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On the existence of a solution for a strongly nonlinear elliptic perturbed anisotropic problem of infinite order with variable exponents

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Abstract

In this work, we shall be interested in the existence of a solution to the following Dirichlet problem for a specific class of elliptical anisotropic equations of the type

$$\begin{cases} A(u) + g(x,u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $A = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha}\big(a_{\alpha}|D^{\alpha}u|^{p_{\alpha}(x)-2}D^{\alpha}u\big)$ is an operator of infinite order and $g(x,s)$ is a non-linear lower order term that verify some natural growth and sign conditions, where the data f is framed in $L^1(\Omega)$.

Keywords: Strongly nonlinear elliptic equations of infinite order, monotonicity condition,variable exponents, sign condition
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1 Introduction

The purpose of this study is to investigate the existence of a weak solution to the nonlinear Dirichlet problem

$$\begin{cases} A(u) + g(x,u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, A is an operator of infinite order defined as:

$$A(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha}\big(a_{\alpha}|D^{\alpha}u|^{p_{\alpha}(x)-2}D^{\alpha}u\big)$$

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with $a_\alpha(x,\zeta)$ is a Carathéodory function for all α satisfying the non polynomial growth and coercivity conditions, without supposing a monotonicity condition in anisotropic Sobolev spaces with variable exponents. Where $p_\alpha(x)$ are continous functions on $\bar{\Omega}$, such that $p_\alpha(x) > 1$ for any $x \in \bar{\Omega}$ and for any multi-indices α.

The solvability of the problem (1.1) has been studied by many authors. For example, M. Chrid et al in [5, 8, 9], demonstrated this result in the particular case when $p_\alpha(x) = p_\alpha$. In setting, especially, in the isotropic $L^{p(x)}$ and $W_0^{m,p(x)}(\Omega)$, its has also been used other authors in different articles [11, 14, 15, 16, 18, 20, 21, 25, 27, 28, 29, 30, 31], The mathematical modeling of physical processes in space of variable exponents has generated a particular interest in the study of such equations see for example [1, 2, 7, 10].

In this study, we study the presence of a weak solution to problem (1.1) in anisotropic Sobolev spaces of infinte order $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$, without supposing a monotonicity condition and we assume that the second member belongs to $L^1(\Omega)$.

This paper is organized as follows. In Section 2 we introduce some notation, functional spaces, and certain technical results that will be needed in the sequel. Section 3 covers the solvability of the main result.

2 Preliminaries

We can begin by recalling some definitions and properties of the variable exponent Lebesgue Sobolev spaces $L^{p(x)}(\Omega)$, where Ω is a bounded subset of $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\},$$

for any $h \in C_+(\overline{\Omega})$. We define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(x)} = \{u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(x)} = \inf\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\},$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [22].

Lemma 2.1. (see Fan and Zhao [17] and Zhao et al. [31])

(1) The space $(L^{p(x)}(\Omega), |u|_{p(x)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}.$$

(2) If $p_1,\ p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ for any $x \in \overline{\Omega}$, then

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega).$$

and the imbedding is continuous.

Lemma 2.2. (see Fan and Zhao [17] and Zhao et al. [31]) If we denote

$$\rho(u) = \int_\Omega |u|^{p(x)} dx\ \forall\ u \in L^{p(x)},$$

then

(1) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;
(2) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;
(3) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^-} \geq \rho(u) \geq |u|_{p(x)}^{p^+}$;
(4) $|u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$; $\quad |u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

Lemma 2.3. (see Fan and Zhao [17] and Zhao et al. [31])
If $u,\ u_n \in L^{p(x)}(\Omega)$, $n = 0, 1, 2, ...$, then the following statements are equivalent each other:
(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$;
(2) $\lim_{n\to\infty} \rho(u_n - u) = 0$;
(3) $u_n \to u$ in measure in Ω and $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Finally, we introduce a naturel generalization of the variable exponent Sobolev space $W_0^{m,p(x)}(\Omega)$, that will enable us to study with sufficient accuracy anisotropic problem in section 3. For this purpose, let us denote by $\vec{p}(x)$ the vectorial function
$$\vec{p}(x) = \{p_\alpha(x),\ |\alpha| \leq m\},$$
where m is a positive integer such that $m \geq 1$ and $p_\alpha(.) \in C_+(\overline{\Omega})$ for all multi-indices α such that $|\alpha| \leq m$.

