. Indeed, since

$$Fx = [0, 1] \prec_1 [y, 1] = Gy \text{ for all } y \in Fx$$

and

$$Gx = [x, 1] \prec_1 [0, 1] = Fy \text{ for all } y \in Gx$$

so F and G are weakly increasing with respect to $\prec_1$ but $G1 = 1 \not\prec_2 0, 1 = F1$ for $1 \in F1$, so F and G are not weakly increasing with respect to $\prec_2$.

Theorem 3.12. Let $(X, \preceq, \lambda)$ be a partially ordered complete symmetric Branciari S_b-metric space. Suppose that $T : X \to 2^X$ are set-valued mappings and satisfies the following conditions:
(i)

$$\begin{aligned} &H_\lambda(Tx, Tx, Ty) \leq \beta_1(\lambda(x, x, y))\psi(\lambda(x, x, y)) + \beta_2(\lambda(x, x, y))\psi(\Lambda(x, x, Tx)) \\ &+\beta_3(\lambda(x, x, y))\psi(\Lambda(y, y, Ty)) + \beta_4(\lambda(x, x, y)) \min\{\psi(\Lambda(x, x, Ty), \psi(\Lambda(y, y, Tx))\}. \end{aligned} \tag{3.20}$$

(ii) T and i_x be a weakly increasing pair on X w.r.t $\prec_1$;
(iii) there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$ and $\{x_0\} \prec_1 T^2x_0$;
(iv) X has the property $\alpha-$regular generalized metric space.

Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ converges to the fixed point of T.

Proof . Define the sequence x_n in X by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a fixed point for T. Using that the pair of set-valued mappings T and i_x is weakly increasing and by define $\alpha : X \times X \times X \to [0, +\infty)$

$$\alpha(x, x, y) = \begin{cases} 1 & if x \preceq y \\ 0 & if x \succ y. \end{cases}$$

It can be easily shown that the sequence x_n is nondecreasing w.r.t, $\preceq$ i.e; and

$$\alpha_*(\{x_0\}, \{x_0\}, Tx_0) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_0, x_0, x_1) \geq 1 \Rightarrow x_0 \preceq x_1.$$

Now since T and i_x is weakly increasing with respect to $\prec_1$, we have $x_1 \in Tx_0 \prec_1 Tx_1$. Thus there exist some $x_2 \in Tx_1$ such that $x_1 \preceq x_2$. Again since T and i_x is weakly increasing with respect to $\prec_1$, we have $x_2 \in Tx_1 \prec_1 Tx_2$. Thus there exist some $x_3 \in Tx_2$ such that $x_2 \preceq x_3$. Continue this process, we will get a nondecreasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{n+1} \in Tx_n, n = 0, 1, 2, 3, \cdots$ We get

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq x_{n+2} \preceq \cdots .$$

In particular x_n, x_{n+j} are comparable for all $j \in \mathbb{N}$. $\alpha(x_n, x_{n+j}) \geq 1$ for all $n \in \mathbb{N}_0$ and by equation (2.1) and (2.3) we have $\lim_{n\to\infty} \lambda(x_n, x_n, x_{n+j}) = 0$. Following the proof of Theorem (3.1), we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete symmetric Branciari S_b-metric space $(X, \preceq, \lambda)$. There exists $x^* \in X$ such that $\lim_{n\to+\infty} \lambda(x_n, x_n, x^*) = 0$. and condition (iv), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\alpha(x_{n_j+1}, x_{n_j+1}, x^*) \geq \alpha_*(Tx_{n_j}, Tx_{n_j}, Tx^*) \geq 1$ for all j. Thus,

