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Õrder-norm continuous operators and õrder weakly compact operators

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Abstract

Let E be a sublattice of a vector lattice F. A continuous operator T from E into a normed vector space X is said to be õrder-norm continuous if $x_\alpha \xrightarrow{Fo} 0$ implies $T(x_\alpha) \xrightarrow{\|\cdot\|} 0$ for every $(x_\alpha)_{\alpha\in A} \subseteq E$. This paper aims to investigate the properties of this new class of operators and explore their relationships with existing classifications of operators. We introduce a new class of operators called õrder weakly compact operators. A continuous operator $T : E \to X$ is considered õrder weakly compact if $T(A)$ in X is a relatively weakly compact set for every Fo-bounded $A \subseteq E$. In this manuscript, we examine various properties of this class of operators and explore their connections with õrder-norm continuous operators.

Keywords: Vector lattice, property (F), õ-convergence, order-to-norm continuous operator, õrder-norm continuous operator, õrder weakly compact
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1 Introduction and Preliminaries

Our motivation for writing this article is to disseminate and expand upon the concepts introduced in the articles [4] and [8]. These papers have introduced and studied concepts such as õ-convergence, the property (F), and order-to-norm continuous operators, exploring their properties and relationships with other lattice properties.

In this article, we introduce a new class of operators known as õrder-norm continuous operators, and examine some of their properties and relationships with other known operators.

To state our results, we need to fix some notations and recall some definitions. A net $(x_\alpha)_{\alpha\in A}$ in a vector lattice E is said to be order convergent to $x \in E$ if there is a net $(y_\beta)_{\beta\in B}$ in E such that $y_\beta \downarrow 0$ and for every $\beta \in B$, there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \le y_\beta$ whenever $\alpha \ge \alpha_0$. For short, we will denote this convergence by $x_\alpha \xrightarrow{o} x$ and write that x_α is o-convergent to x. A net $(x_\alpha)_{\alpha\in A}$ in vector lattice E is unbounded order convergent to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$. We denote this convergence by $x_\alpha \xrightarrow{uo} x$ and write that x_α uo-convergent to x. It is clear that for order bounded nets, uo-convergence is equivalent to o-convergence. A net $(x_\alpha)_{\alpha\in A} \subseteq E$ is said to be õrder convergent to $x \in E$ if there is a net $(y_\beta)_{\beta\in B} \subseteq F$, possibly over a different index set, such that $y_\beta \downarrow 0$ in F and for every $\beta \in B$, there exists α_0 such that $|x_\alpha - x| \le y_\beta$ whenever $\alpha \ge \alpha_0$. We denote this convergence by $x_\alpha \xrightarrow{Fo} x$

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and write that$(x_\alpha)_{\alpha\in A}$ is õ-convergent to x. It is clear that if E is regular in F and $x_\alpha \xrightarrow{o} x$ in E, then $x_\alpha \xrightarrow{Fo} x$. The converse is not true in general. For example, c_0 is a sublattice of ℓ^∞ and $(e_n) \subseteq c_0$. $e_n \xrightarrow{\ell^\infty o} 0$ in c_0, but it is not order convergent to 0 in c_0. A subset A of E is said to be F-order bounded (in short, Fo-bounded), if there exist $x, y \in F$ that $A \subseteq [x, y]$. A vector lattice E is said to have the property (F) if every Fo-bounded set $A \subseteq E$ is also order bounded (see [4]). A Banach lattice E is called an AM-space if for every $x, y \in E$ such that $|x| \wedge |y| = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. Similarly, a Banach lattice E is called an AL-space if for every $x, y \in E$ such that $|x| \wedge |y| = 0$, we have $\|x + y\| = \|x\| + \|y\|$. Furthermore, a Banach lattice E is referred to as a KB-space if every increasing, norm-bounded sequence in E^+ is norm convergent. Let E and G be vector spaces. $L(E, G)$ will denote the space of all operators from E into G. $L_b(E, G)$ is the all of order bounded operators in this manuscript. An operator T from a Banach space X into a Banach space Y is weakly compact if $\overline{T(B_X)}$ is weakly compact where B_X is the closed unit ball of X. A continuous operator T from Banach lattice E into Banach space X is called M-weakly compact if $\lim \|T(x_n)\| = 0$ holds for every norm bounded disjoint sequence $(x_n)_n$ of E. An operator $T : E \to F$ from Banach lattice E into Banach lattice F is said to preserve disjointness whenever for each $x, y \in E$ such that $x \perp y$ in E implies $T(x) \perp T(y)$ in F. A subset A of a vector lattice E is called b-order bounded in E if it is order bounded in $E^{\sim\sim}$. If each b-order bounded subset of E is order bounded in E, then E is said to have the property (b). Jalili, Haghnejad and Moghimi characterized $L_{o\tau}(E, G)$ and $L^\sigma_{o\tau}(E, G)$ spaces in [8]. An operator T from a vector lattice E into topological vector space G is said to be order-to-topology continuous whenever $x_\alpha \xrightarrow{o} 0$ implies $T(x_\alpha) \xrightarrow{\tau} 0$ for each $(x_\alpha)_{\alpha\in A} \subseteq E$. For each sequence $(x_n)_n \subseteq E$, if $x_n \xrightarrow{o} 0$ implies $T(x_n) \xrightarrow{\tau} 0$, then T is called σ-order-to-topology continuous operator. The collection of all order-to-topology continuous operators from a vector lattice E into topological vector space G will be denoted by $L_{o\tau}(E, G)$; the subscript $o\tau$ is justified by the fact that the order-to-topology continuous operators; that is,

