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Qualitative study for the p-Laplacian type diffusion-convection equation with the effect of an absorption source

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Abstract

Within this paper, we study the initial growth of interfaces and asymptotic locally weak solutions by the construction of its self-similar solution to the Cauchy Problem for a parabolic p-Laplacian type diffusion-convection equation with the effect of absorption source. The significant methods are applied in this work, techniques of blowing up, rescaling, and comparison principles in non-smooth domains. The importance of this model came through its applications in a variety of fields such as chemical process design, biophysics, plasma physics, quantum physics, and others.

Keywords: p-Laplacian diffusion equation, rescaling technique, self-similarity, weak solution, interface
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1 Introduction

The heat equation served as the primary focus of mathematical research on heat transmission and diffusion for a very long time. Throughout the past 200 years, both conceptually and practically, there has been a substantial advancement in a mathematical model of heat dispersion and propagation. The hypothesis was inspired by engineering and physics. Nowadays, this impact has spread to disciplines as diverse as biology, economics, and social sciences. The concept of diffusion is derived from the meaning "to spread out" by moving from a high-concentration area to a low-concentration one, as described by [8]. Over the years, a number of researchers has become interested in the investigation of the mathematical equation describing diffusion problems. In [9],the non-linear diffusion equation has a fresh analysis and precise solutions. Also, some modulation equations are derived in [19] for hexagonal pattern in system of reaction-diffusion equations. In comparison to Smith-Hohenberg-type models or Rayleigh-Bernard convection, these systems have additional nonlinearities. The diffusion equation with non-homogeneous reaction force were explored by Nakamura et al.[20],The existence and uniqueness of the solutions to a self-similarity diffusion equations was investigated by Ayeni and Agusto [5].The study looked at models for microwave heating of various materials and fast gas flow through a porous media.

Let us think about the turbulent, polytrophic, and the one-dimensional flow of gas in porous medium equation, see [14].The following laws

$$P_r = c\Gamma^k \tag{1.1}$$

$$n\Gamma_t + (\Gamma s)_t = 0 \tag{1.2}$$

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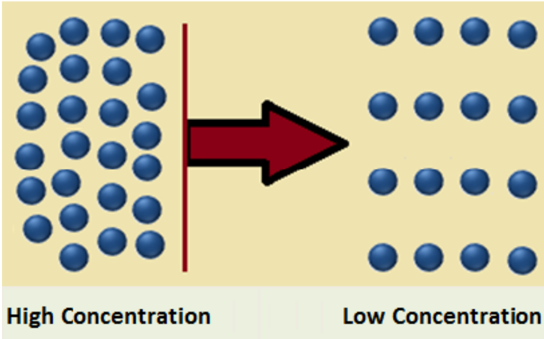

Figure 1: The particles randomly move around ("diffuse") to satisfy the equilibrium

$$\Gamma s = -M|\psi_x|^{p-2}\psi_x \tag{1.3}$$

$$\psi = P_r^{(k+1)/k}. \tag{1.4}$$

Equations (1.1)-(1.3) are called polytropic state process, the continuity equation and the flux under turbulent condition, respectively. Where P_r is the pressure, Γ is a density of the gas, and s is the gas's velocity at space point x at time instant t. Also, the physical parameters c, n, M have positive sign, and $p \geq 3/2, k \geq 1$
To combine equations (1.1)-(1.4), we get

$$n\Gamma_t = Mc^{(p-1)(k+1)/k}(|(\Gamma^{k+1})_x|^{p-2}(\Gamma^{k+1})_x)_x. \tag{1.5}$$

By rescaling the parameters in (1.5), we get porous medium as follows

$$v_t = (|(v^m)_x|^{p-2}(v^m)_x)_x.$$

where,$v = v(x,t), m(p-1) > 1, m = k+1.\ if\ m = 1$ 1Consequently, it is now known as the non-Newtonian elastic filtering equation.

$$v_t - (|v_x|^{p-2}v_x)_x = 0 \tag{1.6}$$

where $p > 2$ which represents slow diffusion. The behaviour of the interfaces and the emergence of local solutions close to the boundary to the p-Laplacian type slow diffusion equations was introduced in [18].The existence, uniqueness, and regularity of solutions to certain starting and boundary value problems to the general p-Laplacian-type diffusion equation (1.6) that has been established in the literature [2, 3].Moreover, Several survey studies introduced the evolution of interfaces for the slow and fast diffusion equations with absorption of the p-Laplacian type. This paper is one of contributions to the theory of qualitative to the non-linear PDEs in irregular domain. It represents the extension of the previous work in [1]

2 Preliminary and Statement of Problem

Due to its vast mathematical applications in areas including chemical reaction architecture, plasma physics, biophysics, and quantum physics, as well as its rich mathematical content, the parabolic non-linear p-Laplacain equation (1.6) has attracted a lot of attention (see [2, 3, 6, 10, 15]). Our study takes into consideration the following problem:

$$Lv \equiv v_t - (|(v)_x|^{p-2}(v)_x)_x + a(v^\beta)_x + bv^\beta = 0, \tag{2.1}$$

$$v(x,0) = v_0(x), \tag{2.2}$$

for $x \in R, 0 < t < T; p > 2, a < 0, b > 0, \beta > 0$ with continuity of the function $v_0 \geq 0$. Cauchy problem CP(2.1)-(2.2)is defined on the parabolic p-Laplacain type diffusion and nonlinear convection-reaction terms with the initial data. Because the non-positive sign of the advection coefficient a. The nonlinearity quality of the process (2.1) is one of its primary properties and because of the gap produced by nonlinear factors and irregular domains, forces of the equation type (2.1) are occasionally known as equations with peculiar growth conditions.

