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An application of mixed monotone operator on a fractional differntial equation on an unbounded domain

Maral Sangi, Somayeh Saiedinezhad*

School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran

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Abstract

This paper provides sufficient conditions that guarantee the existence of positive solutions to a boundary value problem of a nonlinear fractional differential equation on the half line. Our analysis takes advantage of a theory on cones and mixed monotone operators combined with the diagonalization method. The paper also contains some examples that are numerically solved by the Adomian Decomposition Method.

Keywords: Fractional differential equation, Mixed monotone operator, Positive solutions, Diagonalization method
2020 MSC: 34A08, 47H07, 34B18, 34B40

1 Introduction

Fractional calculus has received very much momentum in depth as well as in scope in the last few decades. It is an applicable branch of mathematics which involves fractional integration and differentiation. Fractional differential equations [11, 24, 25, 30] have gained a bold platform in many areas of science and engineering, see [12, 13, 22, 27, 29, 23, 33].

Mathematics has emerged as a powerful tool for modeling the physically complicated problems. The accuracy of the approximate solution is a major problem which depends on two factors: the accuracy of model and method. Classical derivatives are local and not able to model some physical problems because of their limited domain in order and lack of diverse types of derivatives. In this regard, fractional derivatives, for e.g [5, 10, 9, 20, 24], eliminate the mentioned shortcoming of the classical ones by letting a vast domain of degrees and providing different derivatives for the sake of getting better results.

The discussion about the theory of boundary value problems on infinite intervals was originally began by [2]. After this initial step, results of fractional boundary value problems (FBVP for short) on the interval which is not compact are of paramount importance. Many studies have focused on the solvability of FBVP on the half line, for similar survey on finite intervals one can see [3, 8, 21, 16, 31, 32]. Researchers apply different methods to explore the solvability of FBVP on the half line, as main methods, we can refer to the the extension of continuous solutions on the corresponding finite intervals under a diagonalization process, fixed point theorems in special Banach space or in special Frechet space; see [1, 6, 26, 35] and the references therein.

*Corresponding author
Email addresses: maral_sangi98@mathdep.iust.ac.ir (Maral Sangi), ssaiedinezhad@iust.ac.ir (Somayeh Saiedinezhad)

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In [4], the authors studied the existence of a FBVP with Riemann-Liouville derivative on infinite interval by the classical nonlinear alternative of Leray–Schauder type combined with the diagonalization process. Another study, [34], directly deal with the existence of a class of a FBVP with Riemann-Liouville derivative on an infinite interval using the theory of cone and operators. The method of mixed monotone operator which was introdiced in [18], has wide applications in science and technology and been of great interest in nonlinear differential and integral equations, see [14, 15, 36]. To our knowledge, there is no contribution which obtaine the existence of positive solutions for a FBVP with caputo derivative on an unbounded domain using mixed monotone operator combined with the diagonalization process besid showing numerical results. [7] studied the existence of solutions for the boundary value problem of third order differential equation of the form

$$\begin{cases} u'''(t) + q(t)f(t, u(t), u'(t), u''(t)) = 0, 0 < t < +\infty \\ u(0) = 0, \quad u'(0) = a_1, \quad u'(+\infty) = b_1. \end{cases} \tag{1.1}$$

Inspired by the work above, this paper deals with the existence of solutions to the following boundary value problem for fractional differential equation,

$$\begin{cases} {}^C D_a^\alpha u(t) + q(t)f(u(t), u'(t)) = 0, \quad 0 < t < +\infty \\ u(0) = 0, u'(0) = 0, u'(+\infty) = l \end{cases} \tag{1.2}$$

where ${}^C D^\alpha$ denotes the Caputo derivative of fractional order $3 < \alpha \leq 4$, $\phi \in C[0, +\infty)$ with $\phi(t) > 0$ for $0 < t < +\infty$, the nonlinear $f \in C([0, +\infty) \times (0, +\infty) \times \mathbb{R}^2, \mathbb{R})$ and l is a positive constant.

The rest of this paper is as follows. In section 2, we state some necessary definitions and preliminaries. Section 3 is devoted to the existence on the finite interval, by mixed monotone operator and, and the existence of positive solutions to problem(1.2) by using the diagonalization process. Section 4 contains two supporting examples. In section 5, we have addressed the numerical solution of equations using the Adomian method and neural networks.

2 Preliminaries of fractional calculus

Let $I := (0, +\infty)$ and recall the space $E = C^1[0, b]$, equipped with the norm

$$\|x\| = \max\{\max_{t\in I} |x(t)|, \max_{t\in I} |x'(t)|\}$$

Definition 2.1. [24] The Riemann-Liouville fractional integral of the function $h \in L^1(I)$ of order $\alpha > 0$ is given by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} h(s)ds, \quad t > 0;$$

where Γ is the gamma function.

Definition 2.2. [24] For a function $h \in L^1(I)$ and $\alpha > 0$, the α-th Caputo fractional-order derivative of h is defined by

$${}^C D_0^\alpha h(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} h^{(n)}(s)ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.3. [24] The equality $D_{0+}^\alpha I_{0+}^\alpha = f(t), \alpha > 0$ holds for $f \in L(0, b)$

Lemma 2.4. [24] Let $\alpha > 0$, then

$$I_0^\alpha {}^C D_0^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1};$$

where $n = [\alpha] + 1$ and for $i = 0, 1, \ldots, n - 1$, $c_i \in \mathbb{R}$.

