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Necessary conditions for vector-valued optimization problems on Banach spaces

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Abstract

In this paper, the class of nonconvex vector-valued optimization problems with inequality constraints is considered. We introduce two constraint qualifications and derive the weak and strong Karush-Kuhn-Tucker type of necessary conditions for a (weakly) efficient solution to the considered problem. All results are given in terms of Dini directional derivative and Clarke subdifferential.

Keywords: Vector-valued function, Optimality conditions, Constraint qualification, Clarke subdifferential
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1 Introduction

Let T be a compact metric space and $\mathcal{X}$ be a Banach space. Clearly, the space $\mathcal{Y}$ of all real-valued continuous functions $y : T \to \mathbb{R}$ endowed with the norm

$$\|y\| := \max\left\{\|y(t)\| \mid t \in T\right\}, \quad \forall y \in \mathcal{Y},$$

is a Banach space. Considering $\psi : \mathcal{X} \to \mathcal{Y}$, we have $\psi(x) : T \to \mathbb{R}$ for each $x \in \mathcal{X}$, and so $\psi(x)(t) \in \mathbb{R}$ for all $t \in T$. For the sake of simplicity, we denote the value of $\psi(x)(t)$ by the symbol $\psi_t(x)$, and in this way, for each $t \in T$, the function $\psi_t : \mathcal{X} \to \mathbb{R}$ is defined as

$$\psi_t(x) := \psi(x)(t), \quad \forall x \in \mathcal{X}.$$

We define the nonempty cone $\mathcal{Y}^+ \subseteq \mathcal{Y}$ by

$$\mathcal{Y}^+ := \{y \in \mathcal{Y} \mid y(t) \geq 0, \ \forall t \in T\}.$$

This cone induces a partial order relation on $\mathcal{Y}$ which is defined as follows:

$$y_1 \leq_{\mathcal{Y}} y_2 \quad \iff y_2 - y_1 \in \mathcal{Y}^+.$$

In other word, $y_1 \leq_{\mathcal{Y}} y_2$ if and only if $y_1(t) \leq y_2(t)$, for all $t \in T$.

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Based on the above order on $\mathcal{Y}$, we consider the following vector-valued optimization problem

$$(VP): \quad \min \quad \psi(x) \quad \text{s.t.} \quad x \in S,$$

where S is a nonempty subset of $\mathcal{X}$, defined by

$$S := \{x \in \mathcal{X} \mid g_j(x) \leq 0,\ j \in J\}, \tag{1.1}$$

in which J is an arbitrary nonempty index set and $g_j : \mathcal{X} \to \mathbb{R}$ is a given function for each $j \in J$. Note that if J is a finite set with p members, then $\mathcal{Y}$ is equal to $\mathbb{R}^p$ and the problem (VP) can be written as

$$\begin{aligned} (VP_1): \qquad\qquad & \min \quad \big(\psi_1(x), \cdots, \psi_p(x)\big) \\ & \text{s.t.} \quad g_j(x) \leq 0, \quad j \in J, \end{aligned}$$

If J is a finite set and $\mathcal{X} = \mathbb{R}^n$, the problem (VP_1) turns into the classic multiobjective programming problem (MP, in brief) [2]. In the MP theory, three kinds of necessary optimality conditions are interested, named Fritz-John (FJ), Karush-Kuhn-Tucker (KKT), and strong KKT (SKKT). For the cases that the functions g_j as $j \in J$ and ψ_t as $t \in T$ are differentiable (resp. convex, locally Lipschitz, lower semi-continuous), these necessary conditions are expressed in term of their gradient (resp. convex subdifferential, Clarke subdifferential, Mordukhovich subdifferential); see for instance, [3, 6, 13]. For study the generalizations of these optimality conditions to the case that $\mathcal{X}$ has infinite dimension, see [1, 15].

If $\mathcal{X} = \mathbb{R}^n$ and J is an infinite set, problem (VP_1) is said multiobjective semi-infinite programming problem (MSIP). For study the FJ, the KKT and the SKKT for MSIP, we refer to [5] for differentiable case, to [9] for linear case, to [4] for convex case, and to [7, 8, 10, 11, 12] for locally Lipschitz case.

