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Jensen's inequality for (p-q)-convex functions and related results

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Abstract

In this paper, we establish Jensen's inequality for (p-q)-convex functions. By using Jensen's inequality we obtain some Hermite-Hadamard type inequality and several sharp inequalities. Some examples are given.

Keywords: Jensen's inequality, Integral inequality, MN-convex, H-H inequality
2020 MSC: 26D15, 26D07, 26A51, 26E60

1 Introduction

Let μ be a positive measure on X such that $\mu(X) = 1$. If f is a real-valued function in $L^1(\mu), a < f(x) < b$ for all $x \in X$ and φ is convex on (a, b), then

$$\varphi(\int_X f d\mu) \leq \int_X (\varphi \circ f) d\mu. \tag{1.1}$$

The inequality (1.1) is known as Jensen's inequality. Let $f : [a, b] \to \mathbb{R}$ be a convex function, then the inequality

$$\varphi(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b \varphi(x) dx \leq \frac{\varphi(a) + \varphi(b)}{2} \tag{1.2}$$

which is known in the literature as Hermite-Hadamard inequality (H-H inequality). Jensen's inequality is one of the important inequalities which has been possible with the help of convexity. This inequality has preserved some important structure and also there are lot of inequalities which are the direct consequences of Jensen's inequality, such as Holder, Hermite-Hadamard and Young's inequalities etc. Recently, many of mathematicians investigated generalization of convex functions, such as MN-convex (M, N=A, G, H), p-convex, r-convex, and obtained the H-H inequality for these convexities [1, 3, 6, 9, 12, 8, 10, 11, 14, 15, 16, 20, 22, 34]. In [43, 41] the author obtained Jensen's inequality for GG-convex and HH-convex functions. In this paper, via the defining of (p-q)-convex functions, we obtain the Jensen's and H-H inequalities for (p-q)-convex function and we show that a lot of convexity, such as MN-convexity (M,N=A,G,H), p-convexity and r-convexity are special cases of (p-q)-convexity.

For the statement of the main results we introduce some notations and terminologies.

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Definition 1.1. A function $M:(0,\infty)\times(0,\infty)\to(0,\infty)$ is called a mean if

1. $M(x,y)=M(y,x)$,
2. $M(x,x)=x$,
3. $x<M(x,y)<y, wherever\ x<y$,
4. $M(\alpha x,\alpha y)=\alpha M(x,y),\ for\ all\ \alpha>0$.

Example 1.2.

1. The Arithmetic Mean $A(x,y)=\dfrac{x+y}{2}$,
2. The Geometric Mean $G(x,y)=\sqrt{xy}$,
3. The Harmonic Mean $H(x,y)=\dfrac{1}{A(\frac{1}{x},\frac{1}{y})}$,
4. The Power Mean $M_p(x,y)=\begin{cases}(\dfrac{x^p+y^p}{2})^{\frac{1}{p}} & p\neq 0\\ \sqrt{xy} & p=0\end{cases}\quad (p\in\mathbb{R})$.

Definition 1.3. Let M,N be two means on real interval $I\subset(0,\infty)$ and $\varphi:I\to(0,\infty)$ be a continuous function. φ is called MN-convex if for all $x,y\in I$,

$$\varphi(M(x,y))\leq N(f(x),f(y)).$$

Note that this definition reduces to usual convexity when $M(x,y)=N(x,y)=\dfrac{x+y}{2}$. Since φ is continuous, this the MN-convexity of φ that is

$$\varphi\left(M(x,y);1-t,t\right)\leq N\left(\varphi(x),\varphi(y);1-t,t\right)$$

for all $x,y\in I,t\in[0,1]$ (see [26]). For example

1. φ is AG-convex, if for every $x,y\in I$ and $t\in[0,1]$,
$$\varphi(tx+(1-t)y)\leq\varphi^t(x)\varphi^{1-t}(y)$$
2. φ is GH-convex, if for every $x,y\in I$ and $t\in[0,1]$,
$$\varphi(x^ty^{1-t})\leq\frac{\varphi(x)\varphi(y)}{(1-t)\varphi(x)+t\varphi(y)}$$

Definition 1.4. Let $I\subset(0,\infty)$ be interval and $p,q\in\mathbb{R}$. A continuous function $\varphi:I\to(0,\infty)$ is said to be M_pM_q-convex or $(p-q)$-convex if the following inequality holds

$$\begin{aligned}\varphi\left(tx^p+(1-t)y^p\right)^{\frac{1}{p}}&\leq\left(t\varphi^q(x)+(1-t)\varphi^q(y)\right)^{\frac{1}{q}} && (p\neq 0,q\neq 0)\\ \varphi(x^ty^{1-t})&\leq\left(t\varphi^q(x)+(1-t)\varphi^q(y)\right)^{\frac{1}{q}} && (p=0,q\neq 0)\\ \varphi\left(tx^p+(1-t)y^p\right)^{\frac{1}{p}}&\leq\varphi^t(x)\varphi^{1-t}(y) && (p\neq 0,q=0)\\ \varphi(x^ty^{1-t})&\leq\varphi^t(x)\varphi^{1-t}(y) && (p=0,q=0)\end{aligned}$$

wherever $x,y\in I$ and $t\in[0,1]$. It can be easily seen that for $p=q=1$, $p=q=0$, $p=q=-1$ and $p=1,q=r$, $(p-q)$-convexity reduced respectively to classical convexity, GG-convexity, HH-convexity and r-convexity. In [45] the authers proved that for $0\leq t\leq 1, 0\leq p\leq q$ and $x,y\geq 0$ the following inequalities hold

$$x^ty^{1-t}\leq\left(tx^p+(1-t)y^p\right)^{\frac{1}{p}}\leq\left(tx^q+(1-t)y^q\right)^{\frac{1}{q}} \tag{1.3}$$

By easy calculations we see that for $p\leq q\leq 0$, the inequalities

$$\left(tx^p+(1-t)y^p\right)^{\frac{1}{p}}\leq\left(tx^q+(1-t)y^q\right)^{\frac{1}{q}}\leq x^ty^{1-t} \tag{1.4}$$

hold. Inequalities (1.3) and (1.4) show that $\varphi(x)=x$ is $(p-q)$-convex if $p<q$, $(0-q)$-convex if $q>0$ and $(p-0)$-convex if $p<0$.