The equation (2.1) with $a = 0$ can be understood as a specific situation of the p-Laplacian equation. It has gotten a lot of attention in recent decades and has become a touchstone in the study of parabolic PDEs. Let us introduce

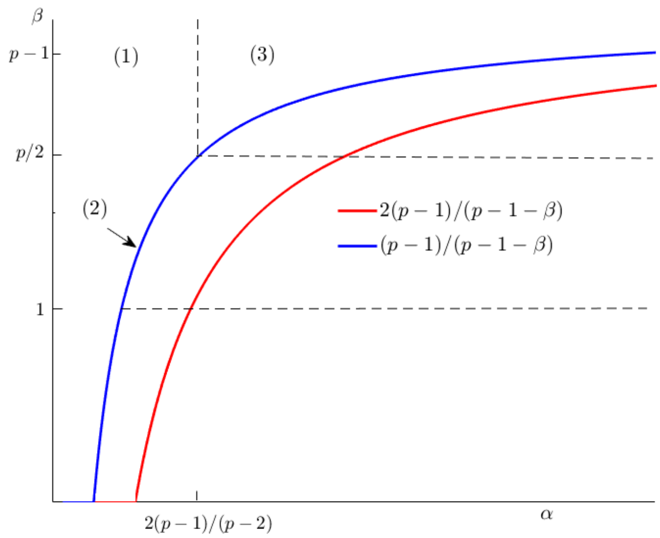

Figure 2: The plane (α, β) to classify qualitative behavior of interface to the CP(2.1)-(2.2) with if $a < 0$.

some important researches to the readers to offer the significant notions, see [7, 12]. Also, the interfaces of the solution to the CP(2.1)-(2.2) are separated regions and for more details to the behaviour of an interface,[1, 2]. The local case for the initial data is

$$v_0(x) \asymp C(-x)_+^{\alpha} \ as\ x \to 0^- \ for\ C > 0, \alpha > 0 \tag{2.3}$$

The conflict between these two powers, p-Laplacian and advection, determines the behavior of interface movement and its direction.v_0 is suitable for satisfying the general theory since it is a bounded initial function that satisfies some parameter restrictions as approaches to zero, see [11]. Furthermore, the global initial data is clearly introduced as

$$v_0(x) = C(-x)_+^{\alpha}. \tag{2.4}$$

The growth rate conditions to nonlinear parabolic PDEs and the porous medium equation (PME) were studied in significant sources, see [16, 4]. The qualitative theory for the equation of the nonlinear diffusion-advection with a source function

$$v_t - (v^m)_{xx} + a(v^{\beta})_x + b(v^{\beta}) = 0 \tag{2.5}$$

is being investigated in the works,[16, 11]. The general theory of the Cauchy problem (2.5)-(2.2) and qualitative qualities are established irregular domains with compactly supported initial data, [12, 13].

In the our scenario where p-Laplacian type dispersion dominates over convection or reaction forces, this study is considered a categorization of the evolution of the interfaces. We focus on the scenario where diffusion of the p-Laplacian type predominates. The main results are introduced to estimate the local weak solution near expanding, shrinking or waiting time interface under some restrictions.

Definition 2.1. The weak solution $v(x,t)$ to the equation (2.1) satisfies the identity

$$I(v,\phi,D) = \int_{\Im_0}^{\Im_1}\int_{u_1(t)}^{u_2(t)} (-v\phi_t + |v_x|^{p-2}v_x\phi_x - av^{\beta}\phi_x + bv^{\beta}\phi)dxdt + \int_{u_1(t)}^{u_2(t)} v\phi dx|_{\Im_0}^{\Im_1} \tag{2.6}$$

where $f \in C_{x,t}^{2,1}(\bar{\psi})$ and $f|_{x=n_1(t)} = f|_{x=n_2(t)} = 0$ for, $\Im_0 \le t \le \Im_1$ and

$$\psi = \{(x,t) : n_1(t) < x < n_2(t),\ \Im_0 < t < \Im_1\}.$$

Also, the solution $v(x,t)$ satisfies a local subsolution (resp. supersolution) of equation (2.1) if $I(v,\phi,D) \le 0$ $(resp\ . \ge 0)$. Several studies in [1, 4, 11], discussed the general theory to the IVP for (2.1). In the article [14], the authors used the energy method to prove the qualitative properties. The technique of the comparison theorem (theorem 2.2, comparison principle in [17]) is very significant in this work. Also, the equation (2.1) by eliminating the advection term with $v = 0$ has a property of the optimal growth rate.