$$\begin{aligned}
\Lambda(x^*, x^*, Tx^*) \leq & k[\lambda(x^*, x^*, x_{n_j+1}) + \lambda(x^*, x^*, x_{n_j+1}) + \Lambda(Tx^*, Tx^*, x_{n_j+2}) + \lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2})] \\
= & 2k\lambda(x^*, x^*, x_{n_j+1}) + k\Lambda(Tx^*, Tx^*, x_{n_j+2}) + k\lambda(x_{n_j+1}, x_{n_{jk}+1}, x_{n_{jk}+2}) \\
= & 2k\lambda(x^*, x^*, x_{n_j+1}) + j\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k\Lambda(Tx^*, Tx^*, Tx_{n_j+1}) \\
\leq & 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_{jk}+2}) + kH_\lambda(Tx^*, Tx^*, Tx_{n_j+1}) \\
\leq & 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_{jk}+1}, x_{n_{jk}+2}) + k[\beta_1(\lambda(x^*, x^*, x_{n_j+1}))\psi(\lambda(x^*, x^*, x_{n_j+1})) \\
& + \beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) \\
& + \beta_4(\lambda(x^*, x^*, x_{n_j+1})) \min\{\psi(\Lambda(x^*, x^*, Tx_{n_j+1})), \psi(\Lambda(x_{n_j+1}, x_{n_j+1}, Tx^*))\}] \\
\leq & 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k[\beta_1(\lambda(x^*, x^*, x_{n_j+1}))\psi(\lambda(x^*, x^*, x_{n_j+1})) \\
& + \beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_{jk}+1}))\psi(\Lambda(x^*, x^*, Tx^*)) \\
& + \beta_4(\lambda(x^*, x^*, x_{n_k+1})) \min\{\psi(\lambda(x^*, x^*, x_{n_j+2})), \psi(\Lambda(x_{n_j+1}, x_{n_j+1}, Tx^*))\}] \\
\leq & k[\beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*))] \\
\leq & k[\beta_2(\lambda(x^*, x^*, x_{n_j+1}))\Lambda(x^*, x^*, Tx^*) + \beta_3(\lambda(x^*, x^*, x_{n_{jk}+1}))\Lambda(x^*, x^*, Tx^*)] \\
< & k[\beta_2(\lambda(x^*, x^*, x_{n_j+1})) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))]\Lambda(x^*, x^*, Tx^*)
\end{aligned}$$

for all $j \in \mathbb{N}$ and $k \geq 1$. Hence, $\Lambda(x^*, x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. $\square$

Theorem 3.13. Let $(X, \preceq, \lambda)$ be a partially ordered complete symmetric Branciari S_b-metric space. Suppose that $T : X \to 2^X$ are set-valued mappings and satisfies the following conditions:

(i)

$$\begin{aligned} &H_\lambda(Tx, Tx, Ty) \leq \beta_1(\lambda(x, x, y))\psi(\lambda(x, x, y)) + \beta_2(\lambda(x, x, y))\psi(\Lambda(x, x, Tx)) \\ &+\beta_3(\lambda(x, x, y))\psi(\Lambda(y, y, Ty)) + \beta_4(\lambda(x, x, y)) \min\{\psi(\Lambda(x, x, Ty), \psi(\Lambda(y, y, Tx))\}. \end{aligned} \tag{3.21}$$

(ii) T and i_x be a weakly increasing pair on X w.r.t $\prec_2$;

(iii) there exists $x_0 \in X$ such that $Tx_0 \prec_2 \{x_0\}$ and $T^2x_0 \prec_2 \{x_0\}$;

$(iv)X$ has the property $\alpha-$regular generalized metric space.

Then T has fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ converges to the fixed point of T.

Proof . Define the sequence x_n in X by $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x^* = x_n$ is a fixed point for T. Using that the pair of set-valued mappings T and i_x is weakly increasing and by define $\alpha : X \times X \times X \to [0, +\infty)$

$$\alpha(x, x, y) = \begin{cases} 1 & if x \succeq y \\ 0 & if x \prec y. \end{cases}$$

It can be easily shown that the sequence x_n is non-increasing w.r.t, $\preceq$ i.e; and

$$\alpha_*(Tx_0, Tx_0, \{x_0\}) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_1, x_1, x_0) \geq 1 \Rightarrow x_1 \preceq x_0;$$