$$L_{o\tau}(E, G) = \{T \in L(E, G) : \ T \text{ is order-to-topology continuous } \}.$$

Similarly, $L^\sigma_{o\tau}(E, G)$ represents the collection of all σ-order-to-topology continuous operators, that is,

$$L^\sigma_{o\tau}(E, G) = \{T \in L(E, G) : \ T \text{ is } \sigma - \text{order-to-topology continuous } \}.$$

For a normed space G, $L_{on}(E, G)$ is collection of order-to-norm topology continuous operators.

Let E and G be two normed vector lattices. Recall that from [9], a continuous operator $T : E \to G$ is said to be σ-*uon*-continuous if every norm-bounded uo-null sequence $(x_n)_n \subseteq E$ implies $T(x_n) \xrightarrow{\|.\|} 0$. Furthermore, an operator T from a Banach lattice E into a Banach space X is referred to as a wun-Dunford-Pettis operator if $x_n \xrightarrow{wun} 0$ in E implies $T(x_n) \xrightarrow{|.|} 0$ in X for every sequence $(x_n)_n \subseteq E$ (See [10] for more information).

Recall that a Banach lattice E is said to have the property (P) if there exists a positive contractive projection $P : E^{**} \to E$, where E is identified as a sublattice of its topological bidual E^{**}.

In a Banach lattice E, a subset A is considered almost order bounded if, for any $\epsilon > 0$, there exists $u \in E^+$ such that $A \subseteq [-u, u] + \epsilon B_E$, where B_E denotes the closed unit ball of E. A useful fact to note is that $A \subseteq [-u, u] + \epsilon B_E$ if and only if $\sup_{x\in A} \|(|x| - u)^+\| = \sup_{x\in A} \||x| - |x| \wedge u\| \leq \epsilon$. This fact can be easily verified using the Riesz decomposition theorem. According to Theorems 4.9 and 3.44 in [1], every almost order bounded subset in an order continuous Banach lattice is relatively weakly compact. Furthermore, it is known that a subset $A \subseteq L_1(\mu)$ is relatively weakly compact if and only if it is almost order bounded (see [7]).

A sublattice G of a vector lattice E is called majorizing if, for every $x \in E$, there exists $y \in G$ such that $x \leq y$. On the other hand, a sublattice G of a vector lattice E is said to be order dense in E if, for each $0 < x \in E$, there exists $y \in G$ such that $0 < y \leq x$. Recall that a Banach lattice E is said to have the positive Schur property if every positive w-null sequence in E is norm null. Furthermore, it is said to have the dual positive Schur property if every positive w^*-null sequence in E^* is norm null. Moreover, a vector lattice is considered laterally complete if every subset of pairwise disjoint positive vectors has a supremum.

Throughout this paper, unless otherwise stated, we consider F and G as two vector lattices. Additionally, we assume that E is a sublattice of F, and X and Y are two normed vector spaces.

2 The property (F) in vector lattices

In this section, we focus on investigating the property (F) as defined in [4]. Our aim is to derive new insights and results based on this property.