Recently, in [16, 17], the problem (VP) with feasible set S defined by (1.1), has been investigated for infinite T and finite J. It is noteworthy that in this case $\mathcal{Y}$ is an infinite-dimensional Banach space and (VP) can not be shown as (VP_1). Note that in these articles, some FJ type necessary optimality conditions are presented for (VP). Since the KKT and the SKKT types optimality conditions for (VP) have not been investigated so far (even for finite J), one of the goal of the present paper is to fill this gap. In this way, we consider the net $(\psi_t(x) :\ t \in T)$ in $\mathbb{R}$, and we define the concept of *optimality* for the problem

$$\begin{aligned} (NP): \qquad\qquad & \min \quad \big(\psi_t(x) :\ t \in T\big) \\ & s.t. \quad g_j(x) \leq 0, \qquad\qquad j \in J, \end{aligned}$$

in such a way that it is a generalization of the concept of optimality for (VP_1) (in fact, if we put $|T| = p$ in (NP), the problem turns into problem (VP_1)). The next step will be to shown that the concept of optimality for (NP) coincides with the concept of optimality for (VP). Finally, we prove the KKT and SKKT types necessary optimality conditions for (NP) and\or (VP).

2 Preliminaries

In this section, we briefly overview some notions of convex analysis and nonsmooth analysis widely used in formulations and proofs of main results of the paper. For more details, discussion, and applications see [1].

Throughout this section, we assume that $\mathbb{X}$ is a real Banach space. The dual space of $\mathbb{X}$ is denoted by $\mathbb{X}^*$ and is equipped with weak* topology. The zero vectors of $\mathbb{X}$ and $\mathbb{X}^*$ are respectively denoted by $0_{\mathbb{X}}$ and $0_{\mathbb{X}^*}$, and the value of functional $\xi \in \mathbb{X}^*$ at $x \in \mathbb{X}$ is referred by $\langle \xi, x \rangle$. The closure, the convex hull, and the convex cone of $A \subset \mathbb{X}$ are denoted respectively by $cl(A)$, $conv(A)$, and $cone(A)$. The weak* closure of $A_* \subset \mathbb{X}^*$ is referred by $cl^{w^*}(A_*)$.

Theorem 2.1. [6] If Ω is an arbitrary index set and A_γ is a convex subset of $\mathbb{X}$ for each $\gamma \in \Omega$, then

$$conv\Big(\bigcup_{\gamma\in\Omega} A_\gamma\Big) = \bigcup \left\{ \sum_{i=1}^{k} a_{\gamma_i} A_{\gamma_i} \mid a_{\gamma_i} \geq 0,\ \sum_{i=1}^{k} a_{\gamma_i} = 1,\ \gamma_i \in \Omega,\ k \in \mathbb{N} \right\},$$

$$cone\Big(\bigcup_{\gamma\in\Omega} A_\gamma\Big) = \bigcup \left\{ \sum_{i=1}^{k} a_{\gamma_i} A_{\gamma_i} \mid a_{\gamma_i} \geq 0,\ \gamma_i \in \Omega,\ k \in \mathbb{N} \right\}.$$

Theorem 2.2. [15] Let A_* and B_* be two subsets of $\mathbb{X}^*$, and A_* be norm bounded. Then,

$$cl^{w^*}\big(A_*\big) + cl^{w^*}\big(B_*\big) = cl^{w^*}\big(A_* + B_*\big).$$

Let $A \subseteq \mathbb{X}$ and $A_* \subseteq \mathbb{X}^*$ be given. The following sets will be used in the sequel (we write $h_n \downarrow 0$ for a sequence of positive number $\{h_n\}$ with limit 0, and we write $x_n \xrightarrow{A} \hat{x}$ for a sequence $\{x_n\} \subseteq A$ converging to $\hat{x} \in \mathbb{X}$).