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order. We define $W_0^{m,\vec{p}(x)}(\Omega)$, the anisotropic variable exponent Sobolev space, as the closure of $C_0^\infty(\Omega)$ with respect the norm
$$\|u\|_{m,\vec{p}(x)} = \sum_{|\alpha|=0}^{m} |D^\alpha u|_{p_\alpha(x)}.$$

In the case when $p_\alpha(x) \in C_+(\overline{\Omega})$ are constant functions for any $|\alpha| \leq m$, the resulting anisotropic space is denoted by $W_0^{m,\vec{p}}(\Omega)$. Such spaces was developed and considered by authors in [5], [8] and [9] in the study of some anisotropic strongly non linear equations. It was proved that $W_0^{m,\vec{p}}(\Omega)$ is a reflexive Banach space for any $p_\alpha > 1$ for all multi-indice $|\alpha| \leq m$. This result can be easily extend to $W_0^{m,\vec{p}(x)}(\Omega)$. In fact, the following lemma follows

Lemma 2.4. (see [1]) The space $(W_0^{m,\vec{p}(x)}(\Omega),\ \|.\|_{m,\vec{p}(x)})$ is a Banach and reflexive space.

In order to facilitate the manipulation of the space $W_0^{m,\vec{p}(x)}(\Omega)$, we introduce p_+^+ and p_-^- as
$$p_+^+ = \max\{p_\alpha^+(x),\ |\alpha| \leq m\}, \qquad p_-^- = \min\{p_\alpha^-(x),\ |\alpha| \leq m\}.$$

Lemma 2.5. Let Ω be a bounded open subset of $\mathbb{R}^N$. If $mp_-^- > N$, then $W_0^{m,\vec{p}(x)}(\Omega) \subset L^\infty(\Omega) \cap C^k(\overline{\Omega})$ where $k = E(m - \frac{N}{p_-^-})$. Moreover, the embedding is compact.

The proof follows immediately from the corresponding embedding theorems in the isotropic case by using the fact that $W_0^{m,\vec{p}(x)}(\Omega) \subset W_0^{m,p_-^-}(\Omega)$. Now, let $a_\alpha \geq 0$ be a real numbers for multi-indices α. The variable exponent Sobolev space of infinite order is the functional space defined by
$$W^\infty(a_\alpha, p_\alpha(x))(\Omega) = \left\{ u \in C^\infty(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha u|_{p_\alpha(x)}^{p_\alpha^+} < \infty \right\}.$$

Since we shall deal with the Dirichlet problem in this paper, we shall use the functional space $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ defined by
$$W_0^\infty(a_\alpha, p_\alpha(x))(\Omega) = \left\{ u \in C_0^\infty(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha u|_{p_\alpha(x)}^{p_\alpha^+} < \infty \right\}.$$

In contrast with the finite order Sobolev space, the very first question, which arises in the study of the spaces $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$, is the question of their nontriviality (or nonemptiness), i.e. the question of the existence of a function u such that $\sigma(u) < \infty$.

Definition 2.6. (Dubinskii [13]) The space $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ is called nontrivial space if it contains at least one function which not identically equal to zero, i.e. there is a function $u \in C_0^\infty(\Omega)$ such that $\sigma(u) < \infty$.

It turns out that the answer of this question depends not only on the given parameters a_α, p_α of the spaces $W^\infty(a_\alpha, p_\alpha(x))(\Omega)$, but also on the domain Ω. The dual space of $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ is defined as follows

$$W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega) = \Big\{ h : \, h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha h_\alpha, \, \sigma'(h) = \sum_{|\alpha|=0}^{\infty} a_\alpha |h_\alpha|_{p'_\alpha(x)}^{p'^+_\alpha} < \infty \Big\},$$

where $h_\alpha \in L^{p'_\alpha(x)}(\Omega)$ and p'_α is the conjugate of p_α, i.e., $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$. By the definition, the duality pairing between $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ and its dual space $W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega)$ is given by the relation

$$\langle h, v \rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega h_\alpha(x) \, D^\alpha v(x) \, dx,$$

which, as it is not difficult to verify, is correct. In the particular case when $p_\alpha(x) = p_\alpha$ for any multi-indices α, the Sobolev space of infinite order is defined as

$$W_0^\infty(a_\alpha, p_\alpha)(\Omega) = \Big\{ u \in C_0^\infty(\Omega) : \sigma(u) = \sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha u|_{p_\alpha}^{r_\alpha} < \infty \Big\}.$$

$a_\alpha \geq 0$, $p_\alpha > 1$ and $r_\alpha > 1$ are real numbers for all multi-indices α and $|.|_{p_\alpha}$ is the usual norm in the Lebesgue space $L^{p_\alpha}(\Omega)$, (see [13], [12]).