Let E be a closed, order continuous, and regular sublattice of F. Consider a net $(x_\alpha)_{\alpha\in A} \subseteq E$ that is order bounded, and let F be a Dedekind complete Banach lattice. According to Lemma 4.5 in [7], we observe that $x_\alpha \xrightarrow{Fo} x$ in E if and only if $x_\alpha \xrightarrow{o} x$ in E.

Remark 2.1. 1. If E^{**} is lattice isomorphic to a sublattice of F and E has the property (F), then E also has property (b). Moreover, if E^{**} is lattice isomorphic to F, then properties (b) and (F) are equivalent in E.
2. If E is a majorizing sublattice of F, then E has property (F). Let $A \subseteq E$ be an F-order bounded set. Therefore, there exists $u \in F^+$ such that $A \subseteq [-u, u]$. Since E is a majorizing sublattice of F, there exists $v \in E^+$ such that $u \leq v$ and $-v \leq -u$. Thus, $A \subseteq [-v, v]$, and hence A is order bounded in E.
3. Let E be a sublattice of F and F be a sublattice of G. If F has property (G), then necessarily E does not have property (G). For example, c_0 is a sublattice of c and c is a sublattice of ℓ^∞. c has property (ℓ^∞) while c_0 does not have property (ℓ^∞).
4. Let E be a sublattice of F and F be a sublattice of G. It is clear that if E has property (G), then it also has property (F). If $A \subseteq E$ is an F-order bounded set, then it is also order bounded in G. By assumption, A is order bounded in E. Therefore, E has property (F).

Example 2.2. By Remark 2.1, since c_0 does not have property (b), it also does not have property (ℓ^∞).
Note that c_0 does not have property (ℓ^∞). For example, consider the sequence $(e_n) \subseteq c_0$, where e_n denotes the standard unit vector in c_0 with 1 in the n-th position and 0 elsewhere. This sequence is ℓ^∞-order bounded, but it is not order bounded in c_0.

Proposition 2.3. Let E and F be two Banach lattices with order continuous norms. If E has property (b), then E also has property (F).

Proof . Assume that E has property (b). Let $A \subseteq E$ be an F-order bounded set. Then there exists $u \in F^+$ such that $A \subseteq [-u, u]$. Hence, $|A| \subseteq [-u, u]$. Without loss of generality, assume that $A \subseteq E^+$ and A is directed upward. Let $A = (x_\alpha)_{\alpha\in A}$, where $x_\alpha = \alpha$ for all $\alpha \in A$. Clearly, $0 \leq x_\alpha \uparrow\leq u$. Since E has property (b), by Proposition 2.1 of [2], E is a KB-space. Thus, we have $x_\alpha \xrightarrow{\|\cdot\|} x$ for some $x \in E$. Since F has an order continuous norm, it is a Dedekind complete Banach lattice. Hence, there exists $y \in F$ such that $0 \leq x_\alpha \uparrow\leq y$. It is clear that $y - x_\alpha \downarrow 0$ in F. Since F has order continuous norm, we have $x_\alpha \xrightarrow{\|\cdot\|} y$. Therefore, $y = x \in E$ and $A \subseteq [0, x]$. This shows that E has property (F). □

Theorem 2.4. Let E and F be two Banach lattices, and suppose that E has property (F). Then, if $A \subseteq E$ is almost order bounded in F, it is also almost order bounded in E.

Proof . Suppose that E is a sublattice of F and let I_E denote the ideal generated by E in F. It is clear that E is majorizing in I_E. Thus, A is almost order bounded in E if and only if A is almost order bounded in I_E. Without loss of generality, we assume that E is an ideal of F. Let $A \subseteq E$ be an almost order bounded set in F. This means that for every $\varepsilon > 0$, there exists $u \in F^+$ such that $A \subseteq [-u, u] + \varepsilon B_F$. For each $x \in A$, we can write $x = x_1 + x_2$, where $x_1 \in [-u, u]$ and $x_2 \in \varepsilon B_F$. It follows that $|x| \leq |x_1| + |x_2|$. By Decomposition property of [1], there exist $x_3, x_4 \in F^+$ such that $0 \leq x_3 \leq |x_1|$, $0 \leq x_4 \leq |x_2|$, and $|x| = x_3 + x_4$. Since E is an ideal of F, we have $x_3, x_4 \in E$. Therefore, $|x| = x_3 + x_4 \in [-u, u] + \varepsilon B_E$. Thus, x^+, x^-, also, $x \in [-u, u] + \varepsilon B_E$.

