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The multi-decision-making problem in the recruitment process using penta-partitioned neutrosophic distance measure

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Abstract

In this article, we introduce some different distance measures on penta partitioned neutrosophic sets and some distance measures satisfy axioms of metric. We also formulate a new methodological approach to solve the multi-criteria decision-making problems, in which the suitable decision is by ranking the average of the proposed penta partitioned neutrosophic distance measure for the alternatives to the criteria under certain conditional criteria. Further, we apply these distance measures to a multi-criteria decision-making problem (shortly MCDM) for the best employee selection to recruit for a post. The comparison is finally made between the proposed distance measures and the final decisions are the same in all penta partitioned neutrosophic distance measures.

Keywords: Neutrosophic sets, Penta partitioned neutrosophic sets, Penta partitioned neutrosophic distance measures, Multi-criteria decision-making problems, Recruitment problem
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1 Introduction

The fuzzy set theory was developed by Zadeh [31], which is the generalization of classical set theory to deal with the various types of uncertainties by assigning the membership grade ranging from zero to one to each object. The researchers and academicians rapidly increased their attention to studying the fuzzy set theory applications in various fields such as engineering, economy, medicine, and social sciences. The fuzzy distance measure is a distance measure between two fuzzy sets, which plays an important role in real-life applications to pattern recognition, remote sensing, control systems, medical diagnosis, etc. Voxman [27] was the first one to introduce fuzzy distance measures by defining fuzzy numbers. The similarity measure is also a measure to discuss the relationship and the similarities between two different fuzzy sets. Pappis and Karacapilidis [21] defined some similarity measures in fuzzy sets with their applications. Pramanik and Mondal [22] introduced weighted fuzzy tangent similarity measures and framed a medical application by using these fuzzy tangent similarity measures in this fuzzy world.

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In this uncertain world, the behavior of every human being is different of their culture dress, education, habits, etc. Every person has different senses too like as well as, unlike an object. The study of likeness and unlikeness by giving percentages at the same time leads to a new theory called an intuitionistic fuzzy set theory. This intuitionistic fuzzy set theory is a generalized form of Zadeh's fuzzy set theory, which was developed by Atanassov [4]. The intuitionistic fuzzy set is characterized by considering the degrees of both the membership and the non-membership in $[0, 1]$ such that their summation should be less than 1. This theory motivated many researchers to apply it to various research fields such as coding theory, control systems, data analysis, medical diagnosis, pattern recognition, operational research, etc. Recently Dutta and Goala [14] introduced intuitionistic fuzzy distance measure, which is a distance measure between two different intuitionistic fuzzy sets and gave its application in medical diagnosis. Szmidt and Kacprzyk [25] also introduced some distance measures on intuitionistic fuzzy sets and further applied these measures [26] in medical diagnosis as a real-life application.

Most of the organizations, countries, regions, and divisions in this world choose their leader by conducting elections. Every election gives options to all the people to wish to vote the contestant X, to wish to vote the opponent Y and to wish not to vote in the election. The percentage of people who wish not to vote in the election may be deciding the winner of the election by voting to X or Y, such a situation is called 'neutral or indeterminist'. These kinds of situations led to a new era in mathematics called neutrosophic fuzzy set theory by giving percentages for membership, for non-membership, and for indeterminacy. Smarandache [24] imitated the concept of a neutrosophic fuzzy set which is the generalization of intuitionistic fuzzy set and fuzzy set. It consists of three components as truth membership function, indeterminacy membership function, and falsity membership function whose component values lie in the real non-standard unit interval. In real-life applications, scientific or engineering problems face many difficulties in using the neutrosophic set whose values are from real standard or non-standard interval $]0^-, 1^+[$, but it needs the specified neutrosophic set operators. For applying the neutrosophic set more conveniently in a real-life situation, Wang et al. [28] defined a single-valued neutrosophic set which is a neutrosophic fuzzy set whose component values are lying in the unit interval $[0, 1]$. Thus a single-valued neutrosophic set is a special case of the neutrosophic set. The different properties of neutrosophic sets and single-valued neutrosophic sets are extensively studied and applied to various research fields. The distances between two different neutrosophic sets are calculated to study the relationship between them, and called distance measures, similarity measures, and so on. Majumdar and Samanta [18] defined the Euclidean and the hamming distance measures on single-valued neutrosophic sets. Biswas et al. [5] discussed the cosine similarity measure in multi-criteria decision-making problems. The tangent similarity measure was introduced by Mondal and Pramanik [20] with its application. Biswas et al. [6] again defined a variety of distance measures on single-valued neutrosophic sets and compared their method's decisions with the decisions of other existing methods by solving multi-criteria decision-making problems. Shahzadi et al. [23] established an application of single-valued neutrosophic sets in medical diagnosis. Ye and Zhang [30] applied the single-valued neutrosophic similarity measures and single-valued neutrosophic cross-entropy [29] in multi-criterion decision-making problems. Jayaparthasarathy et al. [16] introduced neutrosophic supra topology with an application in the data mining process and they [15] further developed some operators by defining the concept of N-neutrosophic supra topological spaces. Arockia Dasan et al. [1] defined N-neutrosophic supra topological mappings. Recently Chai et al. [12] defined new distance and similarity measures on single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems. Dutta and Goala [13] discussed the application of medical diagnosis using distance measures in picture fuzzy sets. Arockia Dasan et al. [3] developed a method to solve decision-making problems in plant hybridization by using score functions on single-valued neutrosophic sets. Arockia Dasan et al. [2] further introduced sine metric single-valued neutrosophic distance measure and discussed the applications of multi-criteria decision-making problems in career determination. Broumi and Smarandache [9] introduced the concept of similarity measures in single-valued neutrosophic sets and Broumi's team [10, 8, 11, 7] further developed applications of Fermatean neutrosophic graphs and interval-valued Fermatean neutrosophic sets.