2 Main results

Theorem 2.1. Let φ be a $(p-q)$-convex function on (a,b) and let $0<a<s<t<u<b$.

1. If ($p>0$ and $q>0$) or ($p<0$ and $q<0$), then
$$\frac{\varphi^q(t)-\varphi^q(s)}{t^p-s^p} \le \frac{\varphi^q(u)-\varphi^q(t)}{u^p-t^p} \tag{2.1}$$
2. If ($p>0$ and $q<0$) or ($p<0$ and $q>0$), then
$$\frac{\varphi^q(t)-\varphi^q(s)}{t^p-s^p} \ge \frac{\varphi^q(u)-\varphi^q(t)}{u^p-t^p} \tag{2.2}$$
3. If $p=0$ and $q>0$, then
$$\frac{\varphi^q(t)-\varphi^q(s)}{\ln t-\ln s} \le \frac{\varphi^q(u)-\varphi^q(t)}{\ln u-\ln t} \tag{2.3}$$
4. If $p=0$ and $q<0$, then
$$\frac{\varphi^q(t)-\varphi^q(s)}{\ln t-\ln s} \ge \frac{\varphi^q(u)-\varphi^q(t)}{\ln u-\ln t} \tag{2.4}$$
5. If $p>0$ and $q=0$, then
$$\frac{\ln\varphi(t)-\ln\varphi(s)}{t^p-s^p} \le \frac{\ln\varphi(u)-\ln\varphi(t)}{u^p-t^p} \tag{2.5}$$
6. If $p<0$ and $q=0$, then
$$\frac{\ln\varphi(t)-\ln\varphi(s)}{t^p-s^p} \ge \frac{\ln\varphi(u)-\ln\varphi(t)}{u^p-t^p} \tag{2.6}$$
7. If $p=q=0$, then
$$\frac{\ln\varphi(t)-\ln\varphi(s)}{\ln t-\ln s} \le \frac{\ln\varphi(u)-\ln\varphi(t)}{\ln u-\ln t} \tag{2.7}$$

Proof .

1. Let $p>0, q>0$, and $r=\dfrac{u^p-t^p}{u^p-s^p}$, then $0<r<1$ and $t=(rs^p+(1-r)u^p)^{\frac{1}{p}}$. By the $(p-q)$-convexity of φ we have
$$\begin{aligned}
&\varphi(t)\le (r\varphi^q(s)+(1-r)\varphi^q(u))^{\frac{1}{q}}\\
\Rightarrow\ &\varphi^q(t)\le r\varphi^q(s)+(1-r)\varphi^q(u) \quad (q>0)\\
\Rightarrow\ &r\varphi^q(t)+(1-r)\varphi^q(t)\le r\varphi^q(s)+(1-r)\varphi^q(u)\\
\Rightarrow\ &r(\varphi^q(t)-\varphi^q(s))\le (1-r)(\varphi^q(u)-\varphi^q(t))\\
\Rightarrow\ &\frac{u^p-t^p}{u^p-s^p}(\varphi^q(t)-\varphi^q(s))\le \frac{t^p-s^p}{u^p-s^p}(\varphi^q(u)-\varphi^q(t))\\
\Rightarrow\ &\frac{\varphi^q(t)-\varphi^q(s)}{t^p-s^p}\le\frac{\varphi^q(u)-\varphi^q(t)}{u^p-t^p}.
\end{aligned}$$
Now let $p<0$ and $q<0$. Then $s^p>t^p>u^p$, $0<r=\dfrac{u^p-t^p}{u^p-s^p}<1$ and $t=(rs^p+(1-r)u^p)^{\frac{1}{p}}$. Since φ is $(p-q)$-convex,
$$\begin{aligned}
&\varphi(t)\le (r\varphi^q(s)+(1-r)\varphi^q(u))^{\frac{1}{q}}\\
\Rightarrow\ &\varphi^q(t)\ge (r\varphi^q(s)+(1-r)\varphi^q(u)) \quad (q<0)\\
\Rightarrow\ &r\varphi^q(t)+(1-r)\varphi^q(t)\ge r\varphi^q(s)+(1-r)\varphi^q(u)\\
\Rightarrow\ &r(\varphi^q(t)-\varphi^q(s))\ge (1-r)(\varphi^q(u)-\varphi^q(t))\\
\Rightarrow\ &\frac{u^p-t^p}{u^p-s^p}(\varphi^q(t)-\varphi^q(s))\ge \frac{t^p-s^p}{u^p-s^p}(\varphi^q(u)-\varphi^q(t))
\end{aligned}$$

Since $\dfrac{u^p - s^p}{(u^p - t^p)(t^p - s^p)} < 0$, we have

$$\frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p} \leq \frac{\varphi^q(u) - \varphi^q(t)}{u^p - t^p}.$$

2. Follows by similar way and the details are omitted.
3. Let $p = 0, q > 0$ and $0 < r = \dfrac{\ln u - \ln t}{\ln u - \ln s} < 1$. Then $t = u^{1-r}s^r$. By the $(p-q)$-convexity of φ we have

$$\varphi(t) \leq (r\varphi^q(s) + (1-r)\varphi^q(u))^{\frac{1}{q}}.$$

Since $q > 0$,

$$\begin{aligned} & \varphi^q(t) \leq r\varphi^q(s) + (1-r)\varphi^q(u) \\ \Rightarrow\ & r\varphi^q(t) + (1-r)\varphi^q(t) \leq r\varphi^q(s) + (1-r)\varphi^q(u) \\ \Rightarrow\ & r(\varphi^q(t) - \varphi^q(s)) \leq (1-r)(\varphi^q(u) - \varphi^q(t)) \\ \Rightarrow\ & \frac{\ln u - \ln t}{\ln u - \ln s}(\varphi^q(t) - \varphi^q(s)) \leq \frac{\ln t - \ln s}{\ln u - \ln s}(\varphi^q(u) - \varphi^q(t)) \\ \Rightarrow\ & \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \leq \frac{\varphi^q(u) - \varphi^q(t)}{\ln u - \ln t} \end{aligned}$$