In order to obtain the existence result for the problem (1.2), first, we show the existence of solutions to the following problem, which is on bounded domain, by mixed monotone operator.

$$\begin{cases} {}^{C}D_0^{\alpha}u(t) + q(t)f(u(t), u'(t)) = 0, \quad t \in I_b := (0, b), \quad 3 \leq \alpha < 4, \\ u(0) = u'(0) = u'''(0) = 0, \; u''(b) = l. \end{cases} \tag{2.1}$$

In order to obtain the existence of positive solutions to problem (2.1), we will consider the existence of positive solutions to the following modified problem

$$\begin{cases} {}^{C}D_a^{\alpha-1}v(t) + q(t)f(I^1v(t), v(t)) = 0, \; 0 < t < b, \; 2 \leq \alpha - 1 < 3 \\ v(0) = v''(0) = 0, v'(b) = l. \end{cases} \tag{2.2}$$

Indeed, we have results as follows:

Lemma 2.5. Let $u(t) = I^1v(t), \; v \in C^1[0, b]$. Then one can transform (2.1) into (2.2). Moreover, if $v \in C^1([0, b], [0, \infty))$ is a solution of problem (2.2), then the function $u(t) = I^1v(t)$ is a positive solution of problem (2.1).

Proof . Put $u(t) = I^1v(t)$ into (2.1), by the definition of Riemann-Liouville fractional derivative and Lemma (2.3), we can obtain that $u'(t) = DIv(t) = v(t)$. Also, we have $v(0) = u'(0) = 0, \; v(b) = u'(b) = l$. Now, let $v \in C([0, b], [0, \infty))$ be a solution for problem (2.2). Then from the definition of the Caputo fractional derivative and Lemma (2.4), there has

$$\begin{aligned} {}^{C}D_0^{\alpha}u(t) &= \frac{d^3}{dt^3}I^{3-\alpha}u(t) = \frac{d^3}{dt^3}I^{3-\alpha}Iv(t) = \frac{d^3}{dt^3}I^{4-\alpha}v(t) = D^{\alpha-1}v(t) \\ &= q(t)f(Iv(t), v(t)) \\ &= q(t)f(u(t), u'(t)); \end{aligned}$$

moreover

$$u(0) = 0, \; u'(0) = v(0) = 0, \; v'(b) = u''(b) = l, \; v''(0) = u'''(0) = 0.$$

So, $u(t) = I^1v(t)$ is a positive solution of problem (2.1). □

Lemma 2.6. Let $\alpha \in (3, 4]$ and $h : I \to \mathbb{R}$ be continuous. A function $v(t)$ is a unique solution of the fractional boundary value problem

$$\begin{cases} {}^{C}D_0^{\alpha-1}v(t) + h(t) = 0, 0 < t < b, 2 \leq \alpha - 1 < 3 \\ v(0) = v''(0) = 0, v'(b) = l, \end{cases} \tag{2.3}$$

if and only if $v(t)$ is a solution of the fractional integral equation

$$v(t) = lt + \int_0^b G_b(t, s)h(s)ds. \tag{2.4}$$

where the Green's function $G(t, s)$ is defined by

$$G_b(t, s) = \begin{cases} \frac{t}{\Gamma(\alpha-2)}(b-s)^{\alpha-3} - \frac{1}{\Gamma(\alpha-1)}(t-s)^{\alpha-2}; & 0 \leq s \leq t \leq b \\ \frac{t}{\Gamma(\alpha-2)}(b-s)^{\alpha-3} & 0 \leq t \leq s \leq b \end{cases} \tag{2.5}$$

Proof . Assume $v(t)$ satisfies (2.3), in view of Lemma 2.4, by exerting the fractional integral $I_0^{\alpha-1}$ on (2.1) under the boundary conditions, we obtain

$$v(t) = c_0 + c_1t + c_2t^2 - I^{\alpha-1}h(t)$$

where

$$c_0 = 0, \quad c_1 = l + I^{\alpha-2}h(b), \quad c_2 = 0.$$

So,

$$v(t) = lt + \frac{1}{\Gamma(\alpha-2)}\left(\int_0^t \left[\frac{-1}{(\alpha-1)}(t-s)^{\alpha-2} + t(b-s)^{\alpha-3}\right] h(s), ds + \int_t^b t(b-s)^{\alpha-3}h(s), ds\right)$$

Conversely, it is clear that if $v(t)$ satisfies (2.4), then (2.3) holds. □

Lemma 2.7. The function $G(t,s)$ defined by (2.5) satisfies

$$\left(\frac{(b-s)^{\alpha-3}}{\Gamma(\alpha-2)}-\frac{1}{\Gamma(\alpha-1)}\frac{(b-s)^{\alpha-2}}{b}\right)t\leq G_b(t,s)\leq\frac{t}{\Gamma(\alpha-2)}(b-s)^{\alpha-3} \tag{2.6}$$