The recruitment process is a process of finding suitable candidates for various posts in an organization or in a company. A company or organization decides: Which type of candidate is required? Is the candidate suitable for the position or post? What kind of employee should this company select to move on the path of progress? In such situations, the recruitment process helps to identify suitable or eligible candidates for the suitable post. Companies want to successfully run their business if they have to pay careful attention to improving their recruitment process. If they have consistent fear then they will be made bad hiring decisions. Recruitment acts as the link between job providers and job seekers. The reason behind the recruitment is the retirement of an employee, death of an employee, resignation by an employee, disablement of an employee, and so on. Rama Malik and Surpati Pramanik [19] introduced penta partitioned neutrosophic sets by dividing the indeterminacy component into three components such as contradiction, ignorance, and unknown membership function, and their different properties are derived. Smarandache [24] stated that the penta partitioned neutrosophic set is a particular case of the refined neutrosophic set by spliting the membership, non-membership and indeterminancy functions.

Motivation of our work: On neutrosophic sets and single-valued neutrosophic sets, there are many distance measures and similarity measures, score functions [13, 14, 15, 16, 17, 18, 19, 20, 21, 22,23] are introduced and applied in medical diagnosis, data analysis, and pattern recognition to deal with multi-decision-making problems. These neutrosophic measures are always necessary to check and validate the results to obtain more reliable and convenient results. In addition, the obtained results by these measures are not relevant and contradict one another. Rama Malik and Surpati Pramanik [19] introduced penta partitioned neutrosophic set, which is a particular case of the refined neutrosophic set [] and no one discussed the multi-decision-making problem by defining distance measures on these sets so far. With these in mind, we are motivated to define penta partitioned neutrosophic distance measures and discuss the numerical example in the recruitment process in a MCDM problem. The main objectives of this paper are:

1. To define new distance measures on penta partitioned neutrosophic sets.
2. To formulate a methodological approach by using these distance measures.
3. To apply the above methodology in MCDM problems for the recruitment selection process under certain criteria.
4. To observe the final decision in MCDM problems for the penta partitioned neutrosophic distance measures.

The Organization of the paper is as follows: Section 2 discusses some basic preliminaries about fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, penta partitioned neutrosophic sets, and some known neutrosophic distance measures. The third section introduces some new distance measures on penta partitioned neutrosophic sets, and shows some of the proposed distance measures that satisfy the metric axioms. In section 4, a new methodological approach to MCDM problems is developed by using the proposed penta partitioned neutrosophic distance measures. The fifth section introduces a numerical example as a real-life application to select the best employee in a penta partitioned neutrosophic environment. Section 6 states some advantages and limitations of the present work. The conclusion and the future work are stated in the seventh section, also following the reference section.

2 Preliminaries

In this section, we review some basic definitions of neutrosophic fuzzy sets, single-valued neutrosophic fuzzy sets, and neutrosophic distance measures.

Definition 2.1. [31] Let X be a non empty set and a fuzzy set A on X is of the form $A = \{(x, \mu_A(x)) : x \in X\}$, where $0 \leq \mu_A(x) \leq 1$ represents the degree of membership function of each $x \in X$ to the set A.

Definition 2.2. [4] Let X be a non empty set. An intuitionistic set A is of the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A(x)$ and $\gamma_A(x)$ represents the degree of membership and non-membership function respectively of each $x \in X$ to the set A and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.

Definition 2.3. [24] Let X be a non empty set. A neutrosophic set A having the form $A = \{(x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x) \in]0^-, 1^+[$ represent the degree of membership (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of non membership (namely $\gamma_A(x)$) respectively of each $x \in X$ to the set A such that $0^- \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3^+$, for all $x \in X$. $NS(X)$ denotes the collection of all neutrosophic sets of X.

Definition 2.4. [28] A single-valued neutrosophic set (shortly SVNS) A in X is a neutrosophic set which is of the form $A = \{(x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X\}$, that is characterized by the degree of membership (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of non membership (namely $\gamma_A(x)$), where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x) \in [0, 1]$ such that such that $0 \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3$ for all $x \in X$, respectively. $SVNS(X)$ denotes the collection of all single valued neutrosophic sets of X.

Definition 2.5. [18] Let $X = \{x_1, x_2, ..., x_n\}$ be a discrete confined set. A mapping $d : SVNS(X) * SVNS(X) \rightarrow [0, 1]$ is said to be a distance measure between two single-valued neutrosophic sets if it satisfies the following axioms:

1. $d(A, B) \geq 0$, for all $A, B \in SVNS(X)$.

2. $d(A,B)=0$ if and only if $A=B$, for all $A,B\in SVNS(X)$.
3. $d(A,B)=d(B,A)$, for all $A,B\in SVNS(X)$.
4. If $A\subseteq B\subseteq C$, for all $A,B,C\in SVNS(X)$, then $d(A,C)\geq d(A,B)$ and $d(A,C)\geq d(B,C)$.

If the mapping is defined as $d(A,B)=max\{|\mu_A(x_i)-\mu_B(x_i)|,|\sigma_A(x_i)-\sigma_B(x_i)|,|\gamma_A(x_i)-\gamma_B(x_i)|\}$, $\forall x_i\in X$, then $d(A,B)$ satisfies axioms of distance measure and is called the extended Hausdorff distance measure between two single-valued neutrosophic sets A and B.

Definition 2.6. [18] The normalized Hamming distance measure between two single-valued neutrosophic sets A and B is defined as $d_1(A,B)=\frac{1}{3n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\sigma_A(x_i)-\sigma_B(x_i)|+|\gamma_A(x_i)-\gamma_B(x_i)|)$.

Definition 2.7. [18] The normalized Euclidean distance measure between two single-valued neutrosophic sets A and B is defined as $d_2(A,B)=\{\frac{1}{3n}\sum_{i=1}^{n}((\mu_A(x_i)-\mu_B(x_i)^2+(\sigma_A(x_i)-\sigma_B(x_i)^2+(\gamma_A(x_i)-\gamma_B(x_i)^2)\}^{\frac{1}{2}}$.

