Int. J. Nonlinear Anal. Appl. 16 (2025) 4, 103–108
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.32952.4900

Incomplete inverse problems for the Sturm-Liouville type differential equation with the spectral boundary condition

Yasser Khalili[a,*], Nematollah Kadkhoda[b]

[a]Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, 578 Sari, Iran

[b]Department of Mathematics, Faculty of Engineering Science, Quchan University of Technology, Quchan, Iran

(Communicated by Haydar Akca)

Abstract

In this study, we examine the inverse problem for the differential equation of the Sturm-Liouville type with the spectral boundary condition in the finite interval. Using Lieberman-Hochstadt's method, we show that if $p(x)$ is prescribed on the half interval $\left(\frac{\pi}{2}, \pi\right)$ then a single spectrum suffices to determine $p(x)$ on $(0, \pi)$. Moreover, applying Gesztesy-Simon's method, we demonstrate that if $p(x)$ is assumed over the given segment $[\pi/2(1-\theta), \pi]$ where $\theta \in (0,1)$, a finite number of the spectrum is enough to give $p(x)$ on $(0, \pi)$.

Keywords: Sturm-Liouville equation, Inverse problem, Spectrum
2020 MSC: 34K29, 34B24, 47A10

1 Introduction

Inverse spectral problems (ISPs) are recovering the coefficient of the problem from its spectral data. Such problems often rise in electronics, quantum mechanics, geophysics, mathematical physics, etc [2, 13, 14]. ISPs for Sturm-Liouville equations (SLEs) have been firstly studied by Ambarzumyan in 1929 [1] and secondly by Borg in 1946 [3]. The inverse problems for SLEs are investigated in the next years in [10, 15, 19, 23]. In this article, we study the boundary value problem $L(p)$,

$$-u''(x) + p(x)u(x) = \lambda u(x), \quad x \in (0, \pi), \tag{1.1}$$

$$U(u) := u'(0) = 0, \tag{1.2}$$

$$V(u) := \frac{\sin(\varrho\pi)}{\varrho} u'(\pi) + \cos(\varrho\pi)u(\pi) = 0, \tag{1.3}$$

where the real function $p(x) \in L_2(0, \pi)$. The parameter $\lambda = \varrho^2$ is a spectral parameter.

The initial findings regarding the half inverse problems for Sturm-Liouville operators have also been examined by Lieberman and Hochstadt in 1978 [8]. They have shown that the potential $q(x)$ on $(\frac{1}{2}, 1)$ and one spectrum of

$$-u''(x) + q(x)u(x) = \lambda u(x), \qquad x \in (0, 1),$$
$$u'(0) - hu(0) = 0 = u'(1) + Hu(1) = 0,$$

*Corresponding author
Email addresses: y.khalili@sanru.ac.ir (Yasser Khalili), kadkhoda@qiet.ac.ir (Nematollah Kadkhoda)

Received: January 2024 *Accepted:* March 2024

uniquely identify $q(x)$ on $[0,1]$. Then, Hald continued this technique considering impulse conditions inside the interval [7] and these works have been adapted to different types of Sturm-Liouville problems (SLPs) [4, 11, 12, 21, 24]. In the following and with more studies in this field, Gesztesy and Simon have firstly proved that the incomplete data regarding the spectrum and the coefficients of the problem can uniquely establish the problem on all interval [6]. In next years, the authors have applied this technique to survey of the inverse SLPs [9, 10, 18, 23]. Studying these papers, it has made us interested to survey the ISP of the Sturm-Liouville operator (1.1)-(1.3) by these methods. To the best of our knowledge, this problem has not been previously considered. In this work, we would like to discuss on L and prove the uniqueness of the inverse problem from one spectrum plus understanding the potential within the half of the internal by Hochstadt-Lieberman's method. Moreover we study the uniqueness of the inverse problem of L taking the incomplete data on the spectrum and the coefficients of the problem by Gesztesy-Simon's method.

In this paper, we investigate two inverse problems for the SLP (1.1)-(1.3). We give some preliminaries and results in Sec. 2. In Sec. 3, the proof of two uniqueness theorems for (1.1)-(1.3) has been brought.