Theorem 2.2. (Comparison Theorem)Let $\Omega_1 = \{(x,t) : \xi_0(t) < x < +\infty,\ 0 < t < \tau \leq +\infty\}$ be defined. A non-negative function $v(x,t)$ is considered in the space $C(\overline{\Omega_1})$ and ω is in $C^{2,1}_{x,t}(\Omega_1)$ Let the curves $x = \xi_j(t)$ divide Ω_1 into sub-regions Ω^j , where $\xi_j \in C[0.\tau]$, and $\Im_1 \in [\sigma.\tau]$ for arbitrary $\sigma > 0$. Let us consider ω satisfies the following

$$L\omega \equiv \omega_t - (|\omega_x|^{p-2}\omega_x)_x + a(\omega^\beta)_x + b\omega^\beta \geq 0, (or \leq 0)$$

where $\omega \in L^\infty(\Omega_1 \cap (t \leq \tau)), \omega \in C^{2,1}_{x,t}(\Omega_1)$ and $(|\omega_x|^{p-2}\omega_x)_x \in C(\Omega_1)$.Additionally, if the following conditions are satisfied

$$\omega(\xi_0(t),t) \geq (\leq) v(\xi_0(t),t), \omega(x,0) \geq (\leq) v(x,0).$$

Then

$$\omega(x,0) \geq (\leq) v(x,0)\ in\ \overline{\Omega_1},$$

where $v(x,0)$ sometimes grows up as $|x| \to +\infty$, see [2]. The equation (2.1) with $a = b = 0$, and the optimal growth condition was derived in [18]. We try to discuss the qualitative analysis of the CP(2.1)-(2.2) affects the convection and reaction terms. After the basic results are presented the first and second sections, we present the main results in the following sections.

3 Dominated p-Laplacian Term over Reaction & Convection

Through this part, the asymptotically local weak solutions of the CP(2.1)-(2.2) will be estimated near the expending interfaces. The dominated p-Laplacian type diffusion force over the convection and reaction factors is clearly seen in region(1), Figure 2 The following theorem can have two separate sub- regions:

Theorem 3.1. The dominated p-Laplacian type diffusion over the both two forces in the case α is less than the value $(p-2)/(p-1-min\{\beta, p/2\})^{-1}(p-1)$

$$\zeta_* = C^{(p-2)(p+\alpha(2-p))^{-1}}\zeta'_*, \quad \zeta'_* > 0 \tag{3.1}$$

where $\zeta'_* = \zeta'_*(\alpha, p)$. with initially expending interface

$$\eta(t) \sim \zeta_* t^{(p+\alpha(2-p))^{-1}}, \tag{3.2}$$

with self-similar solution along $x = t^{1/(p+\alpha(2-p))}\rho$

$$v(x,t) \sim t^{\alpha/(p+\alpha(2-p))}\varphi(\rho). \tag{3.3}$$

For arbitrarily $\rho < \zeta'_*$, depends $C, \alpha, p, \exists\varphi(\rho) > 0$. The CP(2.1),(2.4) has the following formula along $\zeta = t^{-1/(p+\alpha(2-p))}x$

$$v_*(x,t) = t^{\alpha/(p+\alpha(2-p))}\varphi(\zeta) \tag{3.4}$$

where the shape function φ the non-linear ODE problem satisfies:

$$\begin{cases} (|(\varphi)'|^{p-2}(\varphi)')' + (p+\alpha(2-p))^{-1}\zeta\varphi'(\zeta) - \alpha(p+\alpha(2-p))^{-1}\varphi = 0, \\ \varphi(-\infty) \sim C(-\zeta)^\alpha, \varphi(\zeta) \equiv 0,\ \zeta \geq \zeta_*,\ \varphi(\zeta_*) = 0, \quad \zeta < \zeta_*. \end{cases}$$

The following illustrates how it is dependent on C:

$$\varphi(\zeta) = \varphi_0((p+\alpha(2-p))^{-1}(2-p)\zeta)C^{p(p-\alpha(p-2))^{-1}}, \quad \varphi_0(\zeta) = v(\zeta,1), \tag{3.5}$$

A solution v with the coefficients $a = b = 0$ satisfies the CP(2.1),(2.4) and is represented by explicit formulas (3.1) and (3.4). The following lemma, which proven in [1],is crucial to the proof lemma 3.3 .

Lemma 3.2. If the restriction α is less than $p/(p-2)$, then the CP(2.2),(2.4) fulfils the shape function (3.5) and has the formula (3.4). Moreover, the CP(2.1)-(2.2) satisfies (3.1)-(3.3) if v_0 fulfills (2.3).

Lemma 3.3. The CP(2.1)-(2.2) has a solution v with the initial data (2.3). Under either of the following

(a) $\alpha < (p-1)/(p-1-\beta), 0 < \beta < p/2$

(b) $\alpha < 2(p-1)/(p-2), p/2 \leq \beta < p-1$

(c) $0 < \alpha < 2(p-1)/(p-2), p-1 \leq \beta$

the self similarity (3.3) is satisfied.

Proof . Since $a < 0$ then the sub regions (a) and (b) are true. Let us be considered the initial condition (2.3). If $x_\epsilon < 0$ *and* $\epsilon > 0$ is small enough values,then

$$v_{-\epsilon}(x,0) \leq v(x,0) \leq v_\epsilon(x,0) \text{ for } x_\epsilon \leq x < +\infty, \tag{3.6}$$

where $v_{\pm\epsilon}(x,0) = (C \pm \epsilon)(-x)_+^\alpha$ .Since $v_{\pm\epsilon}(x,t)$ are solutions to the CP(2.1)-(2.2) with the initial conditions $v_{\pm\epsilon}(x,0)$. Additionally, due to the ongoing nature of the fixes for the CP(2.1)-(2.2), $\exists\sigma > 0$ depends on ϵ such that

$$v_{-\epsilon}(x,t) \leq v(x,t) \leq v_{+\epsilon}(x,t) \qquad for\ x = x_\epsilon,\ 0 \leq t \leq \sigma. \tag{3.7}$$