(i) The Bouligand tangent cone of $\emptyset \neq A \subseteq \mathbb{X}$ at $\hat{x} \in cl(A)$ is

$$\Gamma_B(A,\hat{x}) := \Big\{\nu \in \mathbb{X} \mid \exists h_n \downarrow 0,\ \exists \nu_n \to \nu,\ \hat{x} + h_n\nu_n \in A,\quad \forall n \in \mathbb{N}\Big\}.$$

(ii) The Clarke tangent cone of A at $\hat{x} \in cl(A)$ is

$$\Gamma_C(A,\hat{x}) := \Big\{\nu \in \mathbb{X} \mid \forall x_n \xrightarrow{A} \hat{x},\ \forall h_n \downarrow 0,\ \exists \nu_n \to \nu,\ x_n + h_n\nu_n \in A,\quad \forall n \in \mathbb{N}\Big\}.$$

(iii) The polar cone of A and A_* are respectively

$$A^{\preceq} := \{\xi \in \mathbb{X}^* \mid \langle \xi, x\rangle \leq 0,\quad \forall x \in A\},\quad A_*^{\preceq} := \{x \in \mathbb{X} \mid \langle \xi, x\rangle \leq 0,\ \forall \xi \in A_*\}.$$

(iv) The strictly polar set of A and A_* are respectively

$$A^{\prec} := \{\xi \in \mathbb{X}^* \mid \langle \xi, x\rangle < 0,\quad \forall x \in A\},\quad A_*^{\prec} := \{x \in \mathbb{X} \mid \langle \xi, x\rangle < 0,\ \forall \xi \in A_*\}.$$

(v) The Clarke normal cone of A at $\hat{x} \in cl(A)$ is

$$N_C(A,\hat{x}) = \left(\Gamma_C(A,\hat{x})\right)^{\preceq}.$$

It should be noted that $\Gamma_C(A,\hat{x})$, $A^{\preceq}$, $A^{\prec}$, $A_*^{\preceq}$, $A_*^{\prec}$, and $N_C(A,\hat{x})$ are always convex while $\Gamma_B(A,\hat{x})$ is not necessarily convex. Also, $A^{\preceq}$ and $N_C(A,\hat{x})$ are weak* closed sets in $\mathbb{X}^*$; $\Gamma_C(A,\hat{x})$, $\Gamma_B(A,\hat{x})$ and $A_*^{\preceq}$ are closed sets in $\mathbb{X}$, and the following equalities are true if $A^{\prec} \neq \emptyset$ and $A_*^{\prec} \neq \emptyset$

$$cl^{w^*}(A^{\prec}) = A^{\preceq} \qquad \text{and} \qquad cl(A_*^{\prec}) = A_*^{\preceq}. \tag{2.1}$$

For each $A \subseteq \mathbb{X}$ and $\hat{x} \in cl(A)$, the following inclusion holds by the definition:

$$\Gamma_C(A,\hat{x}) \subseteq \Gamma_B(A,\hat{x}). \tag{2.2}$$

The set $A \subseteq \mathbb{X}$ is said to be regular at $\hat{x} \in cl(A)$ if $\Gamma_B(A,\hat{x}) = \Gamma_C(A,\hat{x})$. Note that, if A is convex, then it is regular at all $\hat{x} \in cl(A)$, and we have

$$N_C(A,\hat{x}) = N(A,\hat{x}) := \{\xi \in \mathbb{X}^* \mid \langle \xi, x - \hat{x}\rangle \leq 0,\quad \forall x \in A\}. \tag{2.3}$$

Theorem 2.3. [1] (bipolar) Let the nonempty sets $A \subseteq \mathbb{X}$ and $A_* \subseteq \mathbb{X}^*$ be given. One has

$$(A^{\preceq})^{\preceq} = cl(cone(A)) \qquad \text{and} \qquad (A_*^{\preceq})^{\preceq} = cl^{w^*}(cone(A_*)).$$

Suppose that $\mathbb{W}$ is a Banach space. A function $\phi : \mathbb{X} \to \mathbb{W}$ is said to be locally Lipschitz if ϕ is Lipschitz around all $\hat{x} \in \mathbb{X}$, i.e., for each $\hat{x} \in \mathbb{X}$ there exist a neighborhood U of $\hat{x}$ and a positive real number $L_U > 0$ such that

$$\| \phi(x) - \phi(y) \| \leq L_U \| x - y \|,\quad \forall x, y \in U.$$

It is easy to see that if $\phi : \mathbb{X} \to \mathbb{W}$ is Lipschitz around $\hat{x}$, then

$$\lim_{h\downarrow 0,\, u\to\nu} \frac{\phi(\hat{x} + hu) - \phi(\hat{x} + h\nu)}{h} = 0. \tag{2.4}$$