4. If $p = 0, q < 0$, by similar way we obtain

$$\begin{aligned} & \varphi^q(t) \geq r\varphi^q(s) + (1-r)\varphi^q(u) \\ \Rightarrow\ & \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \geq \frac{\varphi^q(u) - \varphi^q(t)}{\ln u - \ln t} \end{aligned}$$

5. Let $p > 0, q = 0$ and $0 < r = \dfrac{u^p - t^p}{u^p - s^p} < 1$. Then $t = ((1-r)u^p + rs^p)^{\frac{1}{p}}$. By the $(p-0)$-convexity of φ we have

$$\begin{aligned} & \varphi(t) \leq \varphi^r(s)\varphi^{1-r}(u) \\ \Rightarrow\ & \ln\varphi(t) \leq r\ln\varphi(s) + (1-r)\ln\varphi(u) \\ \Rightarrow\ & r\ln\varphi(t) + (1-r)\ln\varphi(t) \leq r\ln\varphi(s) + (1-r)\ln\varphi(u) \\ \Rightarrow\ & r(\ln\varphi(t) - \ln\varphi(s)) \leq (1-r)(\ln\varphi(u) - \ln\varphi(t)) \\ \Rightarrow\ & \frac{u^p - t^p}{u^p - s^p}(\ln\varphi(t) - \ln\varphi(s)) \leq \frac{t^p - s^p}{u^p - s^p}(\ln\varphi(u) - \ln\varphi(t)) \\ \Rightarrow\ & \frac{\ln\varphi(t) - \ln\varphi(s)}{t^p - s^p} \leq \frac{\ln\varphi(u) - \ln\varphi(t)}{u^p - t^p} \end{aligned}$$

6. Let $p < 0$ and $q = 0$. Then $s^p > t^p > u^p$ and $0 < r = \dfrac{u^p - t^p}{u^p - s^p} < 1$. So

$$\begin{aligned} & t = (rs^p + (1-r)u^p)^{\frac{1}{p}} \\ \Rightarrow\ & \varphi(t) \leq \varphi^r(s)\varphi^{1-r}(u) \\ \Rightarrow\ & \ln\varphi(t) \leq r\ln\varphi(s) + (1-r)\ln\varphi(u) \\ \Rightarrow\ & \frac{u^p - t^p}{u^p - s^p}(\ln\varphi(t) - \ln\varphi(s)) \leq \frac{t^p - s^p}{u^p - s^p}(\ln\varphi(u) - \ln\varphi(t)) \\ \Rightarrow\ & \frac{\ln\varphi(t) - \ln\varphi(s)}{t^p - s^p} \geq \frac{\ln\varphi(u) - \ln\varphi(t)}{u^p - t^p} \end{aligned}$$

7. If $p = q = 0$, then φ is GG-convex. For details see [43] .

□

In the following theorem we obtain the Jensen's inequality for $(p-q)$-convex functions.

Theorem 2.2. Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in a set X, so that $\mu(X) = 1$. If f is a real function in $L^1(\mu), 0 < a < f(x) < b$ for all $x \in X$, and if φ is a $(p-q)$-convex function on (a, b), then

1. If $p \neq 0$ and $q \neq 0$, then
$$\varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}}.$$
2. If $p = 0$ and $q \neq 0$, then
$$\varphi(e^{\int_X \ln f d\mu}) \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}}.$$
3. If $p \neq 0$ and $q = 0$, then
$$\varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq e^{\int_X \ln(\varphi \circ f) d\mu}.$$
4. If $p = q = 0$, then
$$\varphi(e^{\int_X \ln f d\mu}) \leq e^{\int_X \ln(\varphi \circ f) d\mu}.$$

Proof .

1. Put $t = (\int_X f^p d\mu)^{\frac{1}{p}}$, then $a < t < b$. We distinguish two cases.
Case 1. ($p > 0$ and $q > 0$) or ($p < 0$ and $q < 0$). Since φ is $(p-q)$-convex, the inequality (2.1) holds. Let
$$M = \sup_{a<s<t} \frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p}.$$
Then M is no larger than any of the quotients on the right side of (2.1), for any $u \in (t, b)$. It follows that
$$\frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p} \leq M.$$
Now let $p > 0$ and $q > 0$, then
$$\begin{aligned} & \varphi^q(t) - \varphi^q(s) \leq M(t^p - s^p) \\ \Rightarrow \quad & \varphi^q(s) \geq \varphi^q(t) + M(s^p - t^p). \end{aligned}$$
Hence for any $x \in X$, we have
$$\varphi^q(f(x)) \geq \varphi^q(t) + M(f^p(x) - t^p).$$
Since φ is continuous, $\varphi \circ f$ and $(\varphi \circ f)^q$ are measarable. By integrating both sides with respect to measure μ, we get
$$\int_X (g \circ f)^q d\mu \geq \varphi^q(t) + \mu(\int_X f^p d\mu - t^p).$$
Now set $t = (\int_X f^p d\mu)^{\frac{1}{p}}$. It follows that
$$\begin{aligned} & \int_X (g \circ f)^q d\mu \geq \varphi^q(\int_X f^p d\mu)^{\frac{1}{p}} \\ \Rightarrow \quad & \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}}. \end{aligned}$$
If $p < 0$ and $q < 0$, then
$$\varphi^q(t) - \varphi^q(s) \geq \mu(t^p - s^p).$$
By the similar way we obtain
$$\begin{aligned} & \int_X (\varphi \circ f)^q d\mu \leq \varphi^q(\int_X f^p d\mu)^{\frac{1}{p}} \\ \Rightarrow \quad & \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}} \end{aligned}$$
Case 2. ($p > 0$ and $q < 0$) or ($p < 0$ and $q > 0$). Since φ is $(p-q)$-convex, the inequality (2.2) holds. Let
$$m = \inf_{a<s<t} \frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p}.$$