From (3.6)-(3.7), and using the comparison rule we get

$$v_{-\epsilon} \leq v \leq v_\epsilon \qquad for\ x_\epsilon \leq x < \infty,\ 0 \leq t \leq \sigma. \tag{3.8}$$

Now we try to rescale the following function

$$v_k^{\mp\epsilon}(x,t) = kv_{\mp\epsilon}(k^{-1/\alpha}x, k^{-(p-\alpha(p-2))/\alpha}t).$$

After the calculation of the CP(2.1)-(2.2) with respect to the rescaling function $v_k^{\mp\epsilon}(x,t)$, solves the problem

$$v_t - (|(v)_x|^{p-2}v_x)_x + ak^{\frac{\alpha(p-1-\beta)-(p-1)}{\alpha}}(v^\beta)_x + bk^{\frac{\alpha(p-1-\beta)-2(p-1)}{\alpha}}(v^\beta) = 0, \tag{3.9}$$

$$v(x,0) = (C \pm \epsilon)(-x)_+^\alpha. \tag{3.10}$$

The CP(3.9)-(3.10) has a unique solution under the restrictions $\alpha(p-1-\beta)-(p-1) < 0$ and $\alpha(p-1-\beta)-2(p-1) < 0$, therefore

$$lim_{k\to+\infty}v_k^{\pm\epsilon}(x,t) = v_{\pm\epsilon} \tag{3.11}$$

exists. Then the lemma 3.2 implies that $v_{\pm\epsilon}$

$$v_{\pm\epsilon}(\xi_\rho(t),t) = \varphi(\rho)t^{\alpha/(p+\alpha(2-p))}, \tag{3.12}$$

for $\rho < \zeta_*$, along $x = \xi_\rho(t)$ and the equation (3.11) implies

$$lim_{k\to+\infty}kv_{\pm\epsilon}(k^{-1/\alpha}x, k^{-(p+\alpha(2-p))/\alpha}) = \varphi(\rho, C \pm \epsilon)t^{\alpha/(p+\alpha(2-p))}, t \geq 0. \tag{3.13}$$

Thus,

$$v_{\pm\epsilon}(\xi_\rho(\Im),\Im) \sim \varphi((\rho, C \pm \epsilon)\Im^{\alpha/(p+\alpha(2-p))}\ as\ \Im \to 0^+ \tag{3.14}$$

where $\Im = k^{-(p+\alpha(2-p))/\alpha}t$.Consequently, from (3.8), (3.13); (3.4) holds. So the discussion of the cases (a) and (b) is done. Now we will consider a little complicated case which is (c). We start the proof by assuming that $v_{\pm\epsilon}$ solve the BVP

$$v_t = (|(v)_x|^{p-2}v_x)_x - a(v^\beta)_x - b(v^\beta), \quad |x| \leq |x_\epsilon|, 0 < t < \sigma \tag{3.15}$$

$$v_0(x) = (C \pm \epsilon)(-x)_+^\alpha, \quad |x| \leq |x_\epsilon| \tag{3.16}$$

$$v(x_\epsilon,t) = (C \pm \epsilon)(-x)_+^\alpha, v(-x_\epsilon,t) = 0,\ 0 \leq t \leq \sigma. \tag{3.17}$$

Now similarly we try to rescale the functions $v_{\pm\epsilon}$ as previous, then we get the rescaling fiction $v_{\pm\epsilon}^k$ satisfies the following new boundary value problem

$$v_t - (|(v)_x|^{p-2}v_x)_x + ak^{\frac{\alpha(p-1-\beta)-(p-1)}{\alpha}}(v^\beta)_x + bk^{\frac{\alpha(p-1-\beta)-2(p-1)}{\alpha}}(v^\beta) = 0, in\ \Omega_\epsilon^k \tag{3.18}$$

$$v(-k^{\frac{1}{\alpha}}x_\epsilon,t) = 0, v(k^{\frac{1}{\alpha}}x_\epsilon,t) = k(C \pm \epsilon)(-x_\epsilon)_+^\alpha,\ 0 \leq k^{\frac{p+\alpha(2-p)}{\alpha}}t \leq \sigma \tag{3.19}$$

$$v_0(x) = (C \pm \epsilon)(-x)_+^{\alpha}, \quad |x|k^{\frac{1}{\alpha}} \leq |x_\epsilon| \tag{3.20}$$

where

$$\Omega_\epsilon^k = \{|x|k^{\frac{1}{\alpha}} \leq |x_\epsilon|,\ 0 \leq k^{\frac{p+\alpha(2-p)}{\alpha}} t \leq \sigma\}.$$

Then the boundary value problems (3.15)-(3.17) and (3.18)-(3.20) have distinct solutions. Considering the finite speed of propagation, it is possible to decide that $\sigma(\epsilon) > 0$, and

$$v(-x_\epsilon, t) = 0,\ 0 \leq t \leq \sigma \tag{3.21}$$

using theorem 2.2 from (3.6)-(3.8); (3.21) follows. Now we try to get $\{v_k^{\pm\epsilon}\}$ is a convergent sequence. Assume that

$$\omega(x,t) = (C+1)(1+x^2)^{\alpha/2} e^t, \quad x \in R, 0 \leq t \leq \sigma$$

is a function. Then we calculate $L_k\omega$

$$L_k\omega \equiv \omega_t - (|\omega_x|^{p-2}\omega_x)_x + ak^{\frac{\alpha(p-1-\beta)-(p-1)}{\alpha}}(\omega^\beta)_x + bk^{\frac{\alpha(p-1-\beta)-2(p-1)}{\alpha}}(\omega^\beta)$$

$$\equiv (C+1)(1+x^2)^{\alpha/2} T\ in\ \Omega_\epsilon^k$$

$$v(x_\epsilon, t) = (C \pm \epsilon)(-x)_+^{\alpha}, v(-x_\epsilon, t) = 0,\ 0 \leq t \leq \sigma$$