2 Preliminaries and Results

For every fixed $x \in (0, \pi)$, the given equation (1.1) has an entire solution $S(x, \varrho)$ in ϱ that satisfies certain conditions $S(0, \varrho) = 1$ and $S'(0, \varrho) = 0$. The characteristic function of L denoted by

$$\Delta(\varrho) = \frac{\sin(\varrho\pi)}{\varrho} S'(\pi, \varrho) + \cos(\varrho\pi) S(\pi, \varrho), \tag{2.1}$$

has the same zeros as the eigenvalues ϱ_n of L [5]. From [16, 22], the given expression represents an integral form of the solution

$$S(x, \varrho) = \cos(\varrho x) + \int_0^x K(x, t) \cos(\varrho t) dt, \tag{2.2}$$

for a function $K(x, t)$ that is continuous. For sufficiently large ϱ, we can infer this solution as follows:

$$S(x, \varrho) = \cos(\varrho x) + \left(2h + \int_0^x p(t) dt\right) \frac{\sin(\varrho x)}{2\varrho} + o\left(\frac{1}{\varrho} \exp(|\Im \varrho| x)\right). \tag{2.3}$$

Now by using (2.1) and (2.3), one gets for enough large ϱ,

$$\Delta(\varrho) = \cos(2\varrho\pi) + O\left(\frac{1}{\varrho} \exp(2|\Im\varrho|\pi)\right), \tag{2.4}$$

and then the eigenvalues ϱ_n,

$$\varrho_n = \frac{n}{2} + \frac{1}{4} + O(n^{-1}),$$

for sufficiently large n. Denote the set E_κ by $E_\kappa := \{\varrho \in \mathbb{C}; \mid \varrho - \varrho_n \mid \geq \kappa, \ n \in \mathbb{N}\}$ as a fix $\kappa > 0$. Applying (2.4), we receive

$$|\Delta(\varrho)| \geq C_\kappa \exp(2|\Im\varrho|\pi), \quad \varrho \in E_\kappa, \tag{2.5}$$

for some positive constant C_κ. To investigate the inverse problem for the SLP (1.1)-(1.3), alongside $L = L(p)$, we will consider a boundary value problem $\widetilde{L} = L(\widetilde{p})$ in which the equation will have a similar form as before, but with a different coefficient. Now we establish the important results of this article. At first we express the half inverse theorem for the SLP (1.1)-(1.3).

Theorem 2.1. If $\lambda_n = \widetilde{\lambda}_n$ as any natural numbers n and $p(x) = \widetilde{p}(x)$ on $\left(\frac{\pi}{2}, \pi\right)$ then $p(x) = \widetilde{p}(x)$ a.e. on $(0, \pi)$.

Another result of this paper, the Gesztesy-Simon's theorem for the SLP (1.1)-(1.3) is now stated.

Theorem 2.2. Assume $\sigma(L)$ as the spectrum of L, $G \subseteq \sigma(L) \cap \widetilde{\sigma}(L)$ considering

$$\sharp\{\varrho \in G; \varrho \leq \varrho_0\} \geq (1-\theta)\sharp\{\varrho \in \sigma(L); \varrho \leq \varrho_0\} + \frac{\theta}{2},$$

as enough large $\varrho_0 \in \mathbb{R}$, and in the case of $\theta \in (0, 1)$,

$$p(x) = \widetilde{p}(x), \ x \in [\pi/2(1-\theta), \pi].$$

Then $p(x) = \widetilde{p}(x)$, a.e. on $(0, \pi)$.

3 Proofs

In this section, we give the proof of theorems in this work. Firstly, we prove the half inverse theorem.

Proof of Theorem 2.1. $S(x,\varrho)$ can be as the solution of

$$\begin{aligned} -S''(x,\varrho)+p(x)S(x,\varrho)&=\lambda S(x,\varrho), \\ S(0,\varrho)=1, \quad S'(0,\varrho)&=0, \end{aligned} \tag{3.1}$$

and $\widetilde{S}(x,\varrho)$ represents the solution of

$$\begin{aligned} -\widetilde{S}''(x,\varrho)+\widetilde{p}(x)\widetilde{S}(x,\varrho)&=\lambda\widetilde{S}(x,\varrho), \\ \widetilde{S}(0,\varrho)=1, \quad \widetilde{S}'(0,\varrho)&=0. \end{aligned} \tag{3.2}$$

We firstly multiply (3.1) by $\widetilde{S}(x,\varrho)$ and (3.2) by $S(x,\varrho)$, and subtract two resulting equations from one another. Therefore

$$\big(p(x)-\widetilde{p}(x)\big)S(x,\varrho)\widetilde{S}(x,\varrho)=S''(x,\varrho)\widetilde{S}(x,\varrho)-S(x,\varrho)\widetilde{S}''(x,\varrho). \tag{3.3}$$