Then m is no smaller than any of the quotients on the right side of (2.2), for any $u \in (t, b)$. It follows that

$$\frac{\varphi^q(t) - \varphi^q(s)}{t^p - s^p} \geq m.$$

Now let $p > 0$ and $q < 0$, then

$$\begin{aligned}
& \varphi^q(t) - \varphi^q(s) \geq m(t^p - s^p) \\
\Rightarrow \quad & \varphi^q(s) \leq \varphi^q(t) + m(s^p - t^p) \\
\Rightarrow \quad & \varphi^q(f(x)) \leq \varphi^q(t) + m(f^p(x) - t^p) \\
\Rightarrow \quad & \int_X (\varphi \circ f)^q d\mu \leq \varphi^q(t) + m(\int_X f^p d\mu - t^p) \quad (\mu(X) = 1).
\end{aligned}$$

Now set $t = (\int_X f^p d\mu)^{\frac{1}{p}}$. It follows that

$$\begin{aligned}
& \int_X (g \circ f)^q d\mu \leq \varphi^q(\int_X f^p d\mu)^{\frac{1}{p}} \\
\Rightarrow \quad & \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq (\int_X (g \circ f)^q d\mu)^{\frac{1}{q}} \quad (q < 0).
\end{aligned}$$

If $p < 0$ and $q > 0$, then

$$\varphi^q(t) - \varphi^q(s) \leq m(t^p - s^p)$$

By the similar way we obtain

$$\begin{aligned}
& \int_X (\varphi \circ f)^q d\mu \geq \varphi^q(\int_X f^p d\mu)^{\frac{1}{p}} \\
\Rightarrow \quad & \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}} \quad (q > 0)
\end{aligned}$$

2. Put $t = e^{\int_X \ln f d\mu}$, then $a < t < b$. Let $q > 0$ since φ is $(p-q)$-convex, the inequality (2.3) holds. Let

$$M = \sup_{a<s<t} \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s}$$

Then

$$\begin{aligned}
& \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \leq M \\
\Rightarrow \quad & \varphi^q(t) - \varphi^q(s) \leq M(\ln t - \ln s) \\
\Rightarrow \quad & \varphi^q(s) \geq \varphi^q(t) - M(\ln t - \ln s)
\end{aligned}$$

So for any $x \in X$, we have

$$\varphi^q(f(x)) \geq \varphi^q(t) - M(\ln t - \ln s)$$

By integrating both sides with respect to measure μ, we obtain

$$\int_X (\varphi \circ f)^q d\mu \geq \varphi^q(t) - M(\ln t - \int_X \ln f d\mu).$$

Now set $t = e^{\int_X \ln f d\mu}$. Thus

$$\begin{aligned}
& \int_X (\varphi \circ f)^q d\mu \geq \varphi^q(e^{\int_X \ln f d\mu}) \\
\Rightarrow \quad & \varphi(e^{\int_X \ln f d\mu}) \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}} \quad (q > 0).
\end{aligned}$$

If $p = 0$ and $q < 0$, by the similar way we get

$$\frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s} \geq m$$

where

$$m = \inf_{a<s<t} \frac{\varphi^q(t) - \varphi^q(s)}{\ln t - \ln s}.$$

Hence

$$\begin{aligned} & \varphi^q(t) - \varphi^q(s) \geq m(\ln t - \ln s) \\ \Rightarrow\ & \varphi^q(s) \leq \varphi^q(t) - m(\ln t - \ln s) \\ \Rightarrow\ & \int_X (\varphi \circ f)^q d\mu \leq \varphi^q(t) - m(\ln t - \int_X \ln f d\mu). \end{aligned}$$

Now set $t = e^{\int_X \ln f d\mu}$. Thus

$$\begin{aligned} & \int_X (\varphi \circ f)^q d\mu \leq \varphi^q(e^{\int_X \ln f d\mu}) \\ \Rightarrow\ & \varphi(e^{\int_X \ln f d\mu}) \leq (\int_X (\varphi \circ f)^q d\mu)^{\frac{1}{q}} \quad (q < 0) \end{aligned}$$

3. Put $t = (\int_X f^p d\mu)^{\frac{1}{p}}$, then $a < t < b$. Let $p > 0$. By the $(p-q)$-convexity of φ the inequality (2.5) holds. Let

$$\begin{aligned} & M = \sup_{a<s<t} \frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p} \\ \Rightarrow\ & \frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p} \leq M \\ \Rightarrow\ & \ln \varphi(t) - \ln \varphi(s) \leq M(t^p - s^p) \\ \Rightarrow\ & \ln \varphi(s) \geq \ln \varphi(t) - M(t^p - s^p) \\ \Rightarrow\ & \ln \varphi(f(x)) \geq \ln \varphi(t) - M(t^p - f^p(x)) \quad (\forall x \in X) \\ \Rightarrow\ & \int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi(t) - M(t^p - \int_X f^p d\mu) \end{aligned}$$

Since $t = (\int_X f^p d\mu)^{\frac{1}{p}}$, we obtain

$$\begin{aligned} & \int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \\ \Rightarrow\ & \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq e^{\int_X \ln(\varphi \circ f) d\mu} \end{aligned}$$