$T = 1 - G(x) + R,\ and\ R = R_1 + R_2\ where$

$$G(x) = -\alpha^{(p-1)}(C+1)^{(p-2)} e^{t(p-2)}(p-1)(1+x^2)^{\frac{\alpha(p-2)-2(p-1)}{2}} x^{(p-2)} \times (1 + (\alpha-2)(1+x^2)^{(-1)}x^2).$$

Since G is continuous on a compact region $|x| \leq k^{\frac{1}{\alpha}}|x_\epsilon|$, so it has a local maximum and $R = R_1 + R_2$

$$R_1 = ak^{\frac{\alpha(p-1-\beta)-(p-1)}{\alpha}}(C+1)^{(\beta-1)} e^{t(\beta-1)}\alpha\beta(1+x^2)^{\frac{\alpha(\beta-1)-2}{2}} x = O(k^{\frac{\alpha(p-1-\beta)-(p-1)}{\alpha}})$$

$$R_2 = bk^{\frac{\alpha(p-1-\beta)-2(p-1)}{\alpha}}(C+1)^{\beta-1}(1+x^2)^{\frac{\alpha(\beta-1)}{2}} e^{t(\beta-1)} = O(k^{\frac{\alpha(p-1-\beta)-2(p-1)}{\alpha}})$$

and hence

$$L_k\omega = (C+1)(1+x^2)^{\alpha/2} T \geq \hbar \text{ in } \Omega_{0\epsilon}^k = \Omega_\epsilon^k \cap \{(x,t) : 0 < t \leq \sigma\}$$

such that R_1 and R_2 are uniformly convergent on $\Omega_{0\epsilon}^k$ *as* $k \to \infty$. Thus, for $0 < \epsilon \ll 1$, we get

$$\omega(x,0) \geq v_k^{\pm\epsilon}(x,0)\ on\ |x| \leq k^{\frac{1}{\alpha}}|x_\epsilon|$$

$$\omega(\pm k^{\frac{1}{\alpha}} x_\epsilon, t) \geq v_k^{\pm\epsilon}(\pm k^{\frac{1}{\alpha}} x_\epsilon, t), 0 \leq t \leq \sigma.$$

Since, $\exists k_0 = k_0(\beta, \alpha)$ thus $\forall k \geq k_0$ and theorem 2.2 implies

$$v_k^{\pm\epsilon} \leq \omega\ in\ \bar{\Omega}_{0\epsilon}^k \tag{3.22}$$

Thus, from (3.22) we get that $\{v_k^{\pm\epsilon}\}$ is uniformly bounded in the compact set, so it is uniformly Holder continuous. By the Arzela-Ascoli theorem [2], then there exists a subsequence $v_{k'}^{\pm\epsilon}$ converges such that

$$\lim_{k' \to +\infty} v_{k'}^{\pm\epsilon} = u_{\pm\epsilon},\ on\ \rho \subset \Omega_{0\epsilon}^k \tag{3.23}$$

and $u_{\pm\epsilon}$ solve the CP(2.1)-(2.2) with the initial conditions $(C \pm \epsilon)(-x)_+^{\alpha}$ and the coefficients $a = b = 0$; then, from (3.12)-(3.14) and (3.8); (3.4) holds. □

Proof of Theorem 3.1 By the theorem's assumptions, α and β are restricted. Since lemma 3.2 is the source of the formula (3.2). The interface is bounded from below and satisfies the following condition.

$$\zeta_* \leq \lim_{t \downarrow 0^+} \inf \eta(t) t^{(\alpha(p-2)-p)^{-1}}. \tag{3.24}$$

However, in order to determine the interface upper bound, we assume that v_ϵ solves the CP(2.1), (2.4) and that $\epsilon > 0$ is arbitrary suitably small with $a = 0, b > 0$. Let us use $C + \epsilon$ in place of C, the first inequality of (3.7) and the second one of (3.6), which were previously provided, shall be taken into consideration. Let us now demonstrate that $\bar{v}_\epsilon$ fulfills a supersolution of (2.1) with $a < 0, b > 0$, then

$$L\bar{v}_\epsilon = a(\bar{v}_\epsilon^\beta)_x + b\bar{v}_\epsilon^\beta.$$

To prove $a(\bar{v}_\epsilon^\beta)_x \geq 0, b\bar{v}_\epsilon^\beta \geq 0$, and since $a < 0$ and $b > 0$, so, we have to prove $(\bar{v}_\epsilon^\beta)_x \leq 0$, then to resolve, we'll employ regularization (2.1) with $a = 0, b > 0$, and $v_\epsilon(x,t) = (C \pm \epsilon)(-x)_+^\alpha$. Let us prove that $(\bar{v}_\epsilon)_x \leq 0$. Assume that $F = v_\epsilon$ and define

$$\bar{v}_\epsilon(x,t) = \max\big(0, F(xe^{at}, \frac{1}{ap}(e^{apt} - 1))\big) = \max(0, F(\zeta,\tau)).$$