Integrating (3.3) on $(0,\pi)$, one gives

$$\int_0^{\pi}\big(p(x)-\widetilde{p}(x)\big)S(x,\varrho)\widetilde{S}(x,\varrho)dx=(S'(x,\varrho)\widetilde{S}(x,\varrho)-S(x,\varrho)\widetilde{S}'(x,\varrho))|_{x=0}^{x=\pi}.$$

Using the assumption of the theorem, we can conclude

$$\begin{aligned} J(\varrho) &:= \int_0^{\frac{\pi}{2}}\big(p(x)-\widetilde{p}(x)\big)S(x,\varrho)\widetilde{S}(x,\varrho)dx \\ &= S'(\pi,\varrho)\widetilde{S}(\pi,\varrho)-S(\pi,\varrho)\widetilde{S}'(\pi,\varrho). \end{aligned} \tag{3.4}$$

The attributes of $S(x,\varrho)$ and $\widetilde{S}(x,\varrho)$, as well as the boundary condition in (1.3) give that $H(\varrho_n)=0$. Following, we will demonstrate that $J(\varrho)=0$ for whole ϱ. Using (2.2), we will have as a continuous function $K'(x,t)$,

$$S(x,\varrho)\widetilde{S}(x,\varrho) = \frac{1}{2}\left(1+\cos\left(2\varrho x\right)\right)+\int_0^x K'(x,t)\cos\left(2\varrho t\right)dt, \tag{3.5}$$

and then from

$$|\cos(2\varrho x)|\le exp(2|\Im\varrho|x),$$

one gives for $x<\frac{\pi}{2}$,

$$|S(x,\varrho)\widetilde{S}(x,\varrho)|\le M\exp(|\Im\varrho|\pi),$$

as a constant $M>0$. Thus as enough large ϱ,

$$|J(\varrho)|\le C\exp(|\Im\varrho|\pi), \tag{3.6}$$

for some constant $C>0$. Put the analytic function

$$\phi(\varrho)=\frac{J(\varrho)}{\Delta(\varrho)}.$$

Taking (2.5) and (3.6), one can see that $\phi(\varrho)=0$ as sufficiently large ϱ. So, we can write that $J(\varrho)=0$ as whole ϱ. Substituting (3.5) in (3.4) and taking $J(\varrho)=0$, we get

$$\int_0^{\frac{\pi}{2}}\big(p(x)-\widetilde{p}(x)\big)\left(1+\cos(2\varrho x)+\int_0^x K'(x,t)\cos(2\varrho t)dt\right)dx=0,$$

and so

$$\int_0^{\frac{\pi}{2}} \big(p(x)-\widetilde{p}(x)\big)dx + \int_0^{\frac{\pi}{2}} \cos(2\varrho t)\bigg(p(t)-\widetilde{p}(t)+\int_t^{\frac{\pi}{2}} K'(x,t)\big(p(x)-\widetilde{p}(x)\big)dx\bigg)dt = 0.$$

By using the Riemann-Lebesgue Lemma, we find that

$$\int_0^{\frac{\pi}{2}} \big(p(x)-\widetilde{p}(x)\big)dx = 0, \quad \int_0^{\frac{\pi}{2}} \cos(2\varrho t)\bigg(p(t)-\widetilde{p}(t)+\int_t^{\frac{\pi}{2}} K'(x,t)\big(p(x)-\widetilde{p}(x)\big)dx\bigg)dt = 0.$$

Taking the completeness of "*cos*", we have

$$p(t)-\widetilde{p}(t)+\int_t^{\frac{\pi}{2}} K'(x,t)\big(p(x)-\widetilde{p}(x)\big)dx = 0, \qquad 0<t<\frac{\pi}{2}.$$

This equation is a type of integral equation called a homogeneous Volterra integral equation, and it has no solution except for the trivial solution. This implies that the functions $p(x)$ and $\widetilde{p}(x)$ are equal for a. e. x in the interval $\big(0,\frac{\pi}{2}\big)$. The proof is now finished. □

The following is the proof of the Gesztesy-Simon's theorem.