Now since T and i_x are weakly increasing with respect to $\prec_2$, we have $Tx_1 \prec_2 Tx_0$. Thus there exist some $x_2 \in Tx_1$ such that $x_2 \preceq x_1$. Again since T and i_x are weakly increasing with respect to $\prec_2$, we have $Tx_2 \preceq_2 Tx_1$. Thus there exist some $x_3 \in Tx_2$ such that $x_3 \preceq x_2$. Continue this process, we will get a non-increasing sequence $\{x_n\}_{n=1}^{\infty}$ which satisfies $x_{n+1} \in Tx_n$ and $x_{n+2} \in Tx_{n+1}$, $n = 0, 1, 2, 3, \cdots$ We get

$$x_0 \succeq x_1 \succeq x_2 \succeq \cdots \succeq x_n \succeq x_{n+1} \succeq x_{n+2} \succeq \cdots .$$

In particular x_{n+j}, x_n are comparable for all $k \in \mathbb{N}$, $\alpha(x_{n+j}, x_n) \geq 1$ for all $j \in \mathbb{N}_0$ and by equation (2.1) and (2.3) we have $\lim_{n\to\infty} \lambda(x_{n+j}, x_{n+j}, x_n) = 0$. Following the proof of Theorem (3.1), we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete symmetric Branciari S_b-metric space. $(X, \prec, \lambda)$. There exists $x^* \in X$ such that $\lim_{n\to+\infty} \lambda(x_n, x_n, x^*) = 0$. Following the proof of Theorem (3.1), we know that $\{x_n\}$ is a Cauchy sequence in the partially ordered complete symmetric Branciari S_b-metric space $(X, \preceq, \lambda)$. There exists $x^* \in X$ such that $\lim_{n\to+\infty} \lambda(x_n, x_n, x^*) = 0$. and condition (iv), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\alpha(x_{n_j+1}, x_{n_j+1}, x^*) \geq \alpha_*(Tx_{n_j}, Tx_{n_j}, Tx^*) \geq 1$ for all j. Thus,

$$\begin{aligned}
\Lambda(x^*, x^*, Tx^*) \leq & k[\lambda(x^*, x^*, x_{n_j+1}) + \lambda(x^*, x^*, x_{n_j+1}) + \Lambda(Tx^*, Tx^*, x_{n_j+2}) + \lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2})] \\
= & 2k\lambda(x^*, x^*, x_{n_j+1}) + k\Lambda(Tx^*, Tx^*, x_{n_j+2}) + k\lambda(x_{n_j+1}, x_{n_{jk}+1}, x_{n_{jk}+2}) \\
= & 2k\lambda(x^*, x^*, x_{n_j+1}) + j\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k\Lambda(Tx^*, Tx^*, Tx_{n_j+1}) \\
\leq & 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_{jk}+2}) + kH_\lambda(Tx^*, Tx^*, Tx_{n_j+1}) \\
\leq & 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_{jk}+1}, x_{n_{jk}+2}) + k[\beta_1(\lambda(x^*, x^*, x_{n_j+1}))\psi(\lambda(x^*, x^*, x_{n_j+1})) \\
& + \beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) \\
& + \beta_4(\lambda(x^*, x^*, x_{n_j+1})) \min\{\psi(\Lambda(x^*, x^*, Tx_{n_j+1})), \psi(\Lambda(x_{n_j+1}, x_{n_j+1}, Tx^*))\}] \\
\leq & 2k\lambda(x^*, x^*, x_{n_j+1}) + k\lambda(x_{n_j+1}, x_{n_j+1}, x_{n_j+2}) + k[\beta_1(\lambda(x^*, x^*, x_{n_j+1}))\psi(\lambda(x^*, x^*, x_{n_j+1})) \\
& + \beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_{jk}+1}))\psi(\Lambda(x^*, x^*, Tx^*)) \\
& + \beta_4(\lambda(x^*, x^*, x_{n_k+1})) \min\{\psi(\lambda(x^*, x^*, x_{n_j+2})), \psi(\Lambda(x_{n_j+1}, x_{n_j+1}, Tx^*))\}] \\
\leq & k[\beta_2(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*)) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))\psi(\Lambda(x^*, x^*, Tx^*))] \\
\leq & k[\beta_2(\lambda(x^*, x^*, x_{n_j+1}))\Lambda(x^*, x^*, Tx^*) + \beta_3(\lambda(x^*, x^*, x_{n_{jk}+1}))\Lambda(x^*, x^*, Tx^*)] \\
< & k[\beta_2(\lambda(x^*, x^*, x_{n_j+1})) + \beta_3(\lambda(x^*, x^*, x_{n_j+1}))]\Lambda(x^*, x^*, Tx^*)
\end{aligned}$$