Proof . It is clear that $G(t,s)\leq\frac{t}{\Gamma(\alpha-2}(b-s)^{\alpha-3}$, for $0\leq t\leq s\leq b$. On the other hand, for $0\leq s\leq t\leq b$, we have

$$\frac{\partial(\frac{G(t,s)}{t})}{\partial t}=-\frac{1}{\Gamma(\alpha-1)}(t-s)^{\alpha-3}t^2\left[t(\alpha-2)-(t-s)\right]<0$$

thus

$$\min\frac{G(t,s)}{t}=\frac{G(b,s)}{b}=\frac{(b-s)^{\alpha-3}}{\Gamma(\alpha-2)}-\frac{1}{\Gamma(\alpha-1)}(b-s)^{\alpha-2}.$$

This completes the proof of the Lemma. □

Let P be a normal cone of a Banach space E, and $e\in P$ with $\|e\|\leq 1, e\neq\theta$ (θ is zero element of E). Define $Q_e=\{x\in P|$ there exist constants $m,M>0$ such that $\frac{1}{M}e(t)\leq x\leq Me(t)\}$. The following definition and theorem are essential in proving the results (see [17, 18, 19]).

Definition 2.8. Assume $T:Q_e\times Q_e\to Q_e$. T is said to be a mixed monotone operator if $T(x,y)$ is non-decreasing in x and non-increasing in y, i.e., if $x_1\leq x_2$ implies $T(x_1,y)\leq T(x_2,y)$ and $y_1\geq y_2(y_1,y_2\in Q_e)$ implies $T(x,y_1)\leq T(x,y_2)$ for any x_1,x_2,y_1,y_2. $x^*\in Q_e$ is called a fixed point of T if $T(x^*,x^*)=x^*$.

Theorem 2.9. Suppose that $T:Q_e\times Q_e\to Q_e$ is a mixed monotone operator and there exists a constant δ, $0<\delta<1$ such that

$$T\left(tx,\frac{1}{t}y\right)\geq t^\delta T\left(x,y\right),\quad\forall x,y\in Q_e,\ t>0.$$

Then T has a unique fixed point $x^*\in Q_e$.

3 Main results

From now on, we suppose:

(H1) $f(x,y)=g(x)+h(y)$ where $g:[0,+)\to[0,+\infty)$ is continuous and non-decreasing and $h:(0,+\infty)\to(0,+\infty)$ is continuous and non-increasing.

(H2) There exists $\delta\in(0,1)$ such that

$$g(tx)\geqslant t^\delta g(x) \tag{3.1}$$

and

$$h(t^{-1}x)\geqslant t^\delta h(x) \tag{3.2}$$

for all $t,x>0$;

(H3) For $b>0$, $I_b:=[0,b]$; let P be a normal cone of $C^1(I_b)$ defined by

$$P=\{v\in C^1(I_b)|v(t)\geqslant 0, 0\leq t\leq b\}$$

also define

$$Q_e=\{v\in P|\frac{1}{M}e(t)\leqslant v(t)\leqslant Me(t), t\in(0,b)\}$$

where $e(t)=t$, M is a positive constant that

$$M>\max\left\{1,\left(l+\frac{\Theta_1(s)}{\Gamma(\alpha-2)}\right)^{(1-\delta)^{-1}},\ \left(l+\frac{\Theta_2(s)}{\Gamma(\alpha-2)}\right)^{-(1-\delta)^{-1}}\right\}, \tag{3.3}$$

in which

$$\Theta_1(s) = \left[b^{2\delta} g(1) \int_0^b q(s)(b-s)^{\alpha-3} ds + h(1) \int_0^b q(s)(b-s)^{\alpha-3} s^{-\delta} ds \right]$$

and

$$\Theta_2(s) = \left[g(1) \int_0^b \left((b-s)^{\alpha-3} - \frac{1}{(\alpha-1)} \frac{(b-s)^{\alpha-2}}{b} \right) q(s) \left(\frac{s^2}{2} \right)^{\delta} ds \right.$$
$$\left. + b^{-\delta} h(1) \int_0^b \left((b-s)^{\alpha-3} - \frac{1}{(\alpha-1)} \frac{(b-s)^{\alpha-2}}{b} \right) q(s)\, ds \right]$$

and

$$0 < \int_0^b q(s)(b-s)^{\alpha-3} s^{-\delta} ds < \infty \tag{3.4}$$
$$0 < \int_0^b ((b-s)^{\alpha-3} - \frac{1}{(\alpha-1)} \frac{(b-s)^{\alpha-2}}{b}) q(s) (\frac{s^2}{2})^{\delta} ds < \infty$$

Remark 3.1. From (3.1) and (3.2), one can derive $g(t) \geq t^{\delta} g(1),\ g(\bar{x}) \leqslant \bar{x}^{\delta} g(1),\ h(t^{-1}) \geq t^{\delta} h(1),\ t^{-\delta} h(y) \geqslant h(ty),\ h(t) \leq t^{-\delta} h(1)$

Proof . with $x = 1$, there is

$$g(t) \geq t^{\delta} g(1). \tag{3.5}$$

and when $t = \frac{1}{\bar{x}}$, letting $x = \bar{x}$, respectively in (3.1), we obtain

$$g(\bar{x}) \leqslant \bar{x}^{\delta} g(1). \tag{3.6}$$

Now, from (3.2) with $x = 1$, we have

$$h(t^{-1}) \geq t^{\delta} h(1) \tag{3.7}$$

let $x = ty$ in (3.2), one has

$$t^{-\delta} h(y) \geq h(ty) \tag{3.8}$$

choosing $y = 1$ in (3.8) yields

$$h(t) \leq t^{-\delta} h(1) \tag{3.9}$$

□

Theorem 3.2. Suppose that **(H1)-(H3)** hold. Then problem (2.2) has a unique positive solution $u(t) = I^1 v(t)$ with $v_b \in Q_e$.