Proof of Theorem 2.2. Considering (3.3), because $p(x)=\widetilde{p}(x),\ x\in[\pi/2(1-\theta),\pi]$, one gets

$$\int_0^{\pi/2(1-\theta)} \big(p(x)-\widetilde{p}(x)\big)S(x,\varrho)\widetilde{S}(x,\varrho)dx = S'(\pi,\varrho)\,\widetilde{S}(\pi,\varrho) - S(\pi,\varrho)\,\widetilde{S}'(\pi,\varrho)\,. \tag{3.7}$$

Putting

$$J_1(\varrho) := \int_0^{\pi/2(1-\theta)} \big(p(x)-\widetilde{p}(x)\big)S(x,\varrho)\widetilde{S}(x,\varrho)dx, \tag{3.8}$$

we will have

$$J_1(\varrho) = S'(\pi,\varrho)\,\widetilde{S}(\pi,\varrho) - S(\pi,\varrho)\,\widetilde{S}'(\pi,\varrho)\,.$$

From the hypothesis of the theorem, one results $J_1(\varrho_n)=0$ as $\varrho_n\in G$. It is enough to demonstrate that $J_1(\varrho)=0$ as other ϱ. Using (3.5) and (3.8), it is inferred that

$$|J_1(\varrho)| \le C_1 \exp(|\Im\varrho|\pi(1-\theta)),$$

as a constant $C_1>0$. Further supposing $\lambda = ib$,

$$|J_1(ib)| \le C_1' \exp(\Im\sqrt{i}|\sqrt{b}|\pi(1-\theta)), \tag{3.9}$$

for some constant $C_1'>0$. Let the entire function

$$\phi_1(\varrho) = \frac{J_1(\varrho)}{\Delta_1(\varrho)}, \tag{3.10}$$

where

$$\Delta_1(\varrho) = \prod_{\lambda_n\in G}\left(1-\frac{\lambda}{\lambda_n}\right).$$

Based on the hypothesis of the theorem, we can give

$$N_{\Delta_1}(\varrho_0) \ge (1-\theta)N_{\Delta}(\varrho_0) + \frac{\theta}{2},$$

in which

$$N_{\Delta_1}(\varrho_0) = \sharp\{\lambda \in G; \lambda \leq \lambda_0\}, \quad N_{\Delta}(\varrho_0) = \sharp\{\lambda \in \sigma(L); \lambda \leq \lambda_0\}.$$

Due to the fact that $\Delta(\varrho)$ is an analytic function, we will have

$$N_{\Delta_1}(\varrho_0) \leq N_{\Delta}(\varrho_0) \leq \mathcal{C}\sqrt{\lambda},$$

as a constant $\mathcal{C} > 0$. Using the known calculations (see [17, 20]), the following result gets

$$ln|\Delta_1(ib)| = (1-\theta)ln|\Delta(ib)| + \frac{\theta}{4}ln(1+b^2). \tag{3.11}$$

From the eigenvalues of L i.e., $\sigma(L)$, we will have

$$|\Delta(ib)| \geq \mathbf{C}\exp\left(2\Im\sqrt{i}|\sqrt{b}|\pi\right), \tag{3.12}$$

for sufficiently large b and some constant $\mathbf{C} > 0$. Thus taking (3.11) and (3.12), we can infer

$$|\Delta_1(ib)| \geq \mathbf{C}_1 b^{\frac{\theta}{2}} \exp\left(2\Im\sqrt{i}|\sqrt{b}|(1-\theta)\pi\right), \tag{3.13}$$

for some constant $\mathbf{C}_1 > 0$. Applying (3.9), (3.10) and (3.13), we can write

$$|\phi_1(ib)| \leq \mathfrak{C}\ b^{\frac{-\theta}{2}} \exp\left(-\Im\sqrt{i}|\sqrt{b}|(1-\theta)\pi\right),$$

for some constant $\mathfrak{C} > 0$. By Phragmn-Lindelf theorem, one gets $\phi_1(\varrho) = 0$ as whole ϱ. Subsequently $J_1(\varrho) = 0$ as whole ϱ. Now the similar discussion done in Theorem 2.1 will result that $p(x) = \widetilde{p}(x)$ a. e. on $[0, \pi/2(1-\theta)]$. The proof is now finished. □

Acknowledgment

The financial support of this study was provided by Sari Agricultural Sciences and Natural Resources University in Iran in the form of research project (Project No. 03-1402-09).

References

[1] V.A. Ambartsumyan, *U¨ ber eine frage der eigenwerttheorie,* Z. Phys. **53** (1929), 690–695.