Lemma 2.7. (see [1])For all nontrivial space $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$, there exists a nontrivial space $W_0^\infty(c_\alpha, 2)(\Omega)$ such that $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega) \subset W_0^\infty(c_\alpha, 2)(\Omega)$.

3 Essential assumptions and main result

Let Ω is an open and bounded set of $\mathbb{R}^N$ and the differential operator $A : W_0^\infty(a_\alpha, p_\alpha(x))(\Omega) \longrightarrow W^{-\infty}(a_\alpha, p'_\alpha(x))(\Omega)$ in divergence form

$$A(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \qquad |\gamma| \leq |\alpha|. \tag{3.1}$$

where $A_\alpha : \Omega \times \mathbb{R}^{\lambda_\alpha} \to I\!\!R$ is a real function and λ_α is the number of multi-indices γ such that $|\gamma| \leq |\alpha|$. We make the following assumptions:

(A_1) $A_\alpha(x, \xi_\alpha)$ is a Carathéodory function for all $\alpha, |\gamma| \leq |\alpha|$.

(A_2) For a.e. $x \in \Omega$, all $m \in I\!\!N^*$, all $\xi_\gamma, \eta_\alpha, |\gamma| \leq |\alpha|$ and some constant $c_0 > 0$, we assume that

$$\Big| \sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma) \eta_\alpha \Big| \leq c_0 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha(x)-1} |\eta_\alpha|,$$

where $a_\alpha \geq 0$, are reals numbers and $(p_\alpha(.))_\alpha$ is a bounded sequence of functions in $C_+(\overline{\Omega})$ for all multi-indices α.

(A_3) There exist constants $c_1 > 0, c_2 \geq 0$ such that for all $m \in I\!\!N^*$, for all $\xi_\gamma, \xi_\alpha; |\gamma| \leq |\alpha|$, we have

$$\sum_{|\alpha|=0}^{m} A_\alpha(x, \xi_\gamma) \cdot \xi_\alpha \geq c_1 \sum_{|\alpha|=0}^{m} a_\alpha |\xi_\alpha|^{p_\alpha(x)} - c_2.$$

(A_4) The space $W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ is nontrivial.

(G_1) The function $g : \Omega \times I\!\!R \mapsto I\!\!R$ is of Carathéodory type such that, for all $\delta > 0$,

$$\sup_{|u|<\delta} |g(x, u)| \leq h_\delta(x) \in L^1(\Omega).$$

(G_2) We assume the "sign condition" $g(x,u)u \geq 0$, for a.e. $x \in \Omega$ and all $u \in I\!R$.

Finally, we assume that

$$f \in L^1(\Omega), \tag{3.1}$$

and we shall prove the existence result without assuming any monotonicity condition.

3.1 Existence results

Our main result is the following theorem.

Theorem 3.1. Let us assume the conditions (A_1) − (A_4), (G_1) and (G_2). Then for all $f \in L^1(\Omega)$, there exists $u \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$ such that

$$\begin{cases} g(x,u) \in L^1(\Omega), \; g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega). \end{cases} \tag{3.2}$$

The proof of Theorem 3.1 is divided into several steps: we show first the existence of solutions to the approximate problem of (3.2) and a priori estimates, the convergence of approximate solution and then passing to the limit in the approximate problems will yield the main result.