Proof . Let $v \in Q_e$, thus, there are $Iv(t) > 0, v(t) > 0$ for $t \in [0, b]$. Thus from the monotonicity of Riemann-Liouville fractional integral $I^{\alpha},\ \alpha > 0$, **(H1)** and (3.6), (3.9), we have for $v \in Q_e$

$$\begin{aligned} g(Iv(t)) &\leq g(IMe(t)) \\ &= g(M\frac{t^2}{2}) \\ &\leq g(Mb^2) \\ &\leq (Mb^2)^{\delta} g(1) \end{aligned}$$

and

$$\begin{aligned} h(v(t)) &\leq h(M^{-1}e(t)) \\ &= h(M^{-1}t) \\ &\leq (M^{-1}t)^{-\delta}h(1) \end{aligned}$$

On the other hand, by **(H1)** and (3.5), (3.7), we have

$$\begin{aligned} g(Iv(t)) &\geq g(IM^{-1}e(t)) \\ &= g(M^{-1}\frac{t^2}{2}) \\ &\geqslant (M^{-1}\frac{t^2}{2})^{\delta}g(1) \end{aligned}$$

and

$$\begin{aligned} h(v(t)) &\geq h(Me(t)) \\ &= h(Mt) \\ &\geq h(Mb) \\ &\geq (Mb)^{-\delta}h(1), \end{aligned}$$

Now, we define the operator $T_b : Q_e \times Q_e \to Q_e$ by

$$T_b(x,y)(t) = lt + \int_0^b G_b(t,s)q(s)[g(Ix(s)) + h(y(s))]ds \tag{3.10}$$

where $G_b(t,s)$ is defined in (2.5). We will show that T_b maps $Q_e \times Q_e \to Q_e$. In fact, for $v, w \in Q_e$, by (2.6,), (3.3) and (3.10), we see that

$$\begin{aligned} T(x,y)(t) &\leq lt + \frac{tM^{\delta}}{\Gamma(\alpha-2)}\Theta_1(s) \\ &\leq tM^{\delta}l + \frac{tM^{\delta}}{\Gamma(\alpha-2)}\Theta_1(s) \\ &= tM^{\delta}\left(l + \frac{\Theta_1(s)}{\Gamma(\alpha-2)}\right) \\ &\leq Mt \end{aligned} \tag{3.11}$$

where $\Theta_1(s)$ is introduced in (3.3) and

$$\begin{aligned} T(x,y)(t) &\geq lt + \frac{tM^{-\delta}}{\Gamma(\alpha-2)}\Theta_2(s) \\ &\geq tM^{-\delta}l + \frac{tM^{-\delta}}{\Gamma(\alpha-2)}.\Theta_2(s) \\ &= tM^{-\delta}\left(l + \frac{1}{\Gamma(\alpha-2)}\Theta_2(s)\right) \\ &\geq M^{-1}t \end{aligned} \tag{3.12}$$

where $\Theta_2(s)$ is introduced in (3.3). From (3.11), (3.12), we see that $T : Q_e \times Q_e \subseteq Q_e$.

Secondly, we show that T is a mixed monotone operator. If $x_1 \leq x_2$, from the monotonocity of Riemann-Liouville integral and **(H1)**, we have $T(x_1, y)(t) \leq T(x_2, y)(t)$ That is T is non-decreasing in x for any $y \in Q_e$. Similarly, If $y_1 \geq y_2$, we have $T(x, y_1)(t) \leq T(x, y_2)(t)$. That is T is non-increasing in y for any $v \in Q_e$ Hence, $T : Q_e \times Q_e \to Q_e$

is a mixed monotone operator. Finally, for $x, y \in Q_e,\ t \in [0,b]$, from **(H2)**, we have

$$\begin{aligned}
T(tx, t^{-1}y)(t) &= lt + \int_0^b G(t,s)q(s)\left[g(tIx(s)) + h(t^{-1}y(s))\right] ds \\
&\geq lt^{\delta} + t^{\delta}\left(\int_0^b G(t,s)q(s)\left[g(Ix(s)) + h(y(s))\right]\right) ds \\
&= t^{\delta}\left(l + \int_0^b G(t,s)q(s)\left[g(Ix(s)) + h(y(s))\right] ds\right) \\
&= t^{\delta} T(x,y)(t).
\end{aligned}$$

Hence,

$$T(tx, t^{-1}y)(t) \geq t^{\delta} T(x,y)(t).$$

Thus, we have shown that all the conditions of Theorem 2.9 hold. So, there exists unique $v^* \in Q_e$ such that $T(v^*, v^*) = v^*$, that is, problem (2.2) has a unique positive solution. Since $v^* \in C([0,1],[0,+\infty)$, there are $I_0 v^* \geq 0, v^* \geq 0$ for all $t \in (0,b)$. Hence $f(Iv(t), v(t))$ is continuous for $v^* \in C([0,1],[0,+\infty),\ t \in (0,b)$. Now, using Lemma (2.5), we see that $u^* = Iv^*$ is a positive solution of (2.1) which is unique. We now use the following diagonalization process. For $k \in \mathbb{N}$, let

$$y_k(t) = \begin{cases} u_k(t) & t \in [0, b_k], \\ u_k(b_k) & t \in [b_k, \infty). \end{cases} \tag{3.13}$$