for all $j \in \mathbb{N}$ and $k \geq 1$. Hence, $\Lambda(x^*, x^*, Tx^*) = 0$ and so $x^* \in Tx^*$. □

3.3 Coupled fixed point

Definition 3.14. ([10]) Let $F : X \times X \to X$ be a mapping, where (X, λ) is a symmetric Branciari S_b-metric space. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if

$$x = F(x, y) \quad y = F(y, x).$$

Note that if (x, y) is a coupled fixed point of F then (y, x) are coupled fixed points of F too. Our results are based on the following simple lemma.

Lemma 3.15. ([20]) Let $F : X \times X \to X$ be a given mapping. Define the mapping $T_F : X \times X \to X \times X$ by $T_F(x, y) = (F(x, y), F(y, x))$ for all $(x, y) \in X \times X$. Then, (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T_F.

Theorem 3.16. Let (X, λ) be a complete symmetric Branciari S_b-metric space and $F : X \times X \to X$ be a given mapping. Assume there are exist nondecreasing functions $\psi_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2$, such that $\psi = \psi_1 + \psi_2$ is convex, $\psi(0) = 0$, $\lim_{n\to+\infty} \psi^n(t) = 0$ for all $t > 0$, a function $\alpha : X^2 \times X^2 \times X^2 \to [0, +\infty)$ and satisfies the following conditions:

(i) for all (x, y), $(u, v) \in X \times X$,

$$\alpha((x, y), (x, y), (u, v))\lambda(F(x, y), F(x, y), F(u, v)) \leq \psi_1(\lambda(x, x, u)) + \psi_2(\lambda(y, y, v));$$

(ii) if for all (x,x,y), $(u,u,v) \in X \times X \times X$,

$$\alpha((x,x,y),(u,u,v)) \geq 1 \Rightarrow \alpha(T_F(x,y),T_F(x,y),T_F(u,v)) \geq 1;$$

(iii) there exists $(x_0,x_0,y_0) \in X \times X \times X$ such that

$$\alpha((x_0,x_0,y_0),T_F(x_0,x_0,y_0)) \geq 1 \text{ and } \alpha(T_F(y_0,x_0),T_F(y_0,x_0),(y_0,x_0) \geq 1; \text{ or}$$

$(iii)^*$ there exists $(x_0,x_0,y_0) \in X \times X \times X$ such that

$$\alpha(T_F(x_0,x_0,y_0),(x_0,x_0,y_0)) \geq 1 \text{ and } \alpha((y_0,y_0,x_0),T_F(y_0,y_0,x_0)) \geq 1;$$

(iv) if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha(x_n,x_{n+1}) \geq 1$, $\alpha(y_n,y_n,y_{n+1}) \geq 1$, for all n, $x_n \to x \in X$, $y_n \to y \in X$ as $n \to \infty$, then there are exist subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(x_{n_k},x_{n_k},x) \geq 1$ and $\alpha(y_{n_k},y_{n_k},y) \geq 1$ for all k; or

$(iv)^*$ if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\alpha(x_{n+1},x_{n+1},x_n) \geq 1$, $\alpha(y_{n+1},y_{n+1},y_n) \geq 1$, for all n, $x_n \to x \in X$, $y_n \to y \in X$ as $n \to \infty$, then there are exist subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(x,x,x_{n_k}) \geq 1$ and $\alpha(y,y,y_{n_k}) \geq 1$ for all k.
Then, F has a coupled fixed point, that is, there exists $(x^*,x^*,y^*) \in X \times X \times X$ such that $x^* = F(x^*,x^*,y^*)$ and $y^* = F(y^*,y^*,x^*)$.