If $p < 0$ and $q = 0$, put $t = (\int_X f^p d\mu)^{\frac{1}{p}}$. By the similar way we obtain

$$\begin{aligned} & \frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p} \geq m \quad (m = \inf_{a<s<t} \frac{\ln \varphi(t) - \ln \varphi(s)}{t^p - s^p}) \\ \Rightarrow\ & \ln \varphi(t) - \ln \varphi(s) \leq m(t^p - s^p) \\ \Rightarrow\ & \ln \varphi(s) \geq \ln \varphi(t) - m(t^p - s^p) \\ \Rightarrow\ & \ln \varphi(f(x)) \geq \ln \varphi(t) - m(t^p - f^p(x)) \\ \Rightarrow\ & \int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi(t) - m(t^p - \int_X f^p d\mu) \\ \Rightarrow\ & \int_X \ln(\varphi \circ f) d\mu \geq \ln \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \\ \Rightarrow\ & \varphi(\int_X f^p d\mu)^{\frac{1}{p}} \leq e^{\int_X \ln(\varphi \circ f) d\mu} \end{aligned}$$

4. For the proof of (4) see [43].

□

Corollary 2.3. Let $X = \{x_1, x_2, \cdots, x_n\}$, $\mu(\{x_i\}) = \dfrac{1}{n}$ and $f(x_i) = a_i > 0$. If φ is a $(p-q)$-convex function on J which includes the image of f, then

1. If $p \neq 0$ and $q \neq 0$, then
$$\varphi\left(\frac{1}{n}\sum_{i=0}^{n} a_i^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{n}\sum_{i=0}^{n} \varphi^q(a_i)\right)^{\frac{1}{q}}$$
2. If $p = 0$ and $q \neq 0$, then
$$\varphi\left(\sqrt[n]{a_1 a_2 \cdots a_n}\right) \leq \left(\frac{1}{n}\sum_{i=0}^{n} \varphi^q(a_i)\right)^{\frac{1}{q}}$$
3. If $p \neq 0$ and $q = 0$, then
$$\varphi\left(\frac{1}{n}\sum_{i=0}^{n} a_i^p\right)^{\frac{1}{p}} \leq \sqrt[n]{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)}$$
4. If $p = q = 0$, then
$$\varphi\left(\sqrt[n]{a_1 a_2 \cdots a_n}\right) \leq \sqrt[n]{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)}$$

Proof is obvious by theorem 2.2 .

Corollary 2.4. Let $f : [a, b] \to (0, \infty)$ $(b > a > 0)$ be a continuous function and $\varphi : J \to (0, \infty)$ be a $(p - q)$-convex function on an interval J which includes the image of f. Then

1. If $p \neq 0$ and $q \neq 0$, then
$$\varphi\left(\frac{p}{b^p - a^p}\int_a^b \frac{f^p(x)}{x^{1-p}}dx\right)^{\frac{1}{p}} \leq \left(\frac{p}{b^p - a^p}\int_a^b \frac{\varphi^q(f(x))}{x^{1-p}}dx\right)^{\frac{1}{q}}$$
2. If $p = 0$ and $q \neq 0$, then
$$\varphi\left(e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(x)}{x}dx}\right) \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\varphi^q(x)}{x}dx\right)^{\frac{1}{q}}$$
3. If $p \neq 0$ and $q = 0$, then
$$\varphi\left(\frac{p}{b^p - a^p}\int_a^b \frac{f^p(x)}{x^{1-p}}dx\right)^{\frac{1}{p}} \leq e^{\frac{p}{b^p - a^p}\int_a^b \frac{\ln \varphi(f(x))}{x^{1-p}}dx}$$
4. If $p = q = 0$, then
$$\varphi\left(e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(x)}{x}dx}\right) \leq e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln \varphi(f(x))}{x}dx}$$

Proof . For the proof of (1) and (3) put $X = [a, b]$ and $d\mu = \dfrac{pdx}{x^{1-p}(b^p - a^p)}$ in the theorem 2.2 . For the proof of (2) and (4) put $X = [a, b]$ and $d\mu = \dfrac{dx}{(\ln b - \ln a)x}$ in theorem2.2 . □

In the following theorem we obtain the Hermite-Hadamard inequality for $(p - q)$-convex functions.

Theorem 2.5. Let $\varphi : [a, b] \to (0, \infty)$ $(b > a > 0)$ be a $(p - q)$-convex function. Then

1. If $p \neq 0$ and $q \neq 0$, then
$$\varphi\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \leq \left(\frac{p}{b^p - a^p}\int_a^b \frac{\varphi^q(x)}{x^{1-p}}dx\right)^{\frac{1}{q}} \leq \left(\frac{\varphi^q(a) + \varphi^q(b)}{2}\right)^{\frac{1}{q}}$$
2. If $p = 0$ and $q \neq 0$, then
$$\varphi(\sqrt{ab}) \leq \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\varphi^q(x)}{x}dx\right)^{\frac{1}{q}} \leq \left(\frac{\varphi^q(a) + \varphi^q(b)}{2}\right)^{\frac{1}{q}}$$
3. If $p \neq 0$ and $q = 0$, then
$$\varphi\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \leq e^{\frac{p}{b^p - a^p}\int_a^b \frac{\ln \varphi(x)}{x^{1-p}}dx} \leq \sqrt{\varphi(a)\varphi(b)}$$

4. If $p = q = 0$, then
$$\varphi(\sqrt{ab}) \le e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln \varphi(x)}{x}dx} \le \sqrt{\varphi(a)\varphi(b)}$$

Proof .