Here $\{b_k\}_k \in \mathbb{N}^*$ is a sequence of numbers satisfying $0 < b_1 < \cdots < b_k < \cdots \uparrow \infty$ Let $S = \{y_k\}_{k=1}^{\infty}$. Notice that $|y_k(t)| \leq Mt$ for $t \in [0, b_1], k \in \mathbb{N}$. Also for $k \in \mathbb{N}$ and $t \in [0, b_1]$, we have

$$y_{n_k}(t) = \frac{l}{b_1}t + \int_0^{b_1} q(s)G_{b_1}(t,s)f(Iy_{b_k}(s), y_{b_k}(s))ds.$$

Thus, for $k \in \mathbb{N}$ and $t, x \in [0, b_1]$, we have

$$y_{b_k}(t) - y_{b_k}(x) = \frac{l}{b_1}(t-x) + \int_0^{b_1} \left[G_{b_1}(t,s) - G_{b_1}(x,s)\right] q(s)f(Iy_{b_k}(s), y_{b_k}(s))ds.$$

and by (3.11)

$$|y_{b_k}(t) - y_{b_k}(x)| \leq M|t-x|.$$

The Arzel'a-Ascoli Theorem guarantees that there is a subsequence N_1^* of $\mathbb{N}$ and a function $z_1 \in C([0,b_1],\mathbb{R})$ with $y_{b_k} \to z_1$ in $C([0,b_1],\mathbb{R})$ as $k \to \infty$ through N_1^*. Let $N_1 = N_1^* \setminus 1$. Notice that $|y_{b_k}(t)| \leq Mt$ for $t \in [0, b_2], k \in \mathbb{N}$. Also for $k \in \mathbb{N}$ and $t, x \in [0, b_2]$, we have

$$|y_{b_k}(t) - y_{b_k}(x)| \leq M|t-x|.$$

The Arzel'a-Ascoli Theorem guarantees that there is a subsequence N_2^* of $\mathbb{N}_1$ and a function $z_2 \in C([0,b_2],\mathbb{R})$ with $y_{b_k} \to z_2$ in $C([0,b_2],\mathbb{R})$ as $k \to \infty$ through N_2^*. Note that $z_1 = z_2$ on $[0, b_2]$ since $N_2^* \subseteq N_1$. Let $N_2 = N_2^* \setminus 2$.

Proceed inductively to obtain for $m \in \{3, 4, \cdots$ a subsequence N_m^* of $N_m \setminus 1$ and a function $z_m \in C([0,b_m],\mathbb{R})$ with $y_{b_k} \to z_m$ in $C([0,b_m],\mathbb{R})$ as $k \to \infty$ through N_m^*. Let $N_m = N_m^* \setminus m$. Define a function u as follows. Fix $t \in (0,\infty)$ and let $m \in \mathbb{N}$ with $s \leq b_m$. Then define $u(t) = z_m(t)$. Then $u \in C([0,\infty],\mathbb{R}), u(0) = 0$ and $|u(t)| \leq M$ for $t \in [0,\infty)$. Again fix $t \in (0,\infty)$ and let $m \in \mathbb{N}$ with $s \leq b_m$. Then for $n \in N_m$, we have

$$y_{b_k}(t) = \frac{l}{b_m}t + \int_0^{b_m} G_{b_m}(t,s)q(s)f(Iy_{b_k}(s), y_{b_k}(s))d.$$

Let $b_k \to \infty$ through N_m to obtain

$$z_m(t) = \frac{l}{b_m}t + \int_0^{b_m} G_{b_m}(x,s)q(s)f(Iz_m(s), z_m(s))ds,$$

that is

$$u(t) = \frac{l}{b_m}t + \int_0^{b_m} G_{b_m}(t,s)q(s)f(Iu(s),u(s))ds \tag{3.14}$$

we can use this method for each $t \in [0, b_m]$, for each $m \in \mathbb{N}$. Thus

$$\begin{cases} {}^C D_a^\alpha u(t) + q(t)f(u(t),u'(t)) = 0, & \text{for } t \in [0,b_m], \\ u(0) = 0, u(b_m) = l. \end{cases} \tag{3.15}$$

for each $m \in \mathbb{N}$ and $\alpha \in (2,3]$. This completes the proof of the theorem. □

4 Example

Example 4.1. Consider the nonlinear problem

$$\begin{cases} {}^C D_0^{\frac{10}{3}} u(t) + t^{-\frac{1}{4}}(u^2 + 2u + \frac{1}{\sqrt[4]{u'}}) = 0, & 0 < t < +\infty, \\ u(0) = 0, u'(0) = 0, u'(+\infty) = l, \end{cases} \tag{4.1}$$