Step 1: Approximate problem

Consider $\varphi \in C_0^\infty(I\!R^N)$ such that $0 < \varphi(x) < 1$ and $\varphi(x) = 1$ for x close to 0. Let f_n be a sequence of regular functions defined by

$$f_n(x) = \varphi(\frac{x}{n})T_n f(x),$$

where T_n is the usual truncation given by

$$T_n\xi = \begin{cases} \xi & \text{if } |\xi| < n \\ \frac{n\xi}{|\xi|} & \text{if } |\xi| \geq n. \end{cases}$$

It is clear that $|f_n| \leq n$ for a.e. $x \in \Omega$. Thus, it follows that $f_n \in L^\infty(\Omega)$.Using Lebesgue's dominated convergence theorem, since $f_n \to f$ a.e. $x \in \Omega$ and $|f_n| \leq |f| \in L^1(\Omega)$, we conclude that f_n strongly converges to f in $L^1(\Omega)$. Define the operator of order $2n+2$ by

$$A_{2n+2}(u) = \sum_{|\alpha|=n+1} (-1)^{n+1} c_\alpha D^{2\alpha}u + \sum_{|\alpha|=0}^{n} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u), \quad |\gamma| \leq n,$$

where c_α are constants small enough such that they fulfill the conditions of the Lemma 2.6. The operator A_{2n+2} is clearly monotone since the term of higher order of derivation is linear and satisfies the monotonicity condition, this follows from the result of [23]. Moreover from assumptions (A_1), (A_2) and (A_3), we deduce that A_{2n+2} satisfies the growth, the coerciveness and the monotonicity conditions. Hence by Theorem 3.1 (see [1]), there exists an approximate solution u_n of the following problem:

$$(\mathrm{Pb_n}) \begin{cases} g(x,u_n) \in L^1(\Omega), \; g(x,u_n)u_n \in L^1(\Omega) \\ \langle A_{2n+2}(u_n), v\rangle + \int_\Omega g(x,u_n)v\,dx = \langle f_n, v\rangle, \quad \forall v \in W_0^{n+1,\vec{p}(x)}(\Omega) \end{cases}$$

with

$$f_n = \sum_{|\alpha|=0}^{n} (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha \,, \qquad f_\alpha \in L^{p'_\alpha(x)}(\Omega).$$

Step 2: Apriori estimates

Set $v = u_n$ and using (A_3), (G_2), Lemma 2.1 and 2.2, we deduce the estimates

$$\sum_{|\alpha|=n+1} c_\alpha |D^\alpha u_n|_2^2 + \sum_{|\alpha|=0}^{n} a_\alpha |D^\alpha u_n|_{p_\alpha}^{\beta_\alpha} \leq K \tag{3.5}$$

and

$$\int_{\Omega} g(x, u_n) u_n \, dx \le K \tag{3.6}$$

for some constant $K = K(f) > 0$, with

$$\beta_\alpha = \begin{cases} p_\alpha^+ & \text{if} \quad |D^\alpha u|_{p_\alpha(x)} < 1 \\ p_\alpha^- & \text{if} \quad |D^\alpha u|_{p_\alpha(x)} > 1 \end{cases}$$

From this and since the summation in estimate (3.5) is finite, we can also write

$$\sum_{|\alpha|=n+1} c_\alpha |D^\alpha u_n|_2^2 + \sum_{|\alpha|=0}^{n} a_\alpha |D^\alpha u_n|_{p_\alpha}^{p_\alpha^+} \le K. \tag{3.7}$$

The estimate (3.7) is equivalent to

$$\sum_{|\alpha|=0}^{n+1} a_\alpha |D^\alpha u_n|_{p_\alpha(x)}^{p_\alpha^+} \le K \tag{3.8}$$

with $a_\alpha = c_\alpha$ and $p_\alpha = 2$ for $|\alpha| = n+1$. Consequently, we have

$$\|u_n\|_{W^{n+1,\vec{p}(x)}} \le K. \tag{3.9}$$

Then via a diagonalization process, there exists a subsequence still, denoted by u_n, which converges uniformly to an element $u \in C_0^\infty(\Omega)$, also for all derivatives there holds $D^\alpha u_n \to D^\alpha u$ (for more details we refer to [5], [13]).

Step 3: Convergence of problem (Pb_n)

There exists a solution u_n of problem (Pb$_n$), $n = 1, 2, \ldots$. Then by passing to the limit, we have

$$\lim_{n\to+\infty} \langle A_{2n+2}(u_n), v\rangle + \lim_{n\to+\infty} \int_{\Omega} g(x, u_n) v \, dx = \lim_{n\to+\infty} \langle f_n, v\rangle,$$

for $v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$. It is clear that

$$\lim_{n\to+\infty} \langle f_n, v\rangle = \langle f, v\rangle \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega).$$

Now, we shall prove that

$$\lim_{n\to+\infty} \langle A_{2n+2}(u_n), v\rangle = \langle Au, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega).$$