Definition 2.8. [12] The distance measures between two single-valued neutrosophic sets A and B are defined as

$$D_1(A,B)=\frac{1}{3n}\sum_{i=1}^{n}(|(\mu_A(x_i))^2-(\mu_B(x_i)^2|+|(\sigma_A(x_i))^2-(\sigma_B(x_i))^2|+|(\gamma_A(x_i))^2-(\gamma_B(x_i))^2|)$$

and

$$D_2(A,B)=\frac{1}{3n}\sum_{i=1}^{n}|((\mu_A(x_i))^2-(\mu_B(x_i))^2)-((\sigma_A(x_i))^2-(\sigma_B(x_i))^2)-((\gamma_A(x_i))^2-(\gamma_B(x_i))^2)|.$$

Definition 2.9. [2] The sine metric distance measure between two single-valued neutrosophic sets A and B is defined as

$$d(A,B)=\frac{5}{3n}\sum_{i=1}^{n}\frac{\sin\{\frac{\pi}{6}(|\mu_A(x_i)-\mu_B(x_i)|)\}+\sin\{\frac{\pi}{6}(|\sigma_A(x_i)-\sigma_B(x_i)|)\}+\sin\{\frac{\pi}{6}(|\gamma_A(x_i)-\gamma_B(x_i)|)\}}{1+\sin\{\frac{\pi}{6}(|\mu_A(x_i)-\mu_B(x_i)|)\}+\sin\{\frac{\pi}{6}(|\sigma_A(x_i)-\sigma_B(x_i)|)\}+\sin\{\frac{\pi}{6}(|\gamma_A(x_i)-\gamma_B(x_i)|)\}}.$$

Definition 2.10. [9] A mapping $S:SVNS(X)*SVNS(X)\rightarrow[0,1]$ is said to be a similarity measure between two single-valued neutrosophic sets if it satisfies the following axioms:

1. $S(A,B)\geq 0$ for all $A,B\in SVNS(X)$.
2. $S(A,B)=1$ if and only if $A=B$ for all $A,B\in SVNS(X)$.
3. $S(A,B)=S(B,A)$ for all $A,B\in SVNS(X)$.
4. If $A\subseteq B\subseteq C$ for all $A,B,C\in SVNS(X)$, then $S(A,C)\leq S(A,B)$ and $S(A,C)\leq S(B,C)$.

Definition 2.11. [19] Let X be a non empty set. A penta partitioned neutrosophic set A having the form $A=\{(x,\mu_A(x),\sigma_{1A}(x),\sigma_{2A}(x),\sigma_{3A}(x),\gamma_A(x)):x\in X\}$, where $\mu_A(x),\sigma_{1A}(x),\sigma_{2A}(x),\sigma_{3A}(x),\gamma_A(x)\in[0,1]$ represent the degree of membership (namely $\mu_A(x)$), the degree of contradiction (namely $\sigma_{1A}(x)$), the degree of ignorance membership (namely $\sigma_{2A}(x)$), unknown membership (namely $\sigma_{3A}(x)$) and the degree of non membership (namely $\gamma_A(x)$) respectively of each $x\in X$ to the set A such that $0\leq\mu_A(x)+\sigma_{1A}(x)+\sigma_{2A}(x)+\sigma_{3A}(x)+\gamma_A(x)\leq 5$ for all $x\in X$. $PNS(X)$ denotes the collection of all penta partitioned neutrosophic sets of X.

Definition 2.12. [19] The following statements are true for penta partitioned neutrosophic sets A and B on X:

1. $\mu_A(x)\leq\mu_B(x),\sigma_{1A}(x)\leq\sigma_{1B}(x),\sigma_{2A}(x)\geq\sigma_{2B}(x),\sigma_{3A}(x)\geq\sigma_{3B}(x)$ and $\gamma_A(x)\geq\gamma_B(x)$ for all $x\in X$ if and only if $A\subseteq B$.
2. $A\subseteq B$ and $B\subseteq A$ if and only if $A=B$.
3. $A\cap B=\{(x,\min\{\mu_A(x),\mu_B(x)\},\min\{\sigma_{1A}(x),\sigma_{1B}(x)\},\max\{\sigma_{2A}(x),\sigma_{2B}(x)\},\max\{\sigma_{3A}(x),\sigma_{3B}(x)\},\max\{\gamma_A(x)\geq\gamma_B(x)\}):x\in X\}$.
4. $A\cup B=\{(x,\max\{\mu_A(x),\mu_B(x)\},max\{\sigma_{1A}(x),\sigma_{1B}(x)\},\min\{\sigma_{2A}(x),\sigma_{2B}(x)\},\min\{\sigma_{3A}(x),\sigma_{3B}(x)\},\min\{\gamma_A(x)\geq\gamma_B(x)\}):x\in X\}$.

3 Penta Partitioned Neutrosophic Distance Measures

This section defines some distance measures on penta partitioned neutrosophic sets. Some properties of these penta partitioned distance measures are derived.

Definition 3.1. Let $X = \{x_1, x_2, ..., x_n\}$ be a discrete confined set. A mapping $d : PNS(X) * PNS(X) \to [0, 1]$ is said to be a penta partitioned neutrosophic distance measure between two penta partitioned neutrosophic sets if it satisfies the following axioms:

1. $d(A, B) \geq 0$ for all $A, B \in PNS(X)$.
2. $d(A, B) = 0$ if and only if $A = B$ for all $A, B \in PNS(X)$.
3. $d(A, B) = d(B, A)$ for all $A, B \in PNS(X)$.
4. If $A \subseteq B \subseteq C$ for all $A, B, C \in PNS(X)$, then $d(A, C) \geq d(A, B)$ and $d(A, C) \geq d(B, C)$.

Definition 3.2. Let $X = \{x_1, x_2, ..., x_n\}$ be a discrete confined set. Let $A = \{(x_j, \mu_A(x_j), \sigma_{1A}(x_j), \sigma_{2A}(x_j), \sigma_{3A}(x_j), \gamma_A(x_j)) : x_j \in X\}$ and $B = \{(x_j, \mu_B(x_j), \sigma_{1B}(x_j), \sigma_{2B}(x_j), \sigma_{3B}(x_j), \gamma_B(x_j)) : x_j \in X\}$ be two penta partitioned neutrosophic sets. Then for $i = 1, 2, 3, 4, 5$, define a mapping $d_i : PNS(X) * PNS(X) \to [0, 1]$ as