We let $q(t) = t^{-\frac{1}{4}}$ and $f(u) = u^2 + 2u + \frac{1}{\sqrt[4]{u'}} = g(u) + h(u')$ where $g(u) = u^2 + 2u$ and $h(u') = \frac{1}{\sqrt[4]{u'}}$. Noting for $l \in (0,1), u > 0, u' > 0$ and $0 < \delta = max\{\delta_1, \delta_2\} < 1$, since for each $\delta_1 \in (0,1)$, $\delta_2 \leq \frac{1}{4}$ and $l \in (0,1)$, we have $l^1 > l^\delta$ and $l^2 \geq l^\delta$, so

$$g(lu) = l^2u^2 + 2lu \geq l^{\delta_1}(u^2 + 2u),$$
$$h(l^{-1}u') = l^{\frac{1}{4}}(u')^{-\frac{1}{4}} \geq l^{\delta_2}(u')^{-\frac{1}{4}}$$

we note that $\delta = \frac{1}{2}$, then $\int_0^b s^{\frac{-3}{4}}(b-s)^{\frac{1}{3}}ds = b^{\frac{7}{12}}\beta(\frac{1}{4}, \frac{4}{3})$ where β is the β function. So, we have

$$0 < \int_0^b s^{\frac{-3}{4}}(b-s)^{\frac{1}{3}}ds < \infty$$

and in the same way, we have

$$\frac{1}{\sqrt{2}}\int_0^b s^{\frac{3}{4}}((b-s)^{\frac{1}{3}} - \frac{3}{7}\frac{(b-s)^{\frac{4}{3}}}{b})ds = \frac{1}{7\sqrt{2}}\left(3\beta(\frac{4}{3}, \frac{11}{4}) + 4\beta(\frac{4}{3}, \frac{7}{4})\right)b^{\frac{25}{12}}$$

so,

$$0 < \int_0^b s^{\frac{-1}{4}}\left((b-s)^{\frac{1}{3}} - \frac{3}{7}\frac{(b-s)^{\frac{4}{3}}}{b}\right)(\frac{s^2}{2})^{\frac{1}{2}}ds < \infty$$

Thus, the assumptions **(H1)-(H3)** hold. Then Theorem(3.2) implies that problem (4.1) has a positive solution in $C^1(0,\infty)$.

Example 4.2. Consider the nonlinear problem

$$\begin{cases} {}^C D_0^{\frac{15}{4}} u(t) + \sqrt{\pi}t(u^{\frac{1}{5}} + u^{\frac{1}{4}} + \frac{1}{\sqrt[8]{u'}}) = 0, & 0 < t < +\infty \\ u(0) = 0, u'(0) = 0, u'(+\infty) = l \end{cases} \tag{4.2}$$

We let $f(u) = u^{\frac{1}{5}} + u^{\frac{1}{4}} + \frac{1}{\sqrt[8]{u'}} = g(u) + h(u')$ where $g(u) = u^{\frac{1}{5}} + u^{\frac{1}{4}}$ and $h(u') = \frac{1}{\sqrt[8]{u'}}$.

Noting for $l \in (0,1), u > 0, u' > 0$ and $\delta = \dfrac{1}{4}$,

$$g(lu) = l^{\frac{1}{5}}u^{\frac{1}{5}} + l^{\frac{1}{4}}u^{\frac{1}{4}} \geq l^{\frac{1}{4}}(u^{\frac{1}{5}} + u^{\frac{1}{4}}),$$
$$h(l^{-1}u') = l^{\frac{1}{8}}u' \geq l^{\frac{1}{4}}u'$$

So $\sqrt{\pi}\int_0^b s^{\frac{3}{4}}(b-s)^{\frac{3}{4}}ds = \sqrt{\pi}\beta(\frac{7}{4},\frac{7}{4})b^{\frac{5}{2}}$, where β is the β function. So, we have

$$0 < \int_0^b s^{\frac{3}{4}}(b-s)^{\frac{1}{2}}ds < \infty$$

and in the same way, we have

$$\frac{\sqrt{\pi}}{\sqrt[4]{2}}\int_0^b s^{\frac{3}{2}}\left((b-s)^{\frac{3}{4}} - \frac{4}{11}\frac{(b-s)^{\frac{7}{4}}}{b}\right)ds = \frac{\sqrt{\pi}}{11\sqrt[4]{2}}\left(4\,\mathrm{B}\left(\frac{7}{4},\frac{7}{2}\right) + 7\,\mathrm{B}\left(\frac{7}{4},\frac{5}{2}\right)\right)b^{\frac{13}{4}}.$$

So,

$$0 < \frac{1}{\sqrt[4]{2}}\int_0^b \sqrt{\pi}s\left((b-s)^{\frac{3}{4}} - \frac{4}{11}\frac{(b-s)^{\frac{7}{4}}}{b}(\frac{s^2}{2})^{\frac{1}{4}}\right)ds < \infty.$$

Thus, the assumptions **(H1)-(H3)** hold. Then Theorem(3.2) implies that problem (4.2) has a positive solution in $C^1(0,\infty)$.

5 Numerical results

The Adomian Decomposition Method (ADM) is a powerful technique for solving nonlinear ordinary and partial differential equations. It was introduced by George Adomian in the 1980s as an alternative to traditional analytical and numerical methods.

The key idea behind the Adomian Decomposition Method is to decompose the solution of a differential equation into a series of components. These components are typically expressed as infinite series of functions. The solution is then approximated by truncating this series at a finite order.