In fact, let n_0 be a fix number sufficiently large ($n > n_0$) and let $v \in W_0^\infty(a_\alpha, p_\alpha)(\Omega)$. Set

$$\langle A(u) - A_{2n+2}(u_n), v\rangle = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \sum_{|\alpha|=0}^{n_0} \langle A_\alpha(x, D^\gamma u) - A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle \\ I_2 &= \sum_{|\alpha|=n_0+1}^{\infty} \langle A_\alpha(x, D^\gamma u), D^\alpha v\rangle \\ I_3 &= -\sum_{|\alpha|=n_0+1}^{n} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle - \sum_{|\alpha|=n+1} c_\alpha \langle D^\alpha u_n, D^\alpha v\rangle, \end{aligned}$$

or in another form,

$$I_3 = -\sum_{|\alpha|=n_0+1}^{n+1} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle.$$

with $A_\alpha(x,\xi_\gamma) = c_\alpha \xi_\alpha$ and $c_\alpha \geq 0$ for $|\alpha| = n+1$. We will go to the limit as $n \to +\infty$ to prove that I_1, I_2 and I_3 tend to 0. Starting by I_1; we have $I_1 \to 0$ since $A_\alpha(x,\xi_\gamma)$ is of Carathéodory type. The term I_2 is the remainder of a convergent series, hence $I_2 \to 0$. For what concerns I_3; in view of (A_2) and Hölder inequality (Lemma 2.1) we have

$$\begin{aligned}
\Big|\sum_{|\alpha|=n_0+1}^{n+1} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle\Big| &\leq \sum_{|\alpha|=n_0+1}^{n+1} |\langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle| \\
&\leq c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha \int_\Omega |D^\alpha u_n|^{p_\alpha(x)-1} |D^\alpha v|\, dx \\
&\leq c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha \big|\ |D^\alpha u_n|^{p_\alpha(x)-1}\big|_{p'_\alpha(x)} |D^\alpha v|_{p_\alpha(x)}.
\end{aligned}$$

Now, in view Lemma 2.3, one get

$$\begin{aligned}
\big|\ |D^\alpha u_n|^{p_\alpha(x)-1}\big|_{p'_\alpha(x)} &\leq \left(\int_\Omega |D^\alpha u_n|^{(p_\alpha(x)-1)p'_\alpha(x)}\, dx\right)^{\nu_\alpha} \\
&\leq \Big(\int_\Omega |D^\alpha u_n|^{p_\alpha(x)}\, dx\Big)^{\nu_\alpha} \\
&\leq |D^\alpha u_n|_{p_\alpha(x)}^{\nu_\alpha\ \beta_\alpha} \\
&\leq |D^\alpha u_n|_{p_\alpha(x)}^{p_\alpha^+-1},
\end{aligned}$$

where ν_α and β_α are real numbers for all multi-indices $|\alpha| \leq n$, defined as

$$\nu_\alpha = \begin{cases} \frac{1}{{p'_\alpha}^+} & \text{if} \quad \big|\ |D^\alpha u_n|^{p_\alpha(x)-1}\big|_{p'_\alpha(x)} < 1 \\ \frac{1}{{p'_\alpha}^-} & \text{if} \quad \big|\ |D^\alpha u_n|^{p_\alpha(x)-1}\big|_{p'_\alpha(x)} > 1 \end{cases}$$

$$\beta_\alpha = \begin{cases} p_\alpha^+ & \text{if} \quad |D^\alpha u_n|_{p_\alpha(x)} < 1 \\ p_\alpha^- & \text{if} \quad |D^\alpha u_n|_{p_\alpha(x)} > 1. \end{cases}$$

It's very easy to verify that for all multi-indices $|\alpha| \leq n$, on has

$$\nu_\alpha\ \beta_\alpha \leq p_\alpha^+ - 1.$$

Indeed, we have $p'_\alpha = \frac{p_\alpha}{p_\alpha - 1}$, then,

case 1: $\nu_\alpha\ \beta_\alpha = \frac{1}{{p'_\alpha}^+} p_\alpha^+ = \frac{p_\alpha^+-1}{p_\alpha^+} p_\alpha^+ = p_\alpha^+ - 1.$

case 2: $\nu_\alpha\ \beta_\alpha = \frac{1}{{p'_\alpha}^-} p_\alpha^- = \frac{p_\alpha^--1}{p_\alpha^-} p_\alpha^- = p_\alpha^- - 1 \leq p_\alpha^+ - 1.$