[2] R.S. Anderssen, *The effect of discontinuous in density and shear velocity on the asymptotic overtone structure of torsional eigenfrequences of the earth,* Geophys. J. Royal Astronom. Soc. **50** (1997), 303–309.

[3] G. Borg, *Eine Umkehrung der Sturm-Liouville schen Eigenwertaufgabe,* Acta Math. **78** (1946), 1–96.

[4] Y. Cakmak and S. Isık, *Half inverse problem for the impulsive diffusion operator with discontinuous coefficient,* Filomat **30** (2016), no. 1, 157–168.

[5] G. Freiling and V.A. Yurko, *Inverse Sturm-Liouville Problems and their Applications,* NOVA Science Publ., New York, 2001.

[6] F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential, II. The case of discrete spectrum*, Trans. Amer. Math. Soc. **352** (2000), 2765–2787.

[7] O.H. Hald, *Discontinuous inverse eigenvalue problems,* Commun. Pure Appl. Math. 37 (1984), no. 5, 539–577.

[8] H. Hochstadt and B. Lieberman, *An inverse Sturm-Liouville problem with mixed given data,* SIAM J. Appl. Math. **34** (1978) 676-680.

[9] Y. Khalili and D. Baleanu, *Recovering differential pencils with spectral boundary conditions and spectral jump conditions,* J. Inequal. and Appl. **2020** (2020), 262, https://doi.org/10.1186/s13660-020-02537-z.

[10] Y. Khalili, N. Kadkhoda, and D. Baleanu, *Inverse problems for the impulsive Sturm–Liouville operator with jump conditions,* Inve. Probl. Sci. Eng. **27** (2019), no. 10, 1442–1450.

[11] Y. Khalili, M. Yadollahzadeh, and M.K. Moghadam, *Half inverse problems for the impulsive operator with eigenvalue-dependent boundary conditions,* Electronic J. Differ. Equ. **190** (2017), 1–5.

[12] H. Koyunbakan and E. Panakhov, *Half-inverse problem for diffusion operators on the finite interval,* J. Math. Anal. Appl. **326** (2007), 1024–1030.

[13] RJ. Kruger, *Inverse problems for nonabsorbing media with discontinuous material properties,* J. Math. Phys. **23** (1982), no. 3, 396–404.

[14] F.R. Lapwood and T. Usami, *Free Oscillation of the Earth,* Cambridge University Press, Cambridge, 1981.

[15] A.S. Ozkan and B. Keskin, *Inverse nodal problems for Sturm-Liouville equation with eigenparameter-dependent boundary and jump conditions,* Inve. Probl. Sci. Eng. **23** (2015), no. 8, 1306–1312.

[16] A.S. Ozkan and B. Keskin, *Uniqueness theorems for an impulsive Sturm-Liouville boundary value problem,* Appl. Math. J. Chinese Univ. **27** (2012), 428–434.

[17] Y.P. Wang, *A uniqueness theorem for Sturm-Liouville operators with eigenparameter dependent boundary conditions,* Tamking J. Math. **43** (2012), 145–152.

[18] Y.P. Wang, *Inverse problems for Sturm-Liouville operators with interior discontinuities and boundary conditions dependent on the spectral parameter,* Math. Meth. Appl. Sci. **36** (2013), 857–868.

[19] Y.P. Wang and C.T. Shieh, *Inverse problems for Sturm-Liouville equations with boundary conditions linearly dependent on the spectral parameter from partial information,* Results. Math. **65** (2014), 105–119.

[20] Y.P. Wang, C.T. Shieh, and Y.T. Ma, *Inverse spectral problems for Sturm-Liouville operators with partial information,* Appl. Math. Lett. **26** (2013), 1175–1181.

[21] Y.P. Wang, C.F. Yang, and Z.Y. Huang, *Half inverse problem for Sturm-Liouville operators with boundary conditions dependent on the spectral parameter,* Turk. J. Math. **37** (2013), no. 3, 445-454.

[22] C.F. Yang, *A uniqueness theorem from partial transmission eigenvalues and potential on a subdomain,* Math. Meth. Appl. Sci. **39** (2016), 527–532.

[23] C.F. Yang, *Inverse problems for the Sturm-Liouville operator with discontinuity,* Inve. Probl. Sci. Eng. **22** (2014), 232–244.

[24] C.F. Yang and A. Zettl, *Half inverse problems for quadratic pencils of Sturm-Liouville operators,* Taiwanese J. Math. **16** (2012), no. 5 1829–1846.