1. Put $f(x) = x$ in corollary 2.4 (1). Hence
$$\varphi\left(\frac{p}{b^p - a^p}\int_a^b \frac{x^p}{x^{1-p}}dx\right) \le \left(\frac{p}{b^p - a^p}\int_a^b \frac{\varphi^q(x)}{x^{1-p}}dx\right)^{\frac{1}{q}}$$
$$\Rightarrow \quad \varphi\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \le \left(\frac{p}{b^p - a^p}\int_a^b \frac{\varphi^q(x)}{x^{1-p}}dx\right)^{\frac{1}{q}}$$

This proves the left side. For the proof of the right side, by the change of variable $x = ((1-t)a^p + tb^p)^{\frac{1}{p}}$ and the $(p-q)$-convexity of φ we have
$$\left(\frac{p}{b^p - a^p}\int_a^b \frac{\varphi^q(x)}{x^{1-p}}dx\right)^{\frac{1}{q}} = \left(\frac{p}{b^p - a^p}\int_0^1 \frac{\varphi^q((1-t)a^p + tb^p)^{\frac{1}{p}}}{((1-t)a^p + tb^p)^{\frac{1}{p}-1}}\frac{1}{p}((1-t)a^p + tb^p)^{\frac{1}{p}-1}(b^p - a^p)dt\right)^{\frac{1}{q}}$$
$$\le \left(\int_0^1 (1-t)\varphi^q(a) + t\varphi^q(b)dt\right)^{\frac{1}{q}} = \left(\frac{\varphi^q(a) + \varphi^q(b)}{2}\right)^{\frac{1}{q}}$$

2. Put $f(x) = x$ in corollary 2.4 (2). Hence
$$\varphi\left(e^{\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln x}{x}dx}\right) \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\varphi^q(x)}{x}dx\right)^{\frac{1}{q}}$$
$$\Rightarrow \quad \varphi(\sqrt{ab}) \le \left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\varphi^q(x)}{x}dx\right)^{\frac{1}{q}}$$

This proves the left side. For the proof of the right side by change of variable $x = a^{1-t}b^t$ and $(p-q)$-convexity of φ we have
$$\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\varphi^q(x)}{x}dx\right)^{\frac{1}{q}} = \left(\frac{1}{\ln b - \ln a}\int_0^1 \frac{\varphi^q(a^{1-t}b^t)}{a^{1-t}b^t}a^{1-t}b^t(\ln b - \ln a)dt\right)^{\frac{1}{q}}$$
$$\le \left(\int_0^1 ((1-t)\varphi^q(a) + t\varphi^q(b))dt\right)^{\frac{1}{q}} = \left(\frac{\varphi^q(a) + \varphi^q(b)}{2}\right)^{\frac{1}{q}}$$

3. Put $f(x) = x$ in corollary 2.4 (3). Hence
$$\varphi\left(\frac{p}{b^p - a^p}\int_a^b \frac{x^p}{x^{1-p}}dx\right)^{\frac{1}{p}} \le e^{\frac{p}{b^p - a^p}\int_a^b \frac{\ln \varphi(x)}{x^{1-p}}dx}$$
$$\Rightarrow \quad \varphi\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \le e^{\frac{p}{b^p - a^p}\int_a^b \frac{\ln \varphi(x)}{x^{1-p}}dx}$$

For the proof of the right side by change of variable $x = ((1-t)a^p + tb^p)^{\frac{1}{p}}$ and $(p-q)$-convexity of φ we have
$$\frac{p}{b^p - a^p}\int_a^b \frac{\ln \varphi(x)}{x^{1-p}}dx = \frac{p}{b^p - a^p}\int_0^1 \frac{\ln \varphi((1-t)a^p + tb^p)^{\frac{1}{p}}}{((1-t)a^p + tb^p)^{\frac{1}{p}-1}}\frac{1}{p}(b^p - a^p)((1-t)a^p + tb^p)^{\frac{1}{p}-1}dt$$
$$\le \int_0^1 \ln \varphi^{1-t}(a)\varphi^t(b)dt = \frac{\ln \varphi(a) + \ln \varphi(b)}{2} = \ln \sqrt{\varphi(a)\varphi(b)}$$

So
$$e^{\frac{p}{b^p - a^p}\int_a^b \frac{\ln \varphi(x)}{x^{1-p}}dx} \le e^{\ln \sqrt{\varphi(a)\varphi(b)}} = \sqrt{\varphi(a)\varphi(b)}$$

4. Put $f(x) = x$ in corollary 2.4 (4). See also [43].

□

Theorem 2.6. (i) If φ is convex (concave) and increasing on (a, b), then φ is $(p-q)$-convex (concave) on (a, b), whenever $p \leq 1 \leq q$ $(p \geq 1 \geq q)$

(ii) If φ is $(p-q)$-convex (concave) and decreasing on (a, b), then φ is convex (concave) on (a, b), whenever $p \leq 1 \leq q$ $(p \geq 1 \geq q)$

Proof . (i) Let φ be convex, increasing and $p \leq 1 \leq q$. We have

$$\begin{aligned}\varphi(tx^p + (1-t)y^p)^{\frac{1}{p}} &\leq \varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)\\ &\leq (t\varphi^q(x) + (1-t)\varphi^q(y))^{\frac{1}{q}} \quad (p \neq 0)\\ \varphi(x^t y^{1-t}) &\leq \varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)\\ &\leq (t\varphi^q(x) + (1-t)\varphi^q(y))^{\frac{1}{q}} \quad (p = 0)\end{aligned}$$

Hence φ is $(p-q)$-convex. The proof of (ii) is simillar and can be omitted. □

Theorem 2.7. Let $\varphi : [a, b] \to (0, \infty)$ $(b > a > 0)$, $q > 0$ and $g(x) = \varphi^q(x^{\frac{1}{p}})$ $(p \neq 0)$, $g(x) = \varphi^q(e^x)$ $(p = 0)$. Then φ is $(p-q)$-convex on (a, b) if and only if g is convex on (a^p, b^p) $(p > 0)$ or (b^p, a^p) $(p < 0)$ or $(\ln a, \ln b)$ $(p = 0)$.