1. $d_1(A, B) = \frac{1}{5n} \sum_{j=1}^{n} (|\mu_A(x_j) - \mu_B(x_j)| + |\sigma_{1A}(x_j) - \sigma_{1B}(x_j)| + |\sigma_{2A}(x_j) - \sigma_{2B}(x_j)| + |\sigma_{3A}(x_j) - \sigma_{3B}(x_j)| + |\gamma_A(x_j) - \gamma_B(x_j)|)$.
2. $d_2(A, B) = \{\frac{1}{5n} \sum_{j=1}^{n} ((\mu_A(x_j) - \mu_B(x_j))^2 + (\sigma_{1A}(x_j) - \sigma_{1B}(x_j))^2 + (\sigma_{2A}(x_j) - \sigma_{2B}(x_j))^2 + (\sigma_{3A}(x_j) - \sigma_{3B}(x_j))^2 + (\gamma_A(x_j) - \gamma_B(x_j))^2\}^{\frac{1}{2}}$.
3. $d_3(A, B) = \frac{1}{5n} \sum_{j=1}^{n} (|(\mu_A(x_j))^2 - (\mu_B(x_j))^2| + |(\sigma_{1A}(x_j))^2 - (\sigma_{1B}(x_j))^2| + |(\sigma_{2A}(x_j))^2 - (\sigma_{2B}(x_j))^2| + |(\sigma_{3A}(x_j))^2 - (\sigma_{3B}(x_j))^2| + |(\gamma_A(x_j))^2 - (\gamma_B(x_j))^2|)$.
4. $d_4(A, B) = \frac{1}{5n} \sum_{j=1}^{n} (|((\mu_A(x_j))^2 - (\mu_B(x_j))^2 + ((\sigma_{1A}(x_j))^2 - (\sigma_{1B}(x_j))^2) - ((\sigma_{2A}(x_j))^2 - (\sigma_{2B}(x_j))^2) - ((\sigma_{3A}(x_j))^2 - (\sigma_{3B}(x_j))^2) - ((\gamma_A(x_j))^2 - (\gamma_B(x_j))^2|)$.
5.
$$d_5(A, B) = \frac{7}{5n} \sum_{j=1}^{n} \frac{\begin{aligned}&(\sin\{\frac{\pi}{6}(|\mu_A(x_j) - \mu_B(x_j)|)\} + \sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j) - \sigma_{1B}(x_j)|)\} \\ &+ \sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j) - \sigma_{2B}(x_j)|)\} + \sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j) - \sigma_{3B}(x_j)|)\} \\ &+ \sin\{\frac{\pi}{6}(|\gamma_A(x_j) - \gamma_B(x_j)|)\})\end{aligned}}{\begin{aligned}&(1 + \sin\{\frac{\pi}{6}(|\mu_A(x_j) - \mu_B(x_j)|)\} + \sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j) - \sigma_{1B}(x_j)|)\} \\ &+ \sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j) - \sigma_{2B}(x_j)|)\} + \sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j) - \sigma_{3B}(x_j)|)\} \\ &+ \sin\{\frac{\pi}{6}(|\gamma_A(x_j) - \gamma_B(x_j)|)\})\end{aligned}}.$$

Theorem 3.3. For each $i = 1, 2, 3, 5, d_i(A, B)$ is a penta partitioned neutrosophic distance measure between two penta partitioned neutrosophic sets A and B.

Proof . Here we only prove $d_3(A, B)$ is a penta partitioned neutrosophic distance measure, and note that the remaining $d_i(A, B)$ have similar proofs for $i = 1, 2, 5$.

1. $d_3(A, B) \geq 0$ is trivially true from the definition of penta partitioned neutrosophic sets.
2. $d_3(A, B) = 0$ if and only if

$$\frac{1}{5n} \sum_{j=1}^{n} (|((\mu_A(x_j))^2 - (\mu_B(x_j))^2)| + |((\sigma_{1A}(x_j))^2 - (\sigma_{1B}(x_j))^2)| - |((\sigma_{2A}(x_j))^2 - (\sigma_{2B}(x_j))^2)|$$
$$- |((\sigma_{3A}(x_j))^2 - (\sigma_{3B}(x_j))^2)| - |((\gamma_A(x_j))^2 - (\gamma_B(x_j))^2|)|) = 0$$

if and only if, for all $x_j \in X$,

$$|((\mu_A(x_j))^2 - (\mu_B(x_j))^2)| + |((\sigma_{1A}(x_j))^2 - (\sigma_{1B}(x_j))^2)| - |((\sigma_{2A}(x_j))^2 - (\sigma_{2B}(x_j))^2)| \\ - |((\sigma_{3A}(x_j))^2 - (\sigma_{3B}(x_j))^2)| - |((\gamma_A(x_j))^2 - (\gamma_B(x_j))^2| = 0,$$

if and only if, for all $x_j \in X$,

$$(\mu_A(x_j))^2 - (\mu_B(x_j))^2 = 0, (\sigma_{1A}(x_j))^2 - (\sigma_{1B}(x_j))^2 = 0, (\sigma_{2A}(x_j))^2 - (\sigma_{2B}(x_j))^2 = 0, \\ (\sigma_{3A}(x_j))^2 - (\sigma_{3B}(x_j))^2 = 0, (\gamma_A(x_j))^2 - (\gamma_B(x_j))^2 = 0,$$

if and only if $\mu_A(x_j) = \mu_B(x_j), \sigma_{1A}(x_j) = \sigma_{1B}(x_j), \sigma_{2A}(x_j) = \sigma_{2B}(x_j), \sigma_{3A}(x_j) = \sigma_{3B}(x_j), \gamma_A(x_j) = \gamma_B(x_j)$, for all $x_j \in X$, if and only if $A = B$.