The Dini directional derivative of $\phi : \mathbb{X} \to \mathbb{W}$ at $\hat{x} \in \mathbb{X}$ in the direction $\nu \in \mathbb{X}$ is

$$\nabla\phi(\hat{x};\nu) := \lim_{h\downarrow 0} \frac{\phi(\hat{x}+h\nu)-\phi(\hat{x})}{h}.$$

By definition, ϕ is said to be Dini directionally derivable at $\hat{x}$ if $\nabla\phi(\hat{x};\nu)$ exists for all $\nu \in \mathbb{X}$. It is known that if ϕ is Frèchet differentiable at $\hat{x}$, and its Frèchet derivative is denoted by $\nabla\phi(\hat{x}) \in \mathbb{X}^*$, we have

$$\nabla\phi(\hat{x};\nu) = \langle \nabla\phi(\hat{x}), \nu\rangle, \qquad \forall \nu \in \mathbb{X}.$$

Let the real-valued function $\phi : \mathbb{X} \to \mathbb{R}$ be Lipschitz around $\hat{x} \in \mathbb{X}$. The upper Dini directional derivative and the upper Clarke directional derivative of ϕ at $\hat{x} \in \mathbb{X}$ in the direction $\nu \in \mathbb{X}$ are respectively defined as:

$$\phi^D(\hat{x};\nu) := \limsup_{h\downarrow 0} \frac{\phi(\hat{x}+h\nu)-\phi(\hat{x})}{h}, \quad \phi^C(\hat{x};\nu) := \limsup_{h\downarrow 0,\ x\to\hat{x}} \frac{\phi(x+h\nu)-\phi(x)}{h}.$$

Remark 2.4. If $\phi : \mathbb{X} \to \mathbb{R}$ is a real-valued function, its Dini directional derivative $\nabla\phi(\hat{x};\nu)$ is introduced in classical books with the name "directional derivative" and is displayed with the symbol $\phi'(\hat{x},\nu)$. Moreover, if ϕ is Lipschitz around $\hat{x} \in \mathbb{X}$, then

$$\phi^D(\hat{x};\nu) \leq \phi^C(\hat{x};\nu), \qquad \forall \nu \in \mathbb{X}. \tag{2.5}$$

Furthermore, if ϕ is directionally derivable at $\hat{x}$, then $\phi^D(\hat{x};\nu) = \nabla\phi(\hat{x};\nu)$, for all $\nu \in \mathbb{X}$.

Let the real-valued function $\phi : \mathbb{X} \to \mathbb{R}$ be Lipschitz around $\hat{x} \in \mathbb{X}$. The Clarke subdifferential of ϕ at $\hat{x}$ is defined as

$$\partial_C\phi(\hat{x}) := \{\xi \in \mathbb{X}^* \mid \langle \xi,\nu\rangle \leq \phi^C(\hat{x};\nu), \quad \forall \nu \in \mathbb{X}\}.$$

Theorem 2.5. [1] Assume that $\phi_1, \cdots, \phi_m$ are locally Lipschitz functions from $\mathbb{X}$ to $\mathbb{R}$ and $\hat{x} \in \mathbb{X}$.

i) $\partial_C\phi(\hat{x})$ is a nonempty convex weak* compact set in $\mathbb{X}^*$.

ii) For all real numbers $\alpha_1, \cdots, \alpha_m$, one has

$$\partial_C\left(\sum_{i=1}^m \alpha_i\phi_i\right)(\hat{x}) \subseteq \sum_{i=1}^m \alpha_i \partial_C\phi_i(\hat{x}).$$

iii) If ϕ is continuously differentiable at $\hat{x}$, then $\partial_C\phi(\hat{x}) = \{\nabla\phi(\hat{x})\}$.

iv) If ϕ is convex on $\mathbb{X}$, then $\partial_C\phi(\hat{x}) = \partial\phi(\hat{x})$, in which $\partial\phi(\hat{x})$ denotes the subdifferential of ϕ in convex analysis sense, defined as

$$\partial\phi(\hat{x}) := \{\xi \in \mathbb{X}^* \mid \phi(x)-\phi(\hat{x}) \geq \langle \xi, x-\hat{x}\rangle,\ \forall x \in \mathbb{X}\}.$$

v) If ϕ attains its minimum on $A \subseteq \mathbb{X}$ at $\hat{x} \in A$, then $\phi^C(\hat{x};\nu) \geq 0$ for all $\nu \in \Gamma_B(A,\hat{x})$. Also, one has