Proof . The idea consists in transporting the problem to the complete symmetric Branciari S_b-metric space (Y,δ), where $Y = X \times X$ and $\delta((x,y),(x,y),(u,v)) = \lambda(x,x,u) + \lambda(y,y,v)$, for all (x,y), $(u,v) \in X \times X$. From condition (i), we have

$$\alpha((x,y),(x,y),(u,v))\lambda(F(x,y),F(x,y),F(u,v)) \leq \psi_1(\lambda(x,x,u)) + \psi_2(\lambda(y,y,v)) \tag{3.22}$$

and

$$\alpha((v,u),(v,u),(y,x))\lambda(F(v,u),F(v,u),F(y,x)) \leq \psi_1(\lambda(v,v,y)) + \psi_2(\lambda(u,u,x)) \tag{3.23}$$

for all $x,y,u,v \in X$. Adding (3.22) to (3.23), we get (note that ψ is super-additive)

$$\begin{aligned}\beta(\xi,\xi,\eta)\delta(T_F\xi,T_F\xi,T_F\eta) \leq&\psi_1(\lambda(\xi_1,\xi_1,\eta_1)) + \psi_2(\lambda(\xi_2,\xi_2,\eta_2)) + \psi_1(\lambda(\eta_2,\eta_2,\xi_2)) + \psi_2(\lambda(\eta_1,\eta_1,\xi_1))\\ \leq&\psi_1(\lambda(\xi_1,\xi_1,\eta_1) + \lambda(\eta_2,\eta_2,\xi_2)) + \psi_2(\lambda(\xi_2,\xi_2,\eta_2) + \lambda(\eta_1,\eta_1,\xi_1))\\ =&\psi(\lambda(\xi_1,\xi_1,\eta_1) + d(\eta_2,\eta_2,\xi_2))\\ =&\psi(\delta(\xi,\xi,\eta))\end{aligned} \tag{3.24}$$

for all $\xi = (\xi_1,\xi_1,\xi_2)$, $\eta = (\eta_1,\eta_1,\eta_2) \in Y$, where $\beta : Y \times Y \to [0,+\infty)$ is the function defined by

$$\beta((\xi_1,\xi_1,\xi_2),(\eta_1,\eta_1,\eta_2)) \quad = \min\{\alpha((\xi_1,\xi_1,\xi_2),(\eta_1,\eta_1,\eta_2)),\alpha((\eta_2,\eta_2,\eta_1),(\xi_2,\xi_2,\xi_1))\} \tag{3.25}$$

and $T_F : Y \to Y$ is given by lemma (3.15). Let $\{(x_n,x_n,y_n)\}$ be a sequence in $Y = X \times X \times X$ such that

$$\beta((x_n,x_n,y_n),(x_{n+1},x_{n+1},y_{n+1})) \geq 1 \text{ and } (x_n,x_n,y_n) \to (x,x,y)$$

as $n \to +\infty$. Using the condition (iv), we obtain easily there exists a subsequence $\{(x_{n_k},x_{n_k},y_{n_k})\}$ of $\{(x_n,x_n,y_n)\}$ such that $\beta((x_{n_k},x_{n_k},y_{n_k}),(x,x,y)) \geq 1$ for all k. Then all the hypotheses of Theorem (3.1) are satisfied. We deduce the existence of a fixed point of T_F that gives us from Lemma (3.15) the existence of a coupled fixed point of F. □