Let $\zeta = xe^{et}; \tau = \frac{1}{ap}(e^{apt} - 1)$,then it can be written as $\bar{v}_\epsilon(x,t) = \max(0, F(\zeta,\tau))$ and

$$L\bar{v}_\epsilon = e^{apt}LF + aF_\zeta[\zeta + \beta\bar{u}_\epsilon^{\beta-1}e^{at}] = aF_\zeta[\zeta + \beta\bar{v}_\epsilon^{\beta-1}e^{at}] \geq 0$$

$$LF = F_\tau - (|F_\zeta|^{p-2}F_\zeta)_\zeta = 0.$$

Under the conditions $b > 0, a < 0$ and $\zeta + \beta\bar{v}_\epsilon^{\beta-1}e^{at} > 0$.Then from [1], $(v_\epsilon^\beta)_x$ must be nonnegative in the region $\Omega = \{(x.t) : x \leq x_\epsilon,\ 0 < t < \sigma\}$, so $\bar{v}_\epsilon \geq 0\ in\ \Omega$ The right-side inequality of (3.8) is true according to the comparison principle and (3.6), (3.7). Hence, from the other side of the interface as follows

$$\zeta_* \geq \lim_{t\downarrow 0^+} \sup \eta(t)t^{(\alpha(p-2)-p)^{-1}}, 0 \leq t < \sigma \tag{3.25}$$

Therefore, by (3.24)-(3.25); (3.2) is done.

4 Both p-Laplacian and Convection Terms in Balance

The parameters range is restricted in this section to satisfy the existing of the solution to CP(2.1)-(2.2) when both p-Laplacian and convection terms are dominated over the absorption force. We will consider this situation in the following theorem and it can be seen clearly in region (2) of Figure 2. The only situation that we focus on where $C > C_*$, and the asymptotic solution has approximated solution in the previous case (Theorem 3.1, region(1) of Figure 2).

Theorem 4.1. If the initial data v_0 satisfies (2.3) and let $0 < \beta < p/2, \alpha = \frac{p-1}{p-1-\beta}$, and

$$C_* = \left[\alpha^{-1}(-a)^{\frac{1}{(p-1)}}\right]^\alpha$$

then the solution has expanding interface for $C > C_*$ and

$$\eta(t) \sim \zeta_* t^{\theta^{-1}(p-1-\beta)} \text{ as } t \to 0^+.$$

The self-similar solution of the CP(2.1)-(2.2) for $\zeta_* > 0$, there exists a shape function $\varphi > 0$,

$$v(x,t) \sim \varphi(\rho)t^{\theta^{-1}(p-1)} \text{ as } t \to 0^+,$$

where,$\theta = \alpha^{-1}p(p-1) - (p-2))$ and $x = \zeta_\rho(t) = \rho t^{\theta^{-1}(p-1-\beta)}$. If $C > C_*$ then the CP(2.1)-(2.2) has the following solution with the global condition (2.4)

$$v = t^{\theta^{-1}(p-1)}\varphi(\zeta), \zeta = t^{-\theta^{-1}(p-1-\beta)}x, \tag{4.1}$$

$$\eta(t) = \zeta_* t^{\theta^{-1}(p-1-\beta)}, 0 \leq t < +\infty \tag{4.2}$$

and

$$C_2 t^{\theta^{-1}(p-1)}(\zeta_2 - \zeta)_+^{\alpha_0} \leq v \leq C_1 t^{\theta^{-1}(p-1)}(\zeta_1 - \zeta)_+^{\frac{p-1}{p-1-\beta}}, \tag{4.3}$$

which suggests

$$\zeta_2 \leq \zeta_* \leq \zeta_1. \tag{4.4}$$

For the details, the following lemma will be discussed to get the self-similar solution and preliminary results to consider the behaviour of the interface.

Lemma 4.2. The CP(2.1)-(2.2) has a solution $v(x,t)$ and $\alpha=(p-1)/p-1-\beta, 0<\beta<p/2$, such that

$$v(x,t)=\varphi(\zeta)t^{\theta^{-1}(p-1)}, \quad \zeta=xt^{-\theta^{-1}(p-1-\beta)}.$$

If $C>C_*$ then $\varphi(0)=A_1(\beta,p,C,a)$ is a positive value. If v_0 satisfies (2.4) with $a<0$, then the solution at the origin value satisfies $v(0,t)=A_1t^{(\theta^(-1)(p-1))}$.

Proof . A rescaling function is considered as follows

$$v_k^{\mp\epsilon}(x,t)=kv_{\mp\epsilon}(k^{\frac{\beta+1-p}{p-1}}x, k^{\frac{-\theta}{p-1}}t), \quad k>0, \tag{4.5}$$

satisfies (2.1), (2.4). Under the condition of this lemma the CP(2.1), (2.4) has a singular global solution. Consequently, by uniqueness of the solution we have

$$v(x,t)=kv(k^{\frac{\beta+1-p}{p-1}}x, k^{\frac{-\theta}{p-1}}t), \quad k>0.$$

By choosing $k=t^{(\theta^(-1)(p-1))}$ the solution (4.1) is done with the shape function $\varphi(\zeta)=v(\zeta,1)$, which is the only nonnegative solution to the boundary value problem for ODEs

$$\begin{cases} (|(\varphi)'|^{p-2}(\varphi)')'+(p-\alpha(p-2))^{-1}(\zeta\varphi'(\zeta)-\alpha\varphi)-a(\varphi^\beta)'+b(\varphi^\beta)=0, \\ \varphi(-\infty)\sim C(-\zeta)^\alpha, \varphi(\zeta_*)=0, \quad \varphi(+\infty)\equiv 0, \ \zeta\geq\zeta_*, \end{cases} \tag{4.6}$$