$$0_{\mathbb{X}^*} \in \partial_C\phi(\hat{x}) + N_C(A,\hat{x}).$$

vi) The function $\nu \to \phi^C(\hat{x},\nu)$ is positively homogeneous and subadditive (and hence convex) on $\mathbb{X}$, and we have

$$\partial\left(\phi^C(\hat{x},\cdot)\right)(0_{\mathbb{X}}) = \partial_C\phi(\hat{x}).$$

vii) $\phi^C(\hat{x};\nu) = \max\{\langle \xi,\nu\rangle \mid \xi \in \partial_C\phi(\hat{x})\}$.

As the final point of this section, suppose that $\theta : \mathbb{X} \rightrightarrows \mathbb{Z}$ is a set-valued function, i.e., θ is a function from $\mathbb{X}$ to power set of $\mathbb{Z}$. The upper limit of θ when x tends to $\tilde{x} \in \{x \in \mathbb{X} \mid \theta(x) \neq \emptyset\}$, in the sense of Painleve-Kuratowski, is defined as

$$\mathbf{Limsup}_{x\to\tilde{x}}\theta(x) = \left\{\tilde{z} \in \mathbb{Z} \mid \exists x_n \to \tilde{x},\ \exists z_n \in \theta(x_n),\ \lim_{n\to\infty} z_n = \tilde{z}\right\}.$$

Remark 2.6. Suppose that the function $f : \mathbb{X} \to \mathbb{R}$ is given and the set-valued function $\theta : \mathbb{X} \rightrightarrows \mathbb{R}$ is defined by $\theta(x) := \{f(x)\}$ for all $x \in \mathbb{X}$. Then, the definitions of $\limsup_{x\to\tilde{x}} f(x)$ and $\mathbf{Limsup}_{x\to\tilde{x}}\theta(x)$ conclude that

$$\limsup_{x\to\tilde{x}} f(x) = \max\left(\mathbf{Limsup}_{x\to\tilde{x}}\theta(x)\right) = \max\left(\mathbf{Limsup}_{x\to\tilde{x}}\{f(x)\}\right) = \max\left\{r \in \mathbb{R} \mid \exists x_n \to \tilde{x},\ \lim_{n\to\infty} f(x_n) = r\right\}.$$

Note that if $\lim_{x\to\tilde{x}} f(x)$ exists, then

$$\mathbf{Limsup}_{x\to\tilde{x}}\theta(x) = \mathbf{Limsup}_{x\to\tilde{x}}\{f(x)\} = \big\{\lim_{x\to\tilde{x}} f(x)\big\}.$$

3 Main Results

As the starting point of this section, we recall the following definition from [2].

Definition 3.1. The feasible point $\hat{x} \in S$ is said to be a local efficient solution for (VP) if there exists a neighborhood U of $\hat{x}$ such that

$$(\psi(S \cap U) - \psi(\hat{x})) \cap (-\mathcal{Y}^+) = \{0_{\mathcal{Y}}\}.$$

Also, $\hat{x} \in S$ is said to be a local weakly efficient solution for (VP) if there exists a neighborhood U of $\hat{x}$ such that

$$(\psi(S \cap U) - \psi(\hat{x})) \cap \left(-int(\mathcal{Y}^+)\right) = \emptyset.$$

The following lemma shows that the above definition expresses a generalization of the concepts local efficient solution and local weakly efficient solution which were considered in [7, 8] for problem (VP_1).

Lemma 3.2. A feasible point $\hat{x} \in S$ is a local efficient solution for (VP) if and only if there is a neighborhood U of $\hat{x}$, in which there is no $x \in S \cap U$ that

$$\begin{cases} \psi_t(x) \le \psi_t(\hat{x}), & \text{for all } t \in T, \\ \psi_{t_0}(x) < \psi_{t_0}(\hat{x}), & \text{for some } t_0 \in T. \end{cases} \tag{3.1}$$

Also, $\hat{x}$ is a local weakly efficient solution iff there is a neighborhood U of $\hat{x}$, in which there is no $x \in S \cap U$ such that $\psi_t(x) < \psi_t(\hat{x})$ for all $t \in T$.