3.4 Application

In this section, an existence result for a fractional integral equation

$$y(t) = \frac{f(t,x(t),y(t))}{\Gamma(\alpha)} \int_0^t \frac{h'(s)g(s,x(s),y(s))}{(h(t)-h(s))^{1-\alpha}}ds, \quad t \in [0,T], \tag{3.26}$$

where $T > 0$, $\alpha \in (0,1)$ and $h : [0,T] \to \mathbb{R}$. We suppose that the following conditions are satisfied.
(i) The function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.
(ii)There exists an upper semi-continuous function $\psi_i : [0,+\infty) \to [0,+\infty)$, $i = 1,2$, are nondecreasing functions such that $\psi = \psi_1 + \psi_2$ is convex, $\psi(0) = 0$, and $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$,

$$|f(t,x(t),y(t)) - f(t,u(t),v(t))| \leq \psi_1(x-u) + \psi_2(y-v), \tag{3.27}$$

for all $(t, x(t), y(t))$ and $(t, u(t), v(t)) \in [0, T] \times \mathbb{R} \times \mathbb{R}$.

(iii) The function $h : [0, T] \to \mathbb{R}$ is C^1 and nondecreasing.

(iv) The function $g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a nondecreasing function $\omega : [0, \infty) \to [0, \infty)$ such that

$$|g(t, x(t), y(t))| \leq \omega(|(x(t), y(t))|) \quad (t, x(t), y(t)) \in [0, T] \times \mathbb{R} \times \mathbb{R}.$$

(v) There exists $r_0 > 0$ such that

$$(\psi(r_0) + F_0)\omega(r_0)(g(T) - g(0)))^{\alpha} \leq r_0\Gamma(\alpha + 1) \text{ and } \frac{\omega(r_0)}{\Gamma(\alpha + 1)} \times (g(T) - g(0))^{\alpha} \leq 1 \tag{3.28}$$

where $F_0 = \frac{1}{2} \max\{|f(t, 0, 0)| : t \in [0, T]\}$.

Example 3.17. Let $X = C([0, T], \mathbb{R})$, $\lambda : X^3 \to \mathbb{R}_0^+$ and $\lambda(x, y, z) = |x(t) - y(t)| + |x(t) - z(t)| + |y(t) - z(t)|$ is a complete symmetric Branciari S_b-metric space for all $x, y, z \in X$ and $t \in [0, T]$.

$$\lambda(x, x, y) = |x(t) - x(t)| + |x(t) - y(t)| + |x(t) - y(t)| = \lambda(y, y, x) \tag{3.29}$$

$$|x - y| \leq |x - a| + |a - b| + |b - y| \tag{3.30}$$

$$|x - y| \leq |x - b| + |b - a| + |a - y|. \tag{3.31}$$

Adding (3.30) to (3.31), we get

$$\begin{aligned} \lambda(x, x, y) =& |x - y| + |x - y| \leq |x - a| + |a - b| + |b - y| + |x - b| + |b - a| + |a - y| \qquad (3.32) \\ \leq& 4k|x - a| + 2k|y - b| + 2k|a - b| + |x - b| \\ =& k[\lambda(x, x, a) + \lambda(x, x, a) + \lambda(y, y, b) + \lambda(a, a, b)] \qquad (3.33) \end{aligned}$$

for all $x, y, z \in X$ and $a, b \in X \setminus \{x, y, z\}$, $a \neq b$, $k \geq 1$.

Theorem 3.18. Consider fractional integral equation (3.26) with $g \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is C^1 and nondecreasing in the third variables. Suppose that for $x \geq u$ and $y \geq v$, we have

$$0 \leq g(t, x, y) - g(t, u, v) \leq \frac{\Gamma(\alpha + 1)}{F_0(h(t) - h(s))^{\alpha}}(\psi_1(x - u) + \psi_2(y - v)). \tag{3.34}$$

Then the fractional integral equation (3.26) with the assumptions $(i - v)$ has at least one solution $y^* \in C([0, T], \mathbb{R})$.