3.

$$\begin{aligned} d_3(A,B) =& \frac{1}{5n}\sum_{j=1}^{n}(|((\mu_A(x_j))^2 - (\mu_B(x_j))^2)| + |((\sigma_{1A}(x_j))^2 - (\sigma_{1B}(x_j))^2)| - |((\sigma_{2A}(x_j))^2 - (\sigma_{2B}(x_j))^2)| \\ & - |((\sigma_{3A}(x_j))^2 - (\sigma_{3B}(x_j))^2)| - |((\gamma_A(x_j))^2 - (\gamma_B(x_j))^2)|)) \\ =& \frac{1}{5n}\sum_{j=1}^{n}(|((\mu_B 0(x_j))^2 - (\mu_A(x_j))^2)| + |((\sigma_{1B}(x_j))^2 - (\sigma_{1A}(x_j))^2)| - |((\sigma_{2B}(x_j))^2 - (\sigma_{2A}(x_j))^2)| \\ & - |((\sigma_{3B}(x_j))^2 - (\sigma_{3A}(x_j))^2)| - |((\gamma_B(x_j))^2 - (\gamma_A(x_j))^2)|) = d_3(B,A). \end{aligned}$$

4. If $A \subseteq B \subseteq C$, then $\mu_A(x_j) \leq \mu_B(x_j) \leq \mu_C(x_j), \sigma_{1A}(x_j) \leq \sigma_{1B}(x_j) \leq \sigma_{1C}(x_j), \sigma_{2A}(x_j) \geq \sigma_{2B}(x_j) \geq \sigma_{2C}(x_j), \sigma_{3A}(x_j) \geq \sigma_{3B}(x_j) \geq \sigma_{3C}(x_j), \gamma_A(x_j) \geq \gamma_B(x_j) \geq \gamma_C(x_j)$, for all $x_j \in X$. Then we have the following inequalities:

$$\begin{aligned} (\mu_A(x_j))^2 - (\mu_C(x_j))^2 \leq& (\mu_A(x_j))^2 - (\mu_B(x_j))^2, (\mu_A(x_j))^2 - (\mu_C(x_j))^2 \\ \leq& (\mu_B(x_j))^2 - (\mu_C(x_j))^2, \\ (\sigma_{1A}(x_j))^2 - (\sigma_{1C}(x_j))^2 \leq& (\sigma_{1A}(x_j))^2 - (\sigma_{1B}(x_j))^2, (\sigma_{1A}(x_j))^2 - (\sigma_{1C}(x_j))^2 \\ \leq& (\sigma_{1B}(x_j))^2 - (\sigma_{1C}(x_j))^2, \\ (\sigma_{2A}(x_j))^2 - (\sigma_{2C}(x_j))^2 \geq& (\sigma_{2A}(x_j))^2 - (\sigma_{2B}(x_j))^2, (\sigma_{2A}(x_j))^2 - (\sigma_{2C}(x_j))^2 \\ \geq& (\sigma_{2B}(x_j))^2 - (\sigma_{2C}(x_j))^2, \\ (\sigma_{3A}(x_j))^2 - (\sigma_{3C}(x_j))^2 \geq& (\sigma_{3A}(x_j))^2 - (\sigma_{3B}(x_j))^2, (\sigma_{3A}(x_j))^2 - (\sigma_{3C}(x_j))^2 \\ \geq& (\sigma_{3B}(x_j))^2 - (\sigma_{3C}(x_j))^2, \\ (\gamma_A(x_j))^2 - (\gamma_C(x_j))^2 \geq& (\gamma_A(x_j))^2 - (\gamma_B(x_j))^2, (\gamma_A(x_j))^2 - (\gamma_C(x_j))^2 \\ \geq& (\gamma_B(x_j))^2 - (\gamma_C(x_j))^2. \end{aligned}$$

From these inequalities we have,

$$\begin{aligned} & \frac{1}{5n}\sum_{j=1}^{n}(|((\mu_A(x_j))^2 - (\mu_C(x_j))^2)| + |((\sigma_{1A}(x_j))^2 - (\sigma_{1C}(x_j))^2)| - |((\sigma_{2A}(x_j))^2 - (\sigma_{2C}(x_j))^2)| \\ & - |((\sigma_{3A}(x_j))^2 - (\sigma_{3C}(x_j))^2)| - |((\gamma_A(x_j))^2 - (\gamma_C(x_j))^2)|) \\ \geq& \frac{1}{5n}\sum_{j=1}^{n}(|((\mu_B(x_j))^2 - (\mu_C(x_j))^2)| + |((\sigma_{1B}(x_j))^2 - (\sigma_{1C}(x_j))^2)| - |((\sigma_{2B}(x_j))^2 - (\sigma_{2C}(x_j))^2)| \\ & - |((\sigma_{3B}(x_j))^2 - (\sigma_{3C}(x_j))^2)| - |((\gamma_B(x_j))^2 - (\gamma_C(x_j))^2)|) \end{aligned}$$

and

$$\frac{1}{5n}\sum_{j=1}^{n}(|((\mu_A(x_j))^2-(\mu_C(x_j))^2)|+|((\sigma_{1A}(x_j))^2-(\sigma_{1C}(x_j))^2)|-|((\sigma_{2A}(x_j))^2-(\sigma_{2C}(x_j))^2)|$$
$$-|((\sigma_{3A}(x_j))^2-(\sigma_{3C}(x_j))^2)|-|((\gamma_A(x_j))^2-(\gamma_C(x_j))^2)|)$$
$$\geq\frac{1}{5n}\sum_{j=1}^{n}(|((\mu_A(x_j))^2-(\mu_B(x_j))^2)|+|((\sigma_{1A}(x_j))^2-(\sigma_{1B}(x_j))^2)|-|((\sigma_{2A}(x_j))^2-(\sigma_{2B}(x_j))^2)|$$
$$-|((\sigma_{3A}(x_j))^2-(\sigma_{3B}(x_j))^2)|-|((\gamma_A(x_j))^2-(\gamma_B(x_j))^2)|)$$

Therefore, $d_3(A,C)\geq d_3(B,C)$ and $d_3(A,C)\geq d_3(A,B)$.

Hence $d_3(A,B)$ is a penta partitioned neutrosophic distance measure. □

Theorem 3.4. For $i=1,2,3,5$, $d_i(A,C)\leq d_i(A,B)+d_i(B,C)$ is true for $A,B,C\in PNS(X)$.