In this section, we employ the Adomian Decomposition Method to numerically approximate the solution of equation (1.2). We know that the solution to this equation is equal to the limit as m approaches infinity of the expression (3.14). Therefore, suppose that b_m is a sufficiently large value. Then, an approximation of u will be as follows

$$u(t) \approx \tilde{u}(t) = \frac{l}{b_m}t + \int_0^{b_m} G_{b_m}(t,s)q(s)f(Iu(s),u(s))ds. \tag{5.1}$$

Now, we apply the Adomian Decomposition method to a similar approach [28] on the right-hand side of the integral equation (5.1). To this end, we assume that

$$\tilde{u}(t) = \sum_{n=0}^{\infty} u_n(t) = \frac{l}{b_m}t + \int_0^{b_m} G_{b_m}(t,s)q(s)f(Iu(s),u(s))ds, \tag{5.2}$$

having identified u_0 as all terms outside the integral sign, the components $u_n(t)$, where $n \geq 1$, of the unknown function $\tilde{u}$ are consequently fully determined in a recurrent manner if we set

$$\begin{cases} u_0(t) = \dfrac{l}{b_m}t, \\ \\ u_n(t) = \int_0^{b_m} G_{b_m}(t,s)q(s)f(Iu_{n-1}(s),u_{n-1}(s))ds, \quad n \geq 1. \end{cases} \tag{5.3}$$

Given that the integral solution in equation (5.3) cannot be explicitly computed, numerical techniques and a suitable neural network can be employed to obtain a good approximation of it.

Using a neural network to approximate the functions $u_n(t)$ involves training a model to learn the mappings between inputs (values of t) and outputs (values of $u_n(t)$). This can be achieved through a supervised learning approach where we provide input-output pairs and let the model learn the underlying relationship. Here's how we can approach this task:

n	Example 1	Example 2
2	2.721E0	9.450E0
3	1.220E0	5.486E0
4	9.789E-1	1.201E0
5	6.175E-1	8.045E-1
6	4.571E-1	6.958E-1
7	2.650E-1	3.881E-1
8	8.968E-2	1.020E-1
9	5.485E-2	7.691E-2
10	1.899E-2	3.742E-2

Table 1: The trend of decreasing absolute error in the approximate solutions of Examples 1 and 2 with increasing n.

- **Data generation:** Generate a dataset consisting of input-output pairs where inputs are values of t and outputs are values of $u_n(t)$ for various values of n. We can use the given equations to generate this data.
- **Neural network architecture:** Design a neural network architecture suitable for function approximation. Since we are dealing with a regression problem (predicting continuous values), a feedforward neural network with appropriate hidden layers can be used.
- **Training the neural network:** Train the neural network using the generated dataset. During training, the network learns to approximate the functions $u_n(t)$ by adjusting its weights and biases to minimize a suitable loss function (e.g., mean squared error).
- **Evaluation:** Evaluate the trained neural network on a separate validation set to assess its performance. Adjust the model architecture and training parameters as needed to improve performance.
- **Prediction:** Use the trained neural network to predict values of $u_n(t)$ for new values of t as required.

In Algorithm 5, we have depicted this pattern.

Algorithm 1 Explanation of the provided Python code

Import Libraries
Import necessary libraries: NumPy for numerical operations and PyTorch for building and training neural networks.
Define Constants
Define constants such as b_m, α, and l which are used throughout the code.
Define Functions
Define functions $q(t)$ and $f(v, w)$ using lambda functions. These functions are used in the problem's equations.
Define G_b Function
Define function $G_b(t, s)$ which implements the piecewise definition provided in the problem statement.
Generate Training Data
Generate training data for $u_n(t)$ for $n = 0$ to 10 by numerically integrating the provided equations.
Convert Data to PyTorch Tensors
Convert the NumPy arrays generated in the previous step into PyTorch tensors for further processing.

In Example 4.1, let $l = 1/2$, and in Example 4.2, let $l = 1/3$. In both examples, we take $b_m = 100$, and consider the first n terms of the series (5.2) as an approximation of $\tilde{u}$, denoted by $\tilde{u}_n$. The numerical results of this approximation for both examples are reported in Figure 1 with $n = 15$. Additionally, we present the reduction of the absolute error of the approximate solution $\tilde{u}_n$ with $\tilde{u}_{15}$ in $[0, b_m]$ in Table 5 for both examples.

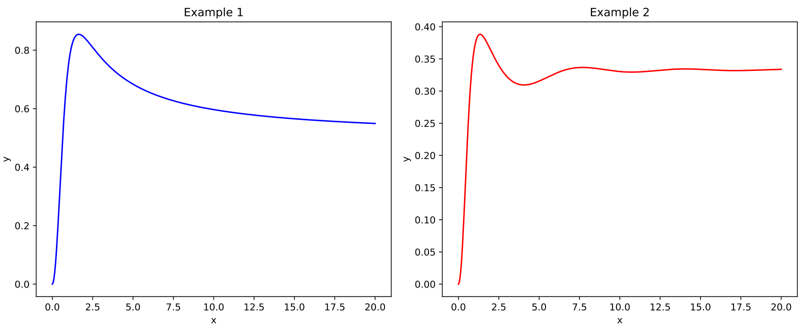

Figure 1: Numerical simulation of the solutions of Examples 1 and 2 using the Adomian decomposition method

As illustrated in this table, the numerical method provides a good approximation that converges to the exact solution.

References

[1] R.P. Agarwal and D. O'Regan, *Non-linear boundary value problems on the semi-infinite interval: an upper and lower solution approach*, Mathematika **49** (2002), no. 1-2, 129–140.