Proof . We prove for local weakly efficient solutions, and proof of local efficient solutions is similar. Suppose that $\hat{x} \in S$ is a local efficient solution for (VP). By definition, we can find a neighborhood U of $\hat{x}$ such that

$$(\psi(S \cap U) - \psi(\hat{x})) \cap \left(-\mathcal{Y}^+ \setminus \{0_{\mathcal{Y}}\}\right) = \emptyset.$$

This implies that

$$\psi(x) - \psi(\hat{x}) \notin (-\mathcal{Y}^+ \setminus \{0_{\mathcal{Y}}\}), \quad \text{for all } x \in S \cap U,$$

which concludes that there is no $x \in S \cap U$ satisfying $\psi(x) - \psi(\hat{x}) \in (-\mathcal{Y}^+ \setminus \{0_{\mathcal{Y}}\})$. Since $-\mathcal{Y}^+ = \{y \in \mathcal{Y} \mid y(t) \le 0,\ \forall t \in T\}$, we have

$$-\mathcal{Y}^+ \setminus \{0_{\mathcal{Y}}\} = \{y \in \mathcal{Y} \mid y(t) \le 0,\ \forall t \in T, \quad \text{and} \quad y(t_0) < 0,\ \exists t_0 \in T\}.$$

Consequently, the local efficiency of $\hat{x}$ implies that there is no $x \in S \cap U$ satisfying

$$\begin{cases} (\psi(x) - \psi(\hat{x}))\,(t) \le 0, & \text{for all } t \in T, \\ (\psi(x) - \psi(\hat{x}))\,(t_0) < 0, & \text{for some } t_0 \in T, \end{cases}$$

which can be considered as a rewrite of (3.1). Conversely, if for some neighborhood U of $\hat{x}$ there is no $x \in S \cap U$ that satisfies (3.1), by reversing the above proof, we can see $\hat{x}$ is a local efficient solution of the problem (VP). □

Now, we define the upper Dini directional derivative for the vector-valued function $\psi : \mathcal{X} \to \mathcal{Y}$. We recall that this concept was defined in Section 2 only for real-valued functions. In order to extend this definition to the function ψ, we take help from set-valued functions. In fact, a natural extension to the vector-valued functions case is obtained by using upper limits of set-valued functions in sense of Painleve-Kuratowski instead of the upper limits of single-valued scalar functions as follows.

Definition 3.3. The upper Dini directional derivative of $\Psi : \mathcal{X} \to \mathcal{Y}$ at $x_0 \in \mathcal{X}$ in the direction $\nu \in \mathcal{X}$ is defined as

$$\begin{aligned} \mathfrak{D}\psi(x_0;\nu) &:= \textbf{Limsup}_{h\downarrow 0} \left\{ \frac{\psi(x_0+hu)-\psi(x_0)}{h} \right\} \\ &= \left\{ y \in \mathcal{Y} \mid \exists h_n \downarrow 0,\ y = \lim_{n\to\infty} \frac{\psi(x_0+h_n\nu)-\psi(x_0)}{h_n} \right\}. \end{aligned}$$

As a necessary optimality condition for (VP), the following theorem presents an important result.

Theorem 3.4. Assume that $\hat{x} \in S$ is a local weakly efficient solution for (VP) and $\Psi : \mathcal{X} \to \mathcal{Y}$ is Lipschitz around $\hat{x}$. Then,

$$\mathfrak{D}\psi(\hat{x};\nu) \subseteq \mathcal{Y} \setminus \left(-int(\mathcal{Y}^+)\right), \quad \forall \nu \in \Gamma_C(S,\hat{x}).$$

Proof . On the contrary, suppose there exists a vector $\nu \in \Gamma_C(S,\hat{x})$ that $\mathfrak{D}\psi(\hat{x};\nu) \nsubseteq \mathcal{Y} \setminus (-int(\mathcal{Y}^+))$. This means $\mathfrak{D}\psi(\hat{x};\nu) \cap (-int(\mathcal{Y}^+)) \neq \emptyset$. Let y be an element in this intersection, i.e.,

$$y \in \mathfrak{D}\psi(\hat{x};\nu) \cap \left(-int(\mathcal{Y}^+)\right).$$

Since $y \in \mathfrak{D}\psi(\hat{x};\nu)$, we can find a sequence $\{h_n\} \downarrow 0$ such that

$$\lim_{n\to\infty} \frac{\psi(\hat{x}+h_n\nu)-\psi(\hat{x})}{h_n} = y. \tag{3.2}$$