E has property(F), then, there exists $v \in E^+$ such that $x \in [-v, v] + \varepsilon B_E$. Since $x \in A$ is arbitrary, $A \subseteq [-v, v] + \varepsilon B_E$. Hence, A is almost order bounded in E. □

Corollary 2.5. Let $(x_n)_n \subseteq E$ be a disjoint and almost order bounded sequence in F. If E has an order continuous norm, then $x_n \xrightarrow{\|\cdot\|} 0$.

Proof . Since $(x_n)_n$ is a disjoint sequence, by Corollary 3.6 of [6], we have $x_n \xrightarrow{uo} 0$ in E. By Theorem 2.4, $(x_n)_n$ is almost order bounded in E. By Proposition 3.7 of [7], we conclude that $x_n \xrightarrow{\|\cdot\|} 0$. □

3 Õrder-norm continuous operators

A continuous operator $T : E \to X$ is said to be õrder-norm continuous (or õn-continuous for short) if $(x_\alpha)_{\alpha\in A} \subseteq E$ is õ-null in E, then $(T(x_\alpha))_{\alpha\in A}$ in X converges to 0 in norm. Similarly, a continuous operator $T : E \to X$ is said to be σ-õrder-norm continuous (or σ-$\tilde{o}n$-continuous for short) if $(x_n)_n \subseteq E$ is õ-null in E, then $(T(x_n))_n$ in X converges to 0 in norm.

The collection of all $\tilde{o}n$-continuous operators from a vector lattice E into a Banach space X (resp. σ-$\tilde{o}n$-continuous operators) will be denoted by $L_{\tilde{o}n}(E,X)$ (resp. $L^{\sigma}_{\tilde{o}n}(E,X)$).

It is clear that if $T : E \to X$ is $\tilde{o}n$-continuous, then T is order-to-norm topology continuous. However, the converse is not true in general, as shown in the following example.

Example 3.1. The identity operator $I : c_0 \to c_0$ is order-to-norm topology continuous. Let $(x_\alpha)_{\alpha\in A} \subseteq c_0$ be an order-null net. Since c_0 has order continuous norm, we have $x_\alpha \xrightarrow{\|\cdot\|} 0$. However, consider the sequence $(e_n)_n \subseteq c_0$. We have $e_n \xrightarrow{\ell^\infty o} 0$ in c_0. But $(I(e_n))_n$ is not convergent to zero in norm in c_0. Hence, $I : c_0 \to c_0$ is not $\ell^\infty on$-continuous.

Obviously, $L_{\tilde{o}n}(E,X)$ is a subspace of $L_{on}(E,X)$. Here are some examples of $\tilde{o}n$-continuous operators.

Example 3.2.

1. If E has the property (F), E^* has order continuous norm, and G has the Schur property, then every continuous operator T from E to G is σ-õn-continuous. Let $(x_n)_n \subseteq E$ be a õ-null sequence. Therefore, $(x_n)_n$ is order-null in F and thus order-bounded in F. Since E has the property (F), $(x_n)_n$ is also order-bounded in E. Moreover, $(x_n)_n$ is uo-null in F, and by Lemma 4.5 of [7], it is also uo-null in E. Since E^* has order continuous norm, by Theorem 6.4 of [5], we have $x_n \xrightarrow{w} 0$ in E. By the continuity of T, we have $T(x_n) \xrightarrow{w} 0$ in G. Since G has the Schur property, we conclude that $T(x_n) \xrightarrow{\|\cdot\|} 0$ in G.
   The Banach lattice c has the property (ℓ^∞), c^* has order continuous norm, and ℓ^1 has the Schur property. Therefore, every continuous operator $T : c \to \ell^1$ is σ-$\ell^\infty on$-continuous.
2. Let F be a Dedekind complete Banach lattice. If E has the property (F) and order continuous norm, then every continuous operator T from E to X is σ-$\tilde{o}n$-continuous. Let $(x_n)_n \subseteq E$ be a õ-null sequence. Therefore, $(x_n)_n$ is order-null in F and thus order-bounded in F. Since E has the property (F), $(x_n)_n$ is also order-bounded in E. Moreover, $(x_n)_n$ is uo-null in F, and by Lemma 4.5 of [7], it is also uo-null in E. Since $(x_n)_n$ is order-bounded, it is also order-null in E. Because E has order continuous norm, we have $x_n \xrightarrow{\|\cdot\|} 0$ in E. Thus, $T(x_n) \xrightarrow{\|\cdot\|} 0$ in X.
3. If $T : F \to X$ is a uon-continuous operator, then $T|_E : E \to X$ is a õn-continuous operator. Let $(x_\alpha)_{\alpha\in A} \subseteq E$ be a õ-null net. It is clear that $x_\alpha \xrightarrow{uo} 0$ in F. By assumption, $T(x_\alpha) \xrightarrow{\|\cdot\|} 0$ in X.