where $\zeta_*>0$ such that φ satisfies (4.6). Then, (4.1) holds. When v_0 satisfies (2.3), then similarly to the proof of lemma 3.2, (3.6) and (3.7) are fulfilled from (2.3). As described previous technique, by (3.6); we get (3.8). Then by taking

$$v_k^{\mp\epsilon}(x,t)=kv_{\mp\epsilon}(k^{(\beta+1-p)(p-1)^{-1}}x, k^{-\theta(p-1)^{-1}}t),$$

thus $v_k^{\mp\epsilon}(x,t)$ solves the CP

$$v_t-(|(v)_x|^{p-2}v_x)_x+a(v^\beta)_x+b(v^\beta)\cong 0, \quad x\in R, t>0, \tag{4.7}$$

$$v(x,0)=(C\pm\epsilon)(-x)_+^{\frac{\beta+1-p}{p-1}}, \quad x\in R. \tag{4.8}$$

Since the existence and uniqueness of a solution to the CP(4.7),(4.8) holds, comparison principle implies

$$\lim_{k\to+\infty} v_k^{\pm\epsilon}(x,t)=u_{\pm\epsilon}; \quad k>0, \tag{4.9}$$

where $v_(\pm\epsilon)$ solves the CP(2.1)-(2.4), $a=0$; and $v_0=(C\pm\epsilon)(-x)_+^{\frac{p-1}{p-1-\beta}}$. Thus, $v_(\pm\epsilon)$ satisfies (4.1). For $x=\zeta_\rho(t)$, and $\rho<\zeta_*$, then

$$\lim_{k\to+\infty} kv_{\pm\epsilon}(k^{(\beta+1-p)(p-1)^{-1}}\zeta_\rho(t), k^{-\theta(p-1)^{-1}}t)=\varphi(\rho, C\pm\epsilon)t^{\theta^{-1}(p-1)}. \tag{4.10}$$

Assume that $\Im=k^{-\theta(p-1)^{-1}}t$, the limit form (4.9) becomes

$$v_{\pm\epsilon}(\zeta_\rho(\tau),\tau)\sim\varphi(\rho,C)\tau^{\theta^{-1}(p-1)}. \tag{4.11}$$

The inequality (4.11) and the comparison principle (3.8), for $\epsilon>0$, then (4.5) satisfies. □

Proof of Theorem 4.1 The weak solution of the CP(2.1),(2.4) is existed and unique. In order to focus on the situation $C>C_*$, to consider the following function

$$\omega(x,t)\sim t^{(p-1)\theta^{-1}}\varphi_1(\zeta), \zeta=xt^{(p-1-\beta)\theta^{-1}},$$

then,

$$L\omega=t^{(p(\beta-1)+1)\theta^{-1}}L^0\varphi_1, \tag{4.12}$$

$$L^0\varphi_1=\theta^{-1}(p-1)\varphi_1(\zeta)-\theta^{-1}(p-1-\beta)\zeta\varphi_1'(\zeta)-(|(\varphi_1)'|^{p-2}(\varphi_1)')'+a(\varphi_1^\beta)'+b(\varphi_1^\beta) \tag{4.13}$$

Let us choose φ_1 such that $\varphi_1(\zeta) = C_0(\zeta_0-\zeta)_+^{\alpha_0}, 0<\zeta<+\infty$, where the constants C_0, ζ_0, α_0 where the constants $\alpha_0 = \frac{p-1}{p-1-\beta}$, then(4.6) implies

$$L^0\varphi_1 = -\alpha\beta\theta^{-1}(p-1)C_0^{\beta}(\zeta_0-\zeta)_+^{\frac{p(\beta-1)+1}{p-1-\beta}}\{1-(\frac{C_0}{C_*})^{p-1-\beta}+\frac{p-1-\beta}{(-a\beta)\theta}\zeta_0(\zeta_0-\zeta)_+^{\frac{(p-1)(1-\beta)}{p-1-\beta}}$$
$$+\frac{b}{(-a\beta)}(\zeta_0-\zeta)^{\frac{-1}{p-1-\beta}}t^{\frac{p(\beta-1)}{p-1-\beta}}(\frac{-1}{p-1-\beta})\}.$$

To prove the upper estimation, we choose $C_0 = C_1, \zeta_0 = \zeta_1$.if $\beta > 1$ then

$$L^0\varphi_1 \geq -\alpha\beta\theta^{-1}(p-1)C_1^{\beta}(\zeta_1-\zeta)_+^{\frac{p(\beta-1)+1}{p-1-\beta}}\{1-(\frac{C_1}{C_*})^{p-1-\beta}+\frac{p-1-\beta}{(-a\beta)\theta}\zeta_1^{\frac{2(p-1)\beta p}{p-1-\beta}}$$
$$+\frac{b}{(-a\beta)}(\zeta_1-\zeta)^{\frac{-1}{p-1-\beta}}t^{\frac{p(\beta-1)}{p-1-\beta}}(\frac{-1}{p-1-\beta})\} = 0$$

where $C_1 > C_*$, and

$$C_1 = C_*(\frac{p-1-\beta}{(-a\beta)\vartheta})^{\frac{1}{p-1-\beta}}\zeta_1^{\frac{2(p-1)-\beta p}{p-1-\beta}}.$$