As $\nu \in \Gamma_C(S,\hat{x})$, for the above $\{h_n\}$ and $x_n = \hat{x}$ for all $n \in \mathbb{N}$, there exists a sequence $\{\nu_n\} \to \nu$ such that $\hat{x} + h_n\nu_n \in S$ for all $n \in \mathbb{N}$. Since,

$$\frac{\psi(\hat{x}+h_n\nu_n)-\psi(\hat{x})}{h_n} = \frac{\psi(\hat{x}+h_n\nu_n)-\psi(\hat{x}+h_n\nu)}{h_n} + \frac{\psi(\hat{x}+h_n\nu)-\psi(\hat{x})}{h_n},$$

and the right hand of the above equality tends to y by (2.4) and (3.2), we have

$$\lim_{n\to\infty} \frac{\psi(\hat{x}+h_n\nu_n)-\psi(\hat{x})}{h_n} = y.$$

As $y \in -int(\mathcal{Y}^+)$, the above equality yields, for n large enough,

$$\frac{\psi(\hat{x}+h_n\nu_n)-\psi(\hat{x})}{h_n} \in -int(\mathcal{Y}^+),$$

which implies there is a positive number $K > 0$ such that

$$\psi(\hat{x}+h_n\nu_n)-\psi(\hat{x}) \in -int(\mathcal{Y}^+), \quad \forall n \geq K.$$

Note that when $n \geq K$, we have $\hat{x} + h_n\nu_n \in U \cap S$ for some neighborhood U of $\hat{x}$. Hence, the last inclusion concludes that

$$(\psi(S\cap U)-\psi(\hat{x})) \cap \left(-int(\mathcal{Y}^+)\right) \neq \emptyset,$$

which contradicts the assumption that $\hat{x}$ is a local weakly efficient solution of (VP). The proof is complete. □

Corollary 3.5. Let $\hat{x} \in S$ be a local weakly efficient solution for (VP), and ψ is Lipschitz around $\hat{x}$.

(i) If S is regular at $\hat{x}$, then

$$\mathfrak{D}\psi(\hat{x};\nu) \subseteq \mathcal{Y} \setminus (-int(\mathcal{Y}^+)), \quad \forall \nu \in \Gamma_B(S,\hat{x}).$$

(iii) If $\hat{x} \in int(S)$, then

$$\mathfrak{D}\psi(\hat{x};\nu) \subseteq \mathcal{Y} \setminus (-int(\mathcal{Y}^+)), \quad \forall \nu \in \mathcal{X}.$$

(iv) If ψ is Dini directionally derivable at $\hat{x}$, then

$$-\nabla\psi(\hat{x};\nu) \notin int(\mathcal{Y}^+), \quad \forall \nu \in \Gamma_C(S,\hat{x}).$$

Proof . The definition of regular sets and Theorem 3.4 imply (i). Since the condition $\hat{x} \in int(S)$ implies that $\Gamma_C(S,\hat{x}) = \mathcal{X}$, (ii) is a corollary of Theorem 3.4. For prove of (iii), we first mention that if ψ is Dini directionally derivable at $\hat{x}$, by Remark 2.6 we have $\mathfrak{D}\psi(\hat{x};\nu) = \{\nabla\psi(\hat{x};\nu)\}$, for all $\nu \in X$. So, Theorem 3.4 implies that $\nabla\psi(\hat{x};\nu) \notin -int(\mathcal{Y}^+)$, for all $\nu \in \Gamma_C(S,\hat{x})$, the result is immediate. □

The following example shows that the assumption of "regularity of S at $\hat{x}$" in Corollary 3.5(i) is necessary. In other words, in Theorem 3.4, we can not replace the cone $\Gamma_C(S,\hat{x})$ with the cone $\Gamma_B(S,\hat{x})$.