The class of õn-continuous operators differs from the class of order continuous operators. For example, the identity operator $I : c_0 \to c_0$ is order continuous, but it is not $\ell^\infty on$-continuous (see Example 3.1).

Proposition 3.3.

1. Let $T \in L_{\tilde{o}n}(E,G)$ and $S : E \to G$ be two operators such that $0 \le S \le T$. Then S is a $\tilde{o}n$-continuous operator.
2. If $T \in L_{\tilde{o}n}(E,X)$ and $S : X \to Y$ is a continuous operator, then $S \circ T \in L_{\tilde{o}n}(E,Y)$.

Proof .

1. Let $(x_\alpha)_{\alpha\in A} \subseteq E$ be a õ-null net. It is obvious that $|x_\alpha| \xrightarrow{Fo} 0$ in E. We have $|S(x_\alpha)| \le |S|(|x_\alpha|) = S(|x_\alpha|) \le T(|x_\alpha|)$. By assumption, $T(|x_\alpha|) \xrightarrow{\|\cdot\|} 0$. Therefore, $|S(x_\alpha)| \xrightarrow{\|\cdot\|} 0$. This shows that S is a õn-continuous operator.
2. Let $(x_\alpha)_{\alpha\in A} \subseteq E$ and $x_\alpha \xrightarrow{Fo} 0$. By assumption, we have $T(x_\alpha) \xrightarrow{\|\cdot\|} 0$. Therefore, $S(T(x_\alpha)) \xrightarrow{\|\cdot\|} 0$. Hence, $S \circ T \in L_{\tilde{o}n}(E,Y)$.

□

Remark 3.4. Let $T : E \to G$ be an order continuous lattice homomorphism from a Dedekind complete vector lattice E to an Archimedean laterally complete normed vector lattice G. If E is order dense in the Archimedean vector lattice F, then by Theorem 2.32 of [1], T can be extended from F to G as an order continuous lattice homomorphism. Furthermore, if G has an order continuous norm, then T is $\tilde{o}n$-continuous.

Theorem 3.5. Let $T : E \to G$ be an order bounded operator. Then the following assertions are true.

1. If G is an Archimedean vector lattice and T preserves disjointness and is $\tilde{o}n$-continuous, then $|T|$ exists and $|T| \in L_{\tilde{o}n}(E,G)$.
2. If E is a projection band in F, G is an atomic Banach lattice with order continuous norm, and T is σ-$\tilde{o}n$-continuous, then $|T|$ exists and $|T| \in L^{\sigma}_{\tilde{o}n}(E,G)$.

Proof .

1. Let $(x_\alpha)_{\alpha\in A} \subseteq E$ be a õ-null net. By the assumption, we have $T(x_\alpha) \xrightarrow{\|\cdot\|} 0$. By Theorem 2.40 of [1], $|T|$ exists and $|T|(|x|) = |T(|x|)| = |T(x)|$ for all $x \in E$. Since $||T|(x_\alpha)| \leq |T|(|x_\alpha|) \xrightarrow{\|\cdot\|} 0$, we have $|T|(x_\alpha) \xrightarrow{\|\cdot\|} 0$. Hence, $|T| \in L_{\tilde{o}n}(E,G)$.
2. Let $(x_n)_n \subseteq E$ and $x_n \xrightarrow{o} 0$ in E. It is clear that $x_n \xrightarrow{Fo} 0$ in E. By the assumption, $T(x_n) \xrightarrow{\|\cdot\|} 0$ in G. Since $(x_n)_n$ is order bounded, $(T(x_n))_n$ is also order bounded. By Lemma 5.1 of [5], $T(x_n) \xrightarrow{o} 0$ in G. Hence, T is a σ-order continuous operator. Note that since G has an order continuous norm, it is Dedekind complete. By Theorem 1.56 of [1], $|T|$ exists and it is σ-order continuous. Let $(x_n)_n \subseteq E$ be a õ-null sequence. Since E is a projection band, we have $|x_n| = P_E(|x_n|) \leq P_E(y_m)$ such that $|x_n| \leq y_m \downarrow 0$ and $(y_m)_m \subseteq F$. Obviously, we have $x_n \xrightarrow{o} 0$ in E. By the assumption, $|T|(x_n) \xrightarrow{o} 0$ in G. Because G has an order continuous norm, $|T|(x_n) \xrightarrow{\|\cdot\|} 0$ in G. Hence, $|T| \in L^{\sigma}_{\tilde{o}n}(E,G)$.