Therefore, for all $\varepsilon > 0$, there exists $k(\varepsilon) > 0$ (see [6, p. 56]) such that

$$\begin{aligned}
\Big|\sum_{|\alpha|=n_0+1}^{n+1} \langle A_\alpha(x, D^\gamma u_n), D^\alpha v\rangle\Big| &\leq \varepsilon c_0 \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha |D^\alpha u_n|_{p_\alpha(x)}^{p_\alpha^+} + c_0 k(\varepsilon) \sum_{|\alpha|=n_0+1}^{n+1} a_\alpha |D^\alpha v|_{p_\alpha(x)}^{p_\alpha^+} \\
&\leq \varepsilon c_0 K + c_0 k(\varepsilon) \sum_{|\alpha|=n_0+1}^{\infty} a_\alpha |D^\alpha v|_{p_\alpha(x)}^{p_\alpha^+},
\end{aligned}$$

where K is the constant given in the estimate (3.8). Since the sequence $(p_\alpha(x))$ is bounded and $\sum_{|\alpha|=n_0+1}^{\infty} a_\alpha |D^\alpha v|_{p_\alpha(x)}^{p_\alpha^+}$ is the remainder of a convergent series, therefore $I_3 \to 0$ holds. Hence $\langle A_{2n+2}(u_n), v\rangle \to \langle A(u), v\rangle$ as $n \to +\infty$ for all $v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$. It remains to show, for our purposes, that

$$\lim_{n\to+\infty} \int_\Omega g(x, u_n) v\, dx = \int_\Omega g(x, u) v\, dx,$$

for all $v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega)$. Indeed, we have $u_n \to u$ uniformly in Ω, hence $g(x, u_n) \to g(x, u)$ for a.e. $x \in \Omega$. In view of (3.6), we deduce by Fatou's lemma that

$$\int_\Omega g(x,u)u\,dx \le \lim_{n\to+\infty} \int_\Omega g(x,u_n)u_n\,dx \le K.$$

This implies that $g(x,u)u \in L^1(\Omega)$. On the other hand, let $\delta > 0$, since $|g(x,t)|\delta \le |g(x,t)t|$ and then $|g(x,t)| \le \delta^{-1}|g(x,t)t|$ for $|t| \ge \delta$, we have

$$\begin{aligned} |g(x,u_n)| &\le \sup_{|t|\le\delta} |g(x,t)| + \delta^{-1}|g(x,u_n).u_n| \\ &\le h_\delta(x) + \delta^{-1}|g(x,u_n)u_n|. \end{aligned}$$

It follows that

$$\int_E |g(x,u_n)|\,dx \le \int_E h_\delta(x)\,dx + \delta^{-1}K,$$

for some measurable subset E of Ω and for some $\varepsilon > 0$. Here, K is the constant of (3.2) which is independent of n. For $|E|$ sufficiently small and $\delta = \frac{2K}{\varepsilon}$, we obtain $\int_E |g(x,u_n)|\,dx < \varepsilon$. Then, using Vitali's, we get theorem $g(x,u_n) \to g(x,u)$ in $L^1(\Omega)$. Hence it follows that $g(x,u) \in L^1(\Omega)$.

Step 4: Passing to the limit

By passing to the limit, we obtain

$$\langle Au, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle, \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega).$$

Consequently,

$$\begin{cases} g(x,u) \in L^1(\Omega), g(x,u)u \in L^1(\Omega) \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x,u)v\,dx = \langle f, v\rangle \quad \text{for all } v \in W_0^\infty(a_\alpha, p_\alpha(x))(\Omega) \end{cases}$$

This completes the proof.

Remark 3.2. Note that the existence result is given with no monotonicity condition on the operator.

4 Conclusions

We have studied a strongly nonlinear elliptic problem in the framework anisotropic Sobolev spaces of infinite order with variable exponents. The order term in elleptic equation is defined by a nonlinear operator of infinite order and a nonlinear lower order term that verify some natural growth and sign condition and the second term f belongs in $L^1(\Omega)$. Under the usual assumptions on the data, we have demonstrated the existence of a weak solution to this problem. The proof of this result is developed through several steps. The existence result is given with no monotonicity condition on the operator.

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