Proof . Here we prove the triangle inequality only for $i=5$, and in the similar manner we can prove for other cases $i=1,2,3$. Let $A,B,C\in PNS(X)$ then the following inequalities are true for the real numbers:

$$|\mu_A(x_j)-\mu_C(x_j)|\leq|\mu_A(x_j)-\mu_B(x_j)|+|\mu_B(x_j)-\mu_C(x_j)|,\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_C(x_j)|)\}$$
$$\leq\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\},\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1C}(x_j)|)\}$$
$$\leq\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\},\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2C}(x_j)|)\}$$
$$\leq\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\},\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3C}(x_j)|)\}$$
$$\leq\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\},\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_C(x_j)|)\}$$
$$\leq\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}.$$

Then,

$$\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_C(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1C}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2C}(x_j)|)\}$$
$$+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3C}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_C(x_j)|)\}$$
$$\leq\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}$$
$$+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\}$$
$$+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\}$$
$$+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}$$

So,

$$\frac{1}{\begin{array}{c}1+\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_C(x_j)|)\}\end{array}}\geq\frac{1}{\begin{array}{c}1+\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}\end{array}}$$

Then

$$1-\frac{1}{\begin{array}{r}1+\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_C(x_j)|)\}\end{array}}\leq 1-\frac{1}{\begin{array}{r}1+\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}\end{array}}.$$

Hence,

$$\frac{\begin{array}{r}\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_C(x_j)|)\}\end{array}}{\begin{array}{r}1+\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_C(x_j)|)\}\end{array}}\leq\frac{\begin{array}{r}\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}\end{array}}{\begin{array}{r}1+\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}\end{array}}\leq$$

$$\frac{\begin{array}{r}\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}\}\end{array}}{\begin{array}{r}1+\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}\}\end{array}}+\frac{\begin{array}{r}\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}\end{array}}{\begin{array}{r}1+\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}\end{array}}\Rightarrow$$

$$\frac{7}{5n}\sum_{j=1}^{n}\frac{\begin{array}{r}\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_C(x_j)|)\}\end{array}}{\begin{array}{r}1+\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_C(x_j)|)\}\end{array}}\leq\frac{7}{5n}\sum_{j=1}^{n}\frac{\begin{array}{r}\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}\}\end{array}}{\begin{array}{r}1+\sin\{\frac{\pi}{6}(|\mu_A(x_j)-\mu_B(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1A}(x_j)-\sigma_{1B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2A}(x_j)-\sigma_{2B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3A}(x_j)-\sigma_{3B}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_A(x_j)-\gamma_B(x_j)|)\}\}\end{array}}+$$

$$\frac{7}{5n}\sum_{j=1}^{n}\frac{\begin{array}{r}\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}\end{array}}{\begin{array}{r}1+\sin\{\frac{\pi}{6}(|\mu_B(x_j)-\mu_C(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{1B}(x_j)-\sigma_{1C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{2B}(x_j)-\sigma_{2C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\sigma_{3B}(x_j)-\sigma_{3C}(x_j)|)\}\\+\sin\{\frac{\pi}{6}(|\gamma_B(x_j)-\gamma_C(x_j)|)\}\end{array}}.$$

Therefore, $d_5(A, C) \leq d_5(A, B) + d_5(B, C)$ is true for $A, B, C \in PNS(X)$. □

Remark 3.5. From the above theorem 3.3 and 3.4, the distance measure $d_i(A, B)$ for $i = 1, 2, 3, 5$, satisfies all the metric axioms[17], but the function $d_4(A, B)$ is not a penta partitioned distance measure as well as not metric on penta partitioned neutrosophic sets A, B, and we call $d_1(A, B)$ is ABHL-type 1 penta partitioned neutrosophic metric distance measure, $d_2(A, B)$ is ABHL-type 2 penta partitioned neutrosophic metric distance measure, $d_3(A, B)$ is ABHL-type 3 penta partitioned neutrosophic metric distance measure, $d_4(A, B)$ is ABHL-type 4 penta partitioned neutrosophic quasi-pseudo distance measure, and $d_5(A, B)$ is ABHL-type 5 penta partitioned neutrosophic sine metric distance measure, where the abbrivation of ABHL is Arockiadasan Bementa Hydarakca Littleflower.

4 Methodologies in Neutrosophic Multi-Criteria Decision-Making Problems

In this section, we propose a methodological approach to neutrosophic MCDM problems by using all ABHL-types penta partitioned neutrosophic distance measures. The following steps are the necessary steps for the proposed methodological approach to select the alternatives with suitable criteria's under different criteria.

Step 1: Problem field Selection: Consider the multi-criteria decision-making problem with l alternatives $E_1, E_2, ..., E_l$, m conditioned criteria $C_1, C_2, ..., C_m$, n decision criteria $S_1, S_2, ..., S_n$ such that $n \leq m$. Here all the criterias e_{pq} and $c_{qr}(p = 1, 2, ..., l; q = 1, 2, ..., m$ and $r = 1, 2, ..., n)$ are all penta partitioned neutrosophic sets.

	C_1	C_2	.	.	.	C_m
E_1	(e_{11})	(e_{12})	.	.	.	(e_{1m})
E_2	(e_{21})	(e_{22})	.	.	.	(e_{2m})
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
E_l	(e_{l1})	(e_{l2})	.	.	.	(e_{lm})

Table 4.1.

	S_1	S_2	.	.	.	S_n
C_1	(c_{11})	(c_{12})	.	.	.	(c_{1n})
C_2	(c_{21})	(c_{22})	.	.	.	(c_{2n})
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
C_m	(c_{m1})	(c_{m2})	.	.	.	(c_{mn})

Table 4.2.

Step 2: The distance measures of alternatives and criterias: Calculate the distance measure of the decision alternatives E_p and the criteria S_r by using the following distance measures $d_i(S_r, E_p)$, where $r = 1, 2, ..., n, p = 1, 2, ..., l$, and for $i = 1, 2, 3, 4, 5$.