□

Corollary 3.6. By the proof of part 2 of Theorem 3.5, if E is a projection band in F and G is an atomic Banach lattice with order continuous norm, then $T : E \to G$ is a σ-$\tilde{o}n$-continuous operator if and only if it is a σ-order continuous operator.

4 Õrder weakly compact operator

A continuous operator $T : E \to X$ is said to be õrder weakly compact (or, õ-weakly compact for short) if, for any Fo-bounded set $A \subseteq E$, the image $T(A)$ in X is a relatively weakly compact set.

The collection of all õ-weakly compact operators from the vector lattice E into the Banach space X will be denoted by $W_{\tilde{o}}(E,X)$.

A subset A in a Banach lattice E is said to be F-almost order bounded if, for any $\epsilon > 0$, there exists $u \in F^+$ such that $A \subseteq [-u,u] + \epsilon B_E$.

As a remark, every weakly compact operator $T : E \to X$ is a õ-weakly compact operator. The converse holds whenever E has an order unit.

Remark 4.1.
1. Let $T : E \to X$ be a weakly compact operator. If A is an F-order bounded set in E, then it is a norm bounded set. Since T is a weakly compact operator, $T(A)$ is a relatively weakly compact set in X. This implies that T is a õ-weakly compact operator.
2. Let E have an order unit and $T : E \to X$ be a õ-weakly compact operator. If A is a norm bounded set in E, then it is an order bounded set in E, and therefore, it is F-order bounded. By the assumption, $T(A)$ is a relatively weakly compact set in X. This implies that T is a weakly compact operator.

Proposition 4.2. If E has order continuous norm with property (F), then the identity operator $I : E \to E$ is õ-weakly compact.

Proof . Let $A \subseteq E$ be a Fo-bounded set. Since E has property (F), A is an order bounded set in E. It is also clear that A is almost order bounded in E. Since E has order continuous norm, by Theorem 4.9(5) and Theorem 3.44 of [1], A is a relatively weakly compact set in E. Hence, $I(A)$ is a relatively weakly compact set in E. This means that I is a õ-weakly compact operator. □

Lemma 4.3. Let E be a vector lattice and $u \in E^+$. For each $x \in E$ such that $|x| < \lambda u$, if $\|x\| \leq M$, then $\lambda \leq \dfrac{M}{\|u\|}$.

Theorem 4.4. A continuous operator $T : E \to X$ is õ-weakly compact if and only if for each disjoint and Fo-bounded sequence $(x_n)_n \subseteq E$, $T(x_n) \xrightarrow{\|\cdot\|} 0$.

Proof . Let $T : E \to X$ be a õ-weakly compact operator. Consider a disjoint and Fo-bounded sequence $(x_n)_n \subseteq E$. There exists $u \in F^+$ such that $(x_n)_n \subseteq [-u, u]$. Let I_u be the ideal generated by u in F. According to Lemma 4.3, we have

$$B_{I_u \cap E} = I_u \cap B_E \subseteq \left[-\frac{1}{\|u\|}u, \frac{1}{\|u\|}u\right] \cap E. \tag{4.1}$$

By assumption, $T(\left[-\frac{1}{\|u\|}u, \frac{1}{\|u\|}u\right] \cap E)$ is relatively weakly compact. Therefore, the operator $T|_{I_u \cap E} : I_u \cap E \to X$ is a weakly compact operator. By Theorem 5.62 of [1], I_u is an AM-space with an order unit. Therefore, $T|_{I_u \cap E} : I_u \cap E \to X$ is M-weakly compact operator. As $(x_n)_n \subseteq I_u$ is a disjoint norm bounded sequence in E, we have $T(x_n) \xrightarrow{\|\cdot\|} 0$.