[2] R.P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Springer Science and Business Media, 2012.

[3] B. Ahmad and S. Ntouyas, *A coupled system of nonlocal fractional differential equations with coupled and uncoupled slit-strips-type integral boundary conditions*, J. Math. Sci. **226** (2017), no. 3.

[4] A. Arara, M. Benchohra, N. Hamidi, and J.J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Anal.: Theory Meth. Appl. **72** (2010), no. 2, 580–586.

[5] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model*, arXiv preprint arXiv:1602.03408 (2016).

[6] C. Bai and C. Li, *Unbounded upper and lower solution method for third-order boundary-value problems on the half-line*, Electronic J. Differ. Equ. **2009** (2009), no. 119, 1–12.

[7] Y. Bao, L. Wang, and M. Pei, *Existence of positive solutions for a singular third-order two-point boundary value problem on the half-line*, Boundary Value Prob.**2022** (2022), no. 1, 1–11.

[8] A. Cabada and G. Wang, *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, J. Math. Anal. Appl. **389** (2012), no. 1, 403–411.

[9] M. Caputo, *Linear models of dissipation whose q is almost frequency independent*, Ann. Geophys.**19** (1966), no. 4, 383–393.

[10] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Prog. Fractional Differ. Appl. **1** (2015), no. 2, 73–85.

[11] K. Diethelm and N.J. Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl. **265** (2002), no. 2, 229–248.

[12] K. Diethelm and A.D. Freed, *On the solution of nonlinear fractional-order differential equations used in the modeling of viscoplasticity*, Scientific computing in chemical engineering II: computational fluid dynamics, reaction engineering, and molecular properties, Springer, 1999, pp. 217–224.

[13] W.G. Glöckle and T.F. Nonnenmacher, *A fractional calculus approach to self-similar protein dynamics*, Biophys. J. **68** (1995), no. 1, 46–53.

[14] J.R. Graef, L. Kong, and Q. Kong, *Application of the mixed monotone operator method to fractional boundary value problems*, Fract. Calc. Differ. Calc. **2** (2011), 554–567.

[15] J.R. Graef, L. Kong, Q. Kong, and M. Wang, *Uniqueness of positive solutions of fractional boundary value problems with non-homogeneous integral boundary conditions*, Fractional Calc. Appl. Anal. **15** (2012), 509–528.

[16] A. Guezane-Lakoud and A. Ashyralyev, *Positive solutions for a system of fractional differential equations with nonlocal integral boundary conditions*, Differ. Equ. Dyn. Syst. **25** (2017), 519–526.

[17] D. Guo, *Fixed points of mixed monotone operators with applications*, Appl. Anal. **31** (1988), no. 3, 215–224.

[18] D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal.: Theory Meth. Appl. **11** (1987), no. 5, 623–632.

[19] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5, Academic Press, 2014.

[20] J. Hadamard, *Essai sur l'étude des fonctions données par leur développement de taylor*, J. Math. Pures Appl. **8** (1892), 101–186.

[21] J. Henderson and R. Luca, *Existence of positive solutions for a system of semipositone coupled discrete boundary value problems*, J. Differ. Equ. Appl. **25** (2019), no. 4, 516–541.

[22] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, 2000.

[23] A.A. Kilbas, O.I. Marichev, and S.G. Samko, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Switzerland, 1993.

[24] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Vol. 204, Elsevier, 2006.

[25] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, CSP, 2009.

[26] H. Lian, P. Wang, and W. Ge, *Unbounded upper and lower solutions method for Sturm–Liouville boundary value problem on infinite intervals*, Nonlinear Anal.: Theory Meth. Appl. **70** (2009), no. 7, 2627–2633.

[27] F. Mainardi, *Fractional Calculus: Some Basic Problems in Continuum and Statistical Mechanics*, Springer, 1997.

[28] H.M. Malaikah, *The Adomian decomposition method for solving Volterra-Fredholm integral equation using maple*, Appl. Math. **11** (2020), 779–787.

[29] R. Metzler, W. Schick, H.-G. Kilian, and T.F. Nonnenmacher, *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys. **103** (1995), no. 16, 7180–7186.

[30] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, 1993.

[31] E. Pourhadi, R. Saadati, and S.K. Ntouyas, *Application of fixed-point theory for a nonlinear fractional three-point boundary-value problem*, Mathematics **7** (2019), no. 6, 526.

[32] M. Sangi, S. Saiedinezhad, and M.B. Ghaemi, *A system of high-order fractional differential equations with integral boundary conditions*, J. Nonlinear Math. Phys.**30** (2023), no. 2, 699–718.

[33] V.E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer Science & Business Media, 2011.

[34] W. Wang and X. Liu, *Properties and unique positive solution for fractional boundary value problem with two parameters on the half-line*, J. Appl. Anal. Comput **11** (2021), no. 5, 2491–2507.

[35] B. Yan, D. O'Regan, and R.P. Agarwal, *Unbounded solutions for singular boundary value problems on the semi-infinite interval: upper and lower solutions and multiplicity*, J. Comput. Appl. Math. **197** (2006), no. 2, 365–386.

[36] C. Zhai and M. Hao, *Fixed point theorems for mixed monotone operators with perturbation and applications to fractional differential equation boundary value problems*, Nonlinear Anal.: Theory Methods Appl. **75** (2012), no. 4, 2542–2551.