1. $d_1(S_r, E_p) = \frac{1}{5n}\sum_{j=1}^{n}(|\mu_{S_r}(x_j) - \mu_{E_p}(x_j)| + |\sigma_{1S_r}(x_j) - \sigma_{1E_p}(x_j)| + |\sigma_{2S_r}(x_j) - \sigma_{2E_p}(x_j)| + |\sigma_{3S_r}(x_j) - \sigma_{3E_p}(x_j)| + |\gamma_{S_r}(x_j) - \gamma_{E_p}(x_j)|)$.
2. $d_2(S_r, E_p) = \{\frac{1}{5n}\sum_{j=1}^{n}((\mu_{S_r}(x_j) - \mu_{E_p}(x_j))^2 + (\sigma_{1S_r}(x_j) - \sigma_{1E_p}(x_j))^2 + (\sigma_{2S_r}(x_j) - \sigma_{2E_p}(x_j))^2 + (\sigma_{3S_r}(x_j) - \sigma_{3E_p}(x_j))^2 + (\gamma_{S_r}(x_j) - \gamma_{E_p}(x_j))^2\}^{\frac{1}{2}}$.
3. $d_3(S_r, E_p) = \frac{1}{5n}\sum_{j=1}^{n}(|(\mu_{S_r}(x_j))^2 - (\mu_{E_p}(x_j))^2| + |(\sigma_{1S_r}(x_j))^2 - (\sigma_{1E_p}(x_j))^2| + |(\sigma_{2S_r}(x_j))^2 - (\sigma_{2E_p}(x_j))^2| + |(\sigma_{3S_r}(x_j))^2 - (\sigma_{3E_p}(x_j))^2| + |(\gamma_{S_r}(x_j))^2 - (\gamma_{E_p}(x_j))^2|)$.
4. $d_4(S_r, E_p) = \frac{1}{5n}\sum_{j=1}^{n}(|((\mu_{S_r}(x_j))^2 - (\mu_{E_p}(x_j))^2) + ((\sigma_{1S_r}(x_j))^2 - (\sigma_{1E_p}(x_j))^2) - ((\sigma_{2S_r}(x_j))^2 - (\sigma_{2E_p}(x_j))^2) - ((\sigma_{3S_r}(x_j))^2 - (\sigma_{3E_p}(x_j))^2) - ((\gamma_{S_r}(x_j))^2 - (\gamma_{E_p}(x_j))^2)|)$.
5.
$$d_5(S_r, E_p) = \frac{7}{5n}\sum_{j=1}^{n}\frac{\begin{array}{r}(\sin\{\frac{\pi}{6}(|\mu_{S_r}(x_j) - \mu_{E_p}(x_j)|)\} + \sin\{\frac{\pi}{6}(|\sigma_{1S_r}(x_j) - \sigma_{1E_p}(x_j)|)\} \\ + \sin\{\frac{\pi}{6}(|\sigma_{2S_r}(x_j) - \sigma_{2E_p}(x_j)|)\} + \sin\{\frac{\pi}{6}(|\sigma_{3S_r}(x_j) - \sigma_{3E_p}(x_j)|)\} \\ + \sin\{\frac{\pi}{6}(|\gamma_{S_r}(x_j) - \gamma_{E_p}(x_j)|)\})\end{array}}{\begin{array}{r}(1 + \sin\{\frac{\pi}{6}(|\mu_{S_r}(x_j) - \mu_{E_p}(x_j)|)\} + \sin\{\frac{\pi}{6}(|\sigma_{1S_r}(x_j) - \sigma_{1E_p}(x_j)|)\} \\ + \sin\{\frac{\pi}{6}(|\sigma_{2S_r}(x_j) - \sigma_{2E_p}(x_j)|)\} + \sin\{\frac{\pi}{6}(|\sigma_{3S_r}(x_j) - \sigma_{3E_p}(x_j)|)\} \\ + \sin\{\frac{\pi}{6}(|\gamma_{S_r}(x_j) - \gamma_{E_p}(x_j)|)\})\end{array}}.$$

Step 3: Tabulation: Tabulate all the calculated distance measures of alternatives and criteria's put in the following table.

Step 4: Final Decision: From the distance measure table, we can conclude that the best criteria S_r is chosen by the largest value in the ascending order of averages $\omega_r, r = 1, 2, ..., n$ for the distance measures.

5 Numerical Example: Applications of ABHL-Types Penta Partitioned Neutrosophic Distance Measures:

The recruitment process is an entire hiring process from inception to the individual recruit's integration into the company. This helps to find and retain talented employees in a vital part of any successful business. Recruitment has been regarded as the most important function of personnel administration. The recruitment should be a sound one. If it is not, the morale of the staff will be very low and the image of the company will be tarnished. Unless the right types of people are hired, even the best plans, organization charts, and control systems will be to no avail. A company cannot prosper, grow or even survive without adequate human resources. It is not done wisely, every other management

E_p \ S_r	E_1	E_2	.	.	.	E_l	$\omega_r = \frac{\sum_{p=1}^{l} d_i(S_r, E_p)}{l}$
S_1	$d_i(S_1, E_1)$	$d_i(S_1, E_2)$	.	.	.	$d_i(S_1, E_l)$	ω_1
S_2	$d_i(S_2, E_1)$	$d_i(S_2, E_2)$	.	.	.	$d_i(S_2, E_l)$	ω_2
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
S_n	$d_i(S_n, E_1)$	$d_i(S_n, E_2)$	.	.	.	$d_i(S_n, E_l)$	ω_n

Table 4.3. Distance measure table

function suffers, and costs increase and bottlenecks get worse. This section demonstrates a numerical example of the recruitment problem as a real-life application for all ABHL types penta partitioned neutrosophic distance measures for the above-proposed methodology virtually.

Step 1: Problem field Selection: Suppose a software company is going to recruit the post of software developer. A selection board has been formed with three experts E_1, E_2, E_3 to examine three candidates S_1, S_2, S_3 under five different qualities or criteria C_1, C_2, C_3, C_4, C_5 for the post of software developer. Table 5.1 shows the information about the expert's expectations to the qualities, for example, the expert E_2 expects to the quality C_3 with the membership value is 0.3, the non-membership value is 0.1, the contradiction value is 0.06, the ignorance value is 0.06, the unknown membership value is 0.06, and so denoted as $(0.3, 0.06, 0.06, 0.06, 0.1)$. Table 5.2 shows the information about the qualities of the candidates, for example, the membership value of the candidate S_1 with the quality C_2 is 0.6, the non-membership value is 0.8, the contradiction value is 0.23, the ignorance value is 0.23, the unknown membership value is 0.23, and so denoted as $(0.6, 0.23, 0.23, 0.23, 0.8)$.