Proof . Let $X = C([0, T], \mathbb{R})$ is partially ordered if we define the following order relation in X:

$$x, y \in X, \quad x \leq y \Leftrightarrow x(t) \leq y(t), \text{ for all } t \in [0, T].$$

It is well-known that (X, λ) is a complete symmetric Branciari S_b-metric space with the metric

$$\lambda(x, y, z) = |x(t) - y(t)| + |x(t) - z(t)| + |y(t) - z(t)|.$$

Suppose $\{x_n\}$ is a nondecreasing sequence in X that converges to $x \in X$. Then for every $t \in [0, T]$, the sequence of the real numbers

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots,$$

converges to $x(t)$. Therefore, for all $t \in I$ and $n \in \mathbb{N}$, we have $x_n(t) \leq x(t)$. Hence $x_n \leq x$, for all $n \in \mathbb{N}$. Also, $X \times X$ is a partially ordered set if we define the following order relation in $X \times X$:

$$(x, y) \leq_r (u, v) \Leftrightarrow x(t) \leq u(t) \text{ and } y(t) \leq v(t), \text{ for all } t \in [0, T],$$

for all (x, y), $(u, v) \in X \times X$. For any x, $y \in X$, $\max\{x(t), u(t)\}$ for all $t \in [0, T]$ is in X and is the upper bound of x, u. Therefore, for every (x, y) and $(u, v) \in X \times X$ $\max\{x(t), u(t)\}$, $\max\{y(t), v(t)\}$, in $X \times X$ for all $t \in [0, T]$ is comparable to (x, y) and (u, v). Define $F : X \times X \to X$ by

$$F(x, y)(t) = \frac{f(t, x(t), y(t))}{\Gamma(\alpha)} \int_0^t \frac{h'(s)g(s, x(s), y(s))}{(h(t) - h(s))^{1-\alpha}} ds, \text{ for all } t \in [0, T].$$

Since f is nondecreasing in the second and third of its variables then F is nondecreasing in each of its variables. Now, for $x \geq u$, $y \geq v$, that is, $x(t) \geq u(t)$, $y(t) \geq v(t)$ for all $t \in [0, T]$. we have

$$\begin{aligned}
\lambda(F(x,y),F(x,y),F(u,v)) =& |F(x,y)(t) - F(x,y)(t)| + |F(x,y)(t) - F(u,v)(t)| + |F(x,y)(t) - F(u,v)(t)| \\
=& 2\left\{\frac{f(t,x(t),y(t))}{\Gamma(\alpha)}\int_0^t \frac{h'(s)g(s,x(s),y(s))}{(h(t)-h(s))^{1-\alpha}}ds\right\} \\
\leq& 2\left\{\frac{F_1}{\Gamma(\alpha)}\int_0^t \frac{h'(s)}{(h(t)-h(s))^{1-\alpha}}(g(s,x(s),y(s)) - g(s,u(s),v(s))ds\right\} \\
\leq& \left\{\frac{F_0}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+1)}{F_0(h(t)-h(s))^{\alpha}}(\psi_1(x-u)+\psi_2(y-v))\int_0^t \frac{h'(s)}{(h(t)-h(s))^{1-\alpha}}ds\right\} \\
\leq& \left\{\frac{F_0}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+1)}{F_1(h(t)-h(s))^{\alpha}}(\psi_1(x-u)+\psi_2(y-v))\frac{(h(t)-h(0))^{\alpha}}{\alpha}\right\} \\
\leq& \left\{\frac{F_0}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha+1)}{F_1(h(t)-h(s))^{\alpha}} \times \frac{(h(t)-h(0))^{\alpha}}{\alpha}(\psi_1(x-u)+\psi_2(y-v))\right\} \\
\leq& \psi_1(d(x,u)) + \psi_2(d(y,v)).
\end{aligned} \tag{3.35}$$

Thus F satisfies the condition of Theorem (3.16). Now, let (x^*, y^*) be a coupled lower solution of the fractional integral equation problem (3.26) then we have $x^* \leq F(x^*, y^*)$ and $y^* \leq F(y^*, x^*)$. Then, Theorem (3.16) gives that F has a unique coupled fixed point (x^*, y^*) with $x^* = y^*$. Then $x^*(t)$ is the solution of the integral equation (3.26). □

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