Conversely, let $A \subseteq E$ be a Fo-bounded set. Then, there exists $u \in F^+$ such that $A \subseteq [-u, u]$. Let I_u be the ideal generated by u in F, and let $(x_n)_n \subseteq I_u \cap E$ be a disjoint norm bounded sequence. Since $(x_n)_n \subseteq I_u$ is norm bounded, by Lemma 1, $(x_n)_n$ is Fo-bounded. By the assumption, we have $T(x_n) \xrightarrow{\|\cdot\|} 0$. Therefore, $T|_{I_u \cap E} : I_u \cap E \to X$ is M-weakly compact. By Theorem 5.61 of [1], $T|_{I_u \cap E} : I_u \cap E \to X$ is a weakly compact operator. Since A is norm bounded in I_u and $T|_{I_u \cap E} : I_u \cap E \to X$ is weakly compact, we conclude that $T(A)$ is a relatively weakly compact set in X. Thus, $T : E \to X$ is a õ-weakly compact operator. □

Corollary 4.5.

1. Let T and S be two operators from E to G such that $0 \leq T \leq S$ and S is a õ-weakly compact operator. If $(x_n)_n \subseteq E$ is a disjoint and Fo-bounded sequence, then by Theorem 4.4, we have $S(x_n) \xrightarrow{\|\cdot\|} 0$. It follows that $T(x_n) \xrightarrow{\|\cdot\|} 0$. Thus, T is a õ-weakly compact operator.
2. Let T be an õ-weakly compact operator from E to X, and let $S \in B(X, Y)$. By Theorem 4.4, it is clear that $S \circ T$ is a õ-weakly compact operator.

It is well-known that if $T : E \to X$ is an õ-weakly compact operator, then it is also order weakly compact. However, the converse is not true in general, as illustrated by the following example.

Example 4.6. The operator $T : \ell^1 \to \ell^\infty$ defined by

$$T(x_1, x_2, \ldots) = \left(\sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \ldots\right)$$

is an order weakly compact operator. Let $(x_n)_n \subseteq \ell^1$ be a disjoint and order bounded sequence. We have $x_n \xrightarrow{uo} 0$ and $(x_n)_n$ is order bounded, therefore, $x_n \xrightarrow{o} 0$. Since ℓ^1 has order continuous norm, $(x_n)_n$ is norm-null. Because T is a continuous operator, we have $T(x_n) \xrightarrow{\|\cdot\|} 0$ in ℓ^∞. Thus, by Theorem 5.57 of [1], T is an order weakly compact operator. If we consider $(e_n)_n \subseteq \ell^1$, we have $e_n \xrightarrow{\ell^\infty o} 0$ in ℓ^1. On the other hand, $T(e_n) = (1, 1, 1, \ldots)$, and therefore, $(T(e_n))_n$ does not converge to zero in the norm topology. Thus, T is not õ-weakly compact.

Theorem 4.7. Let G be a normed vector lattice that is a sublattice of a normed vector lattice H, and let $T : E \to G$ be a õ-weakly compact operator. Under one of the following conditions, the modulus of T exists and is a õ-weakly compact operator.

1. E is an AL-space, and G satisfies both property (P) and property (H).
2. Both E and G have an order unit.
3. G is Archimedean Dedekind complete, and T is an order-bounded operator that preserves disjointness.

Proof .

1. Let $(x_n)_n \subseteq E$ be a disjoint order bounded sequence. It is clear that $(x_n)_n$ is Fo-bounded. By the assumption and Theorem 4.4, we have $T(x_n) \xrightarrow{\|\cdot\|} 0$. Hence, by Theorem 5.57 of [1], T is an order weakly compact operator. Since E is an AL-space and G has property (P), by Theorem 2.2 of [3], the modulus $|T|$ exists and is an order weakly compact operator. Since G has property (H), $|T|$ is a õ-weakly compact operator.
2. Let A be a norm bounded set in E. Since E has an order unit, by Theorem 4.21 of [1], A is order bounded and hence Fo-bounded. By the assumption, $T(A)$ is a relatively weakly compact set in G. Hence, T is a weakly compact operator. Since G has an order unit, by Theorem 2.3 of [11], the modulus of T exists and is a weakly compact operator. It is clear that $|T|$ is also a õ-weakly compact operator.