While if $\beta < 1$, and for $0 \leq \zeta \leq \zeta_1$, then

$$L^0\varphi_1 \geq -a\beta(p-1)C_1^{\beta}(\zeta_1-\zeta)_+^{\frac{p(\beta-1)+1)}{p-1-\beta}}\{1-(\frac{C_1}{C_*})^{p-1-\beta}\} = 0$$

where $C_1 = C_*$. from (4.12) it follows that

$$L\omega = 0 \ for \ x > \zeta_1 t^{\theta^{-1}(p-1-\beta)}, \quad L\omega \geq 0 \ \text{ for } \ x < \zeta_1 t^{\theta^{-1}p-1-\beta}.$$

Then upper bound of the solution ω is defined by (2.1) in $U = \{(x,t) : x > 0, t > 0\}$, from theorem 2.2. Also, since

$$\omega(0,t) = v(0,t) \ for \ 0 \leq t < +\infty \tag{4.14}$$

$$\omega(x,0) = v(x,0) = 0 \ for \ 0 \leq x < +\infty \tag{4.15}$$

the upper bound of (4.4) is proved. Let $\alpha_0 = (p-1)/(p-1-\beta)$ Then, by selecting $C_1 = C_2$,and $\zeta_1 = \zeta_2$, we shall demonstrate the lower estimation. If β is 1, we can get from (3.14) for $0 < \zeta_1 < \zeta_2$ using the same technique. Additionally, it follows from (4.12) that

$$L\omega = 0 \ for \ x > \zeta_2 t^{\theta^{-1}(p-1-\beta)}, \tag{4.16}$$

$$L\omega \leq 0 \ for \ 0 < x < \zeta_2 t^{\theta^{-1}(p-1-\beta)}. \tag{4.17}$$

Let $\alpha_0 = (p-1)/(p-2)$ and then by choosing $\beta > 1, \ for \ 0 \leq \zeta \leq \zeta_2$ we estimate the lower bound from the following calculation

$$L^0\varphi_1 \leq \theta^{-1}(p-1)C_2(\zeta_2-\zeta)_+^{\frac{1}{p-2}}\{\zeta_2 - C_2^{p-2}\frac{\theta(p-1)^{p-1}}{(p-2)^p}+(-a)\beta\theta C_2^{\beta-1}(p-2)^{p-2}(p-1)\zeta_2^{\frac{(\beta-1)(p-1)}{p-2-\beta}}\}$$

where $C_2 > C_*$, and

$$C_2 = \zeta_2^{\frac{(1-\beta)(p-1)}{p-1-\beta}}\{(C_2^{p-2}\theta(p-2)^p(p-1)^{p-1}-\zeta_2)+((-a)\beta(\theta)(p-2)^{p-2}(p-1))^{(1-\beta)}\}^{1/(\beta-1)}$$

which suggests again (4.16)-(4.17). The left-hand side is again deduced from (4.14)-(4.17), and the comparison theorem (Theorem 2.2).

5 Stationary Solution and Waiting Time Interface

This part has slowly dominated of p-Laplacian type diffusion force. The solution of the CP(2.1)-(2.2) will be stationary near a waiting time interface and is clearly seen in region(3), Figure 2.

Theorem 5.1. The solution of CP2.1,(2.4) is stationary with waiting time interface under the conditions $p \leq 2\beta < 2(p-1)$ and $2(p-2)^{-1}(p-1) \leq \alpha < (p-1-\beta)^{-1}(p-1)$.

Proof of Theorem 5.1 Let us consider the super-solution $\omega_{+\epsilon}(x,t) = (C+\epsilon)(-x)_+^{\alpha}$ of $v(x,t)$. Also, let us consider the sub-solution

$$\omega_{-\epsilon}(x,t) = (C-\epsilon)((\Im - t)\Im^{-1})^{\beta}(-x)_+^{(p-1)(p-1-\beta)^{-1}},\ 0 < t < \tau.$$

with non-negative constant β. If $\alpha < (p-1-\beta)^{-1}(p-1)$, we get

$$v(x,t) \geq (C-\epsilon)(-x)_+^{p-1/p-1-\beta}.$$

Since there exist negative values x_ϵ, such that $x_\epsilon < x$, then the continuity of solutions implies $\exists \sigma_\epsilon > 0$ such that

$$v(x_\epsilon,t) \geq (C-\epsilon)((\Im - t)\Im^{-1})^{\beta}(-x_\epsilon)_+^{(p-1-\beta)^{-1}(p-1)},\ 0 < t < \sigma_\epsilon.$$

Then, by substituting it, and by taking $\alpha = (p-1-\beta)^{-1}(p-1)$, we can clearly calculate $L\omega_{-\epsilon}$ to get $L\omega_{-\epsilon} \leq 0$, and combining our results and applying the comparison theorem 2.2, we get $\omega_{-\epsilon} \leq v \leq \omega_\epsilon$ is fulfilled.

6 Conclusion

In this work, we design a self-similar solution to a parabolic p-Laplacian type diffusion-convection equation with the influence of an absorption source, and examine the early growth of interfaces and asymptotic locally weak solutions. We discussed the local weak solution on three significant regions. The dominated The p-Laplacian type diffusion term on these regions is clearly considered. The important techniques include the use of rescaling, comparison theorems, and blowing up techniques in non-smooth domains. This work's conclusion is that the model of this problem can be used in a variety of fields, including chemical process design, plasma physics, and so on.

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