Proof . Let φ be $(p-q)$-convex on (a, b). So

$$\varphi(tx^p + (1-t)y^p)^{\frac{1}{p}} \leq (t\varphi^q(x) + (1-t)\varphi^q(y))^{\frac{1}{q}} \quad (p \neq 0)$$

whenever $x, y \in (a, b)$ and $t \in [0, 1]$. Put $X = x^p$ and $Y = y^p$. Then

$$\begin{aligned}&\varphi(tX + (1-t)Y)^{\frac{1}{p}} \leq (t\varphi^q(X^{\frac{1}{p}}) + (1-t)\varphi^q(Y^{\frac{1}{p}}))^{\frac{1}{q}}\\ \Rightarrow\ &\varphi^q(tX + (1-t)Y)^{\frac{1}{p}} \leq \varphi^q(X^{\frac{1}{p}}) + (1-t)\varphi^q(Y^{\frac{1}{p}}) \quad (q > 0)\\ \Rightarrow\ &g(tX + (1-t)Y) \leq tg(X) + (1-t)g(Y)\end{aligned}$$

It is obvious that $X, Y \in (a^p, b^p)$ $(p > 0)$ or $X, Y \in (b^p, a^p)$ $(p < 0)$. Now let $p = 0$ and φ be $(p-q)$-convex on (a, b), then

$$\begin{aligned}&\varphi(x^t y^{1-t}) \leq (t\varphi^q(x) + (1-t)\varphi^q(y))^{\frac{1}{q}}\\ \Rightarrow\ &\varphi^q(x^t y^{1-t}) \leq t\varphi^q(x) + (1-t)\varphi^q(y) \quad (q > 0)\end{aligned}$$

Put $x = e^X$ and $y = e^Y$. So

$$\begin{aligned}&\varphi^q(e^{tX+(1-t)Y}) \leq t\varphi^q(e^X) + (1-t)\varphi^q(e^Y)\\ \Rightarrow\ &g(tX + (1-t)Y) \leq tg(X) + (1-t)g(Y) \quad (\ln a \leq X, Y \leq \ln b).\end{aligned}$$

□

Theorem 2.8. Let φ be a $(p-q)$-convex function on $[a, b]$ and $q \geq 1$, $p \neq 0$. Then the following inequality holds

$$\frac{1}{b-a}\int_a^b \varphi(x)dx \leq \left[\frac{\varphi^q(a)(pb^{p+1} + a^{p+1} - (p+1)ab^p) + \varphi^q(b)(pa^{p+1} + b^{p+1} - (p+1)ba^p)}{(b-a)(b^p - a^a)(p+1)}\right]^{\frac{1}{q}}.$$

Especially for $p = q = 1$, $\frac{1}{b-a}\int_a^b \varphi(x)dx \leq \frac{\varphi(a) + \varphi(b)}{2}$.

Proof . By change of variable $x = (tb^p + (1-t)a^p)^{\frac{1}{p}}$ and $(p-q)$-convexity of φ we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b \varphi(x)dx &= \frac{1}{b-a}\int_0^1 \varphi(tb^p+(1-t)a^p)^{\frac{1}{p}}\frac{b^p-a^p}{p}(tb^p+(1-t)a^p)^{\frac{1}{p}-1}dt\\ &\le \frac{b^p-a^p}{p(b-a)}\int_0^1 (t\varphi^q(b)+(1-t)\varphi^q(a))^{\frac{1}{q}}(tb^p+(1-t)a^p)^{\frac{1}{p}-1}dt.\end{aligned}$$

Since $\dfrac{b^p-a^p}{p(b-a)}\int_0^1 (tb^p+(1-t)a^p)^{\frac{1}{p}-1}dt = 1$ and $\dfrac{1}{q} \le 1$, by the concavity of $h(x) = x^{\frac{1}{q}}$ and classical Jensen's inequality (1.1) ($d\mu = \dfrac{b^p-a^p}{p(b-a)}(tb^p+(1-t)a^p)^{\frac{1}{p}-1}dt$) we obtain

$$\begin{aligned}&\le \left(\frac{b^p-a^p}{p(b-a)}\int_0^1 (t\varphi^q(b)+(1-t)\varphi^q(a))(tb^p+(1-t)a^p)^{\frac{1}{p}-1}dt\right)^{\frac{1}{q}}\\ &= \left(\frac{b^p-a^p}{p(b-a)}\int_0^1 (t(\varphi^q b)-\varphi^q(a))+\varphi^q(a))(t(b^p-a^a)+a^p)^{\frac{1}{p}-1}dt\right)^{\frac{1}{q}}\end{aligned}$$

Again by change of variable $t(b^p-a^p)+a^p = x^p$ and easy calculations we get

$$\begin{aligned}&= \left(\frac{b^p-a^p}{p(b-a)}\int_a^b [\frac{x^p-a^p}{b^p-a^p}(\varphi^q(b)-\varphi^q(a))+\varphi^q(a)]\frac{p}{b^p-a^a}dx\right)^{\frac{1}{q}}\\ &= \left[\frac{\varphi^q(a)(pb^{p+1}+a^{p+1}-(p+1)ab^p)+\varphi^q(b)(pa^{p+1}+b^{p+1}-(p+1)ba^p)}{(b-a)(b^p-a^a)(p+1)}\right]^{\frac{1}{q}}\end{aligned}$$

□

3 Examples

1. $\varphi(x) = x$ is $(p-q)$-convex $(p<q)$ on $[a,b]$ $(b>a>0)$. Hence by theorem 2.5 we have

$$\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \le \left(\frac{p}{b^p-a^p}\int_a^b \frac{x^q}{x^{1-p}}dx\right)^{\frac{1}{q}} \le \left(\frac{a^q+b^q}{2}\right)^{\frac{1}{q}} \quad (0\ne p<q)$$

$$\sqrt{ab} \le \left(\frac{1}{\ln b-\ln a}\int_a^b \frac{x^q}{x}dx\right)^{\frac{1}{q}} \le \left(\frac{a^q+b^q}{2}\right)^{\frac{1}{q}} \quad (p=0<q)$$

$$\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \le e^{\frac{p}{b^p-a^p}\int_a^b \frac{\ln x}{x^{1-p}}dx} \le \sqrt{ab} \quad (p<0=q)$$