3. By Theorem 2.40 of [1], $|T|$ exists and we have $|T|(|x|) = |T(|x|)| = |T(x)|$ for all x. If $(x_n)_n \subseteq E$ is a Fo-bounded disjoint sequence, then by the assumption, $T(x_n) \xrightarrow{\|\cdot\|} 0$. We have $|T|(|x_n|) = |T(|x_n|)| = |T(x_n)| \xrightarrow{\|\cdot\|} 0$ in G for each n. Now, using the inequality $||T|(x_n)| \leq |T||x_n|$, we have $|T|(x_n) \xrightarrow{\|\cdot\|} 0$. Hence, $|T|$ is a õ-weakly compact operator.

□

The following examples demonstrate that õ-weakly compact operators do not possess the duality property.

Example 4.8. 1. Consider the operator $T : C[0,1] \to c_0$ defined by

$$T(f) = \left(\int_0^1 f(x)\sin(x)dx, \int_0^1 f(x)\sin(2x)dx, ... \right).$$

By Example 3.15 of [10], T is a wun-Dunford-Pettis, and by Theorem 3.11 of [10], T is a weakly compact operator. Therefore, T is an õ-weakly compact operator. We have $T^* : \ell^1 \to (C[0,1])^*$ defined by

$$T^*(x_n)(f) = \sum_{n=1}^{\infty} x_n \left(\int_0^1 f(t)\sin(nt)dt \right).$$

Consider the sequence $(e_n) \subseteq \ell^1$, which is ℓ^∞-order bounded and disjoint. Let $f_n(t) = \sin(nt)$ for all n. We have

$$\|T^*(e_n)\| \geq \|T^*(e_n)(f_n)\| = \int_0^1 (\sin(nt))^2 dt \nrightarrow 0.$$

Thus, by Theorem 4.4, T^* is not a õ-weakly compact operator.

2. consider the functional $f : \ell^1 \to \mathbb{R}$ defined by

$$f(x_1, x_2, ...) = \sum_{n=1}^{\infty} x_n.$$

The sequence $(e_n) \subseteq \ell^1$ is ℓ^∞-order bounded and disjoint, but $f(e_n) \nrightarrow 0$. Therefore, by Theorem 4.4, f is not a õ-weakly compact operator. However, it is obvious that $f^* : \mathbb{R} \to \ell^\infty$ is a õ-weakly compact operator.

In the following, we demonstrate that under certain conditions, if an operator T is õ-weakly compact, then its adjoint T^* is also õ-weakly compact, and vice versa.

Proposition 4.9. Let G be a vector lattice such that $G^* \subseteq F$. Then the following assertions hold:

1. If E has an order unit and $T : E \to G$ is a õ-weakly compact operator, then T^* is also a õ-weakly compact operator.
2. If G^* has an order unit and $T^* : G^* \to E^*$ is a õ-weakly compact operator, then T is also a õ-weakly compact operator.

Proof .

1. Let E have an order unit, and suppose $T : E \to G$ is a õ-weakly compact operator. It is clear that T is a weakly compact operator. By Theorem 5.23 of [1], T^* is also a weakly compact operator, and therefore, T^* is a õ-weakly compact operator.
2. The proof follows a similar argument as in (1).

□

Theorem 4.10. Let $T : F \to X$ be an operator. The restriction $T |_E : E \to X$ is õ-weakly compact if and only if $T(A)$ is relatively weakly compact for every F-almost order bounded subset $A \subseteq E$.

Proof . If $T(A)$ is relatively weakly compact for every F-almost order bounded subset A of E, it is evident that $T |_E$ is a õ-weakly compact operator.

Conversely, let $A \subseteq E$ be an F-almost order bounded set. For every $\epsilon > 0$, there exists $u \in F^+$ such that $A \subseteq [-u, u] + \varepsilon B_E$. Since T is linear, we have $T(A) \subseteq T([-u, u] \cap E) + \epsilon T(B_E)$. As T is õ-weakly compact, $T([-u, u] \cap E)$ is a relatively weakly compact set. By Theorem 3.44 of [1], $T(A)$ is a relatively weakly compact set in X. □

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