	C_1	C_2	C_3	C_4	C_5
E_1	(0.4, 0.16, 0.16, 0.16,	(0.4, 0.16, 0.16, 0.16,	(0.8, 0.2, 0.2, 0.2, 0.1)	(0.9, 0, 0, 00)	(0.1, 0.03, 0.03, 0.03, 0
E_2	(0.8, 0.03, 0.03, 0.03,	(0.6, 0.16, 0.16, 0.16,	(0.3, 0.06, 0.06, 0.06,	(0.8, 0.3, 0.3, 0.3	(0, 0, 0, 0, 1)
E_3	(1, 0, 0, 0, 0)	(0.6, 0.13, 0.13, 0.13,	(0.5, 0.26, 0.26, 0.26,	(1, 0.1, 0.1, 0.1, 0	(0.3, 0.13, 0.13, 0.13, 0

Table 5.1. The relation between experts and criteria's

	S_1	S_2	S_3
C_1	(0.8, 0.06, 0.06, 0.06, 0.6)	(0.9, 0.03, 0.03, 0.03, 0.5)	(1, 0, 0, 0, 0.4)
C_2	(0.6, 0.23, 0.23, 0.23, 0.8)	(0.9, 0.33, 0.33, 0.33, 0)	(0, 0, 0, 0, 1)
C_3	(0, 0, 0, 0, 1)	(0.6, 0.23, 0.23, 0.23, 0.3)	(0.3, 0, 06, 0.060 .06, 0.1)
C_4	(0.8, 0.26, 0.26, 0.26, 0.8)	(0.3, 0.23, 0.23, 0.23, 0.6)	(0.7, 0.26, 0.26, 0.26, 0.1)
C_5	(0.3, 0.06, 0.06, 0.06, 0.1)	(0.2, 0.03, 0.03, 0.03, 0.1)	(0.5, 0.2, 0.2, 0.2, 0.1)

Table 5.2. The relation between criteria's and candidates

Step 2: Distance measure of the Alternatives and Criterias: The distance measure of decision criteria S_r and the alternatives E_p by using the distance measures $d_i(S_r, E_p)$ where $r, p = 1, 2, 3$, and $i = 1, 2, ..., 5$. All the values of distance measures $d_i(S_r, E_p)$ are listed in table 5.3, 5.4, 5.5, 5.6, 5.7.

Step 3: Tabulation: The average of each column by using $\omega_r = \frac{\sum_{p=1}^{3} d_i(S_r, E_p)}{3}$, where $r = 1, 2, 3$.

	E_1	E_2	E_3	ω_r
S_1	0.2192	0.1672	0.182	0.189467
S_2	0.2432	0.2048	0.1708	0.20627
S_3	0.2708	0.2116	0.2192	0.23387

Table 5.3. ABHL - Type 1 $d_1(S_r, E_p)$

	E_1	E_2	E_3	ω_r
S_1	0.31035	0.29825	0.303651	0.30408
S_2	0.34033	0.30394	0.319587	0.3212
S_3	0.331976	0.336131	0.312295	0.3268

Table 5.4. ABHL - Type 2 $d_2(S_r, E_p)$

	E_1	E_2	E_3	ω_r
S_1	0.144392	0.138596	0.122172	0.13505
S_2	0.209364	0.167745	0.13358	0.17021
S_3	0.223044	0.197488	0.148992	0.18984

Table 5.5. ABHL - Type 3 $d_3(S_r, E_p)$

	E_1	E_2	E_3	ω_r
S_1	0.11569	0.132556	0.107124	0.11845
S_2	0.16522	0.079344	0.11162	0.11873
S_3	0.18082	0.114496	0.100064	0.131792

Table 5.6. ABHL - Type 4 $d_4(S_r, E_p)$

	E_1	E_2	E_3	ω_r
S_1	0.507186	0.396117	0.434287	0.44586
S_2	0.491711	0.544554	0.421405	0.48589
S_3	0.574825	0.48779	0.47454	0.512385

Table 5.7. ABHL - Type 5 $d_5(S_r, E_p)$

Rank	Candidates				
	$d_1(A,B)$	$d_2(A,B)$	$d_3(A,B)$	$d_4(A,B)$	$d_5(A,B)$
1	S_3	S_3	S_3	S_3	S_3
2	S_2	S_2	S_2	S_2	S_2
3	S_1	S_1	S_1	S_1	S_1

Table 5.8. Merit List

Step 4: Final Decision From table 5.8, we can observe that the final decisions of all ABHL types penta partitioned neutrosophic distance measures are same. The candidate S_3 is the most suitable candidate for the particular post. The next suitable candidate for the post is S_2 and the last suitable candidate is S_1.

6 Advantages and Limitations

1. The first novelty of this paper is the introduction of distance measures in penta partitioned neutrosophic sets.
2. The distance measures $d_i(A, B)$ for $i = 1, 2, 3, 5$, satisfies all the metric axioms.
3. The function $d_4(A, B)$ is a quasi-pseudo distance measure on penta partitioned neutrosophic sets but does not satisfy metric axioms among all ABHL types penta partitioned neutrosophic distance measures.
4. The proposed method is the first method in a penta partitioned neutrosophic environment for the recruitment process even though the distance measure is a generalization of intuitionistic fuzzy distance measure. This is an advantage of this paper.

5. Another advantage is the real-life application of penta partitioned neutrosophic distance measures in multi-criteria decision-making problems.

6. This method can be used the proposed distance measures in multi-attribute decision-making problems, due to multi-attribute decision-making problems are very similar to multi-criteria decision-making problems.

7. This method did not use any score functions, matrix, etc in multi-criteria decision-making problems.

8. Our method is simple, reliable, and dependable.

7 Conclusion and future works

Penta partitioned neutrosophic set is a generalized form of neutrosophic set, which is a mathematical tool to handle imprecise, incomplete, and inconsistent information in multi decision-making problems. This paper developed some distance measures on penta partitioned neutrosophic sets by satisfying the axioms of distance measures and some of these distance measures are also satisfy metric axioms. Further, we have proposed a real-life application in the recruitment problem of multi-criteria decision-making problem and gave the best decision for the decision-making problem. In multi decision-making problems, penta partitioned neutrosophic distance measures play an important role intake suitable decisions regarding criteria's. These newly developed measures and methods will be an eye-opener for the penta partitioned neutrosophic researchers to implement in other research areas of general topology such as rough topology, digital topology, and so on. We can develop many real-life application models by defining using the penta partitioned neutrosophic score function and the penta partitioned neutrosophic matrix for this method in the future.

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