Thus

$$\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \le \left(\frac{p(b^{p+q}-a^{p+q})}{(p+q)(b^p-a^p)}\right)^{\frac{1}{q}} \le \left(\frac{a^q+b^q}{2}\right)^{\frac{1}{q}} \quad (0\ne p<q)$$

$$\sqrt{ab} \le \left(\frac{b^q-a^q}{q(\ln b-\ln a)}\right)^{\frac{1}{q}} \le \left(\frac{a^q+b^q}{2}\right)^{\frac{1}{q}} \quad (0<q)$$

$$\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \le e^{\frac{b^p\ln b-a^p\ln a}{b^p-a^p}-\frac{1}{p}} \le \sqrt{ab} \quad (p<0)$$

So if $\gamma<0<\alpha<\beta$, then

$$\begin{aligned}\left(\frac{a^\gamma+b^\gamma}{2}\right)^{\frac{1}{\gamma}} &\le e^{\frac{b^\gamma\ln b-a^\gamma\ln a}{b^\gamma-a^\gamma}-\frac{1}{\gamma}} \le \sqrt{ab} \le \left(\frac{b^\alpha-a^\alpha}{\alpha(\ln b-\ln a)}\right)^{\frac{1}{\alpha}}\\ &\le \left(\frac{\alpha(b^{\alpha+\beta}-a^{\alpha+\beta})}{(\alpha+\beta)(b^\alpha-a^\alpha)}\right)^{\frac{1}{\beta}} \le \left(\frac{a^\beta+b^\beta}{2}\right)^{\frac{1}{\beta}}\end{aligned}$$

In the other words by means notations we have

$$M_\gamma(a,b) \le e^{\frac{b^\gamma \ln b - a^\gamma \ln a}{b^\gamma - a^\gamma} - \frac{1}{\gamma}} \le G(a,b) \le T_\alpha(a,b)$$
$$\le L_{\alpha,\beta}(a,b) \le M_\beta(a,b)$$

2. $\varphi(x) = \dfrac{1}{1+x^2}$ is not convex on $(0,\infty)$, but it is $(p-q)$-convex on $(0,\infty)$, where $p=2$, $q=-2$. Because by theorem 2.7, $g(x) = \varphi^{-2}(x^{\frac{1}{2}}) = (1+x)^2$ is convex on $(0,\infty)$.
3. $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ is concave and nondecreasing on $(0,\infty)$ [13]. So by theorem 2.6 (i) ψ is $(p-q)$-concave on $(0,\infty)$, when $p \ge 1 \ge q$. Hence by theorem 2.5 (1) for $b > a > 0$, we have

$$\psi\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}} \ge \left(\frac{p}{b^p-a^p}\int_a^b \frac{\psi^q(x)}{x^{1-p}}\right)^{\frac{1}{q}} \ge \left(\frac{\psi^q(a)+\psi^q(b)}{2}\right)^{\frac{1}{q}}$$

We investigate these inequalities in two special cases.

(i) For $p=q=1$ we have

$$\frac{\psi(a)+\psi(b)}{2} \le \frac{1}{b-a}\int_a^b \psi(x)dx \le \psi\left(\frac{a+b}{2}\right)$$
$$\Rightarrow \frac{\psi(a)+\psi(b)}{2} \le \frac{\ln\Gamma(b)-\ln\Gamma(a)}{b-a} \le \psi\left(\frac{a+b}{2}\right)$$
$$\Rightarrow e^{(b-a)\frac{\psi(a)+\psi(b)}{2}} \le \frac{\Gamma(b)}{\Gamma(a)} \le e^{(b-a)\psi(\frac{a+b}{2})}$$

Especially for $b=a+1$ we get

$$e^{\frac{\psi(a)+\psi(a+1)}{2}} \le \frac{\Gamma(a+1)}{\Gamma(a)} \le e^{\psi(\frac{2a+1}{2})}$$

Finally since $\Gamma(a+1) = a\Gamma(a)$ and $\psi(a+1) = \psi(a) + \dfrac{1}{a}$ $(a>0)$ we obtain

$$e^{\psi(a)+\frac{1}{2a}} \le a \le e^{\psi(a+\frac{1}{2})}$$

(ii) For $p=2$ and $q=1$ we have

$$\frac{\psi(a)+\psi(b)}{2} \le \frac{2}{b^2-a^2}\int_a^b x\psi(x)dx \le \psi\left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow \frac{\psi(a)+\psi(b)}{2} \le \frac{2}{b^2-a^2}\left[x\ln\Gamma(x)\big|_a^b - \int_a^b \ln\Gamma(x)dx\right] \le \psi\left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow \frac{\psi(a)+\psi(b)}{2} \le \frac{2}{b^2-a^2}\left[b\ln\Gamma(b) - a\ln\Gamma(a) - \int_a^b \ln\Gamma(x)dx\right] \le \psi\left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$$

Especially for $b=a+1$ and considering $\int_a^{a+1}\ln\Gamma(x)dx = -a + a\ln a + \sqrt{2\pi}$ [40] we obtain

$$\frac{\psi(a)+\psi(a+1)}{2} \le \frac{2}{2a+1}\left[(a+1)\ln\Gamma(a+1) - a\ln\Gamma(a) + a - a\ln a - \sqrt{2\pi}\right]$$
$$\le \psi\left(a^2+a+\frac{1}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow \psi(a)+\frac{1}{2a} \le \frac{2}{2a+1}\left[\ln a + \ln\Gamma(a) + a - \sqrt{2\pi}\right] \le \psi\left(a^2+a+\frac{1}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow (a+\frac{1}{2})(\psi(a)+\frac{1}{2a}) \le \ln\Gamma(a+1) + a - \sqrt{2\pi} \le (a+\frac{1}{2})\psi\left(a^2+a+\frac{1}{2}\right)^{\frac{1}{2}}$$
$$\Rightarrow e^{\sqrt{2\pi}-a+(a+\frac{1}{2})(\psi(a)+\frac{1}{2a})} \le \Gamma(a+1) \le e^{\sqrt{2\pi}-a+(a+\frac{1}{2})\psi(a^2+a+\frac{1}{2})^{\frac{1}{2}}}$$

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