Example 3.6. Let $T = \{1\}$ (and hence $\mathcal{Y} = \mathbb{R}$), $\mathcal{X} = \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \inf\left\{|x - \frac{1}{2^n}| \mid n \in \mathbb{N}\right\}.$$

Since f is distance function from $A := \left\{\frac{1}{2^n} \mid n \in \mathbb{N}\right\}$, it is locally Lipschitz on $\mathbb{R}$. Take

$$\psi : \mathbb{R} \to \mathbb{R}, \qquad \psi(x) = f(x) - \frac{x}{3},$$
$$S = \left\{\frac{3}{2^{n+2}} \mid n \in \mathbb{N}\right\} \cup \{0\}.$$

Obviously, $\hat{x} = 0$ is a local weakly efficient solution for the problem (VP), ψ is Lipschitz around $\hat{x}$, $\Gamma_C(S,\hat{x}) = \{0\}$, $\Gamma_B(S,\hat{x}) = [0,+\infty)$, and for all $\nu \in \mathbb{R}$ we have $\mathfrak{D}\psi(\hat{x};\nu) = [-\frac{\nu}{3}, 0]$. So, the following condition is not satisfied for all $\nu \in \Gamma_B(S,\hat{x})$, but it is true for all $\nu \in \Gamma_C(S,\hat{x})$:

$$\mathfrak{D}\psi(\hat{x};\nu) = [-\frac{\nu}{3}, 0] \subseteq \mathbb{R} \setminus (-int(\mathbb{R}^+)) = [0,+\infty).$$

Remark 3.7. The function $f : \mathbb{R} \to \mathbb{R}$, defined in Example 3.6, shows that the equality $\mathfrak{D}\phi(\hat{x};\nu) = \{\phi^D(\hat{x};\nu)\}$ is not necessarily true for a locally Lipschitz real-valued function ϕ. In fact, for the mentioned function f, we have $\mathfrak{D}f(0;1) = [0, \frac{1}{3}]$ and $f^D(0;1) = \frac{1}{3}$. Note that for a real-valued function $\phi : \mathcal{X} \to \mathbb{R}$, Remark 2.6 concludes that

$$\phi^D(\hat{x},\nu) = \max\{y \mid y \in \mathfrak{D}\phi(\hat{x};\nu)\}, \quad \forall \nu \in \mathcal{X}. \tag{3.3}$$

Also, if $\phi : \mathcal{X} \to \mathbb{R}$ is Dini directionally derivable at $\hat{x} \in \mathcal{X}$, the equality $\mathfrak{D}\phi(\hat{x};\nu) = \{\phi^D(\hat{x};\nu)\}$ holds by Remark 2.6.

As we know, if $|T| = p$ (hence $\mathcal{Y} = \mathbb{R}^p$) and $\psi := (\psi_1, ..., \psi_p) : \mathcal{X} \to \mathbb{R}^p$ is Dini directionally derivable at $\hat{x} \in \mathcal{X}$ in the direction $\nu \in \mathcal{X}$, then

$$\nabla\psi(\hat{x};\nu) = (\nabla\psi_1(\hat{x};\nu), \ldots, \nabla\psi_p(\hat{x};\nu)).$$

Since the above equality can be written in the form of

$$\mathfrak{D}\psi(\hat{x};\nu) = \mathfrak{D}\psi_1(\hat{x};\nu) \times \ldots, \times\mathfrak{D}\psi_p(\hat{x};\nu),$$

it is natural to ask the question "If $|T| = p$ and the locally Lipschitz function $\psi : \mathcal{X} \to \mathcal{Y}$ is not Dini directionally derivable at $\hat{x} \in \mathcal{X}$, is the above equality still true or not?". The following theorem shows that the answer to this question is "yes", even when T is infinite.

Theorem 3.8. Suppose that $\psi : \mathcal{X} \to \mathcal{Y}$ and $\hat{x} \in \mathcal{X}$ are given. Then, for all $\nu \in \mathcal{X}$, one has:

$$\mathfrak{D}\psi(\hat{x};\nu)(t) = \mathfrak{D}\psi_t(\hat{x};\nu), \quad \forall t \in T,$$

where $\mathfrak{D}\psi(\hat{x};\nu)(t)$ is defined as follows

$$\mathfrak{D}\psi(\hat{x};\nu)(t) = \{y(t) \mid y \in \mathfrak{D}\psi(\hat{x};\nu)\}.$$

Proof . By the definition, $y(t) \in \mathfrak{D}\psi(\hat{x};\nu)(t)$ if and only if $y \in \mathfrak{D}\psi(\hat{x};\nu)$, i.e., $y = \lim_{n\to\infty} \frac{\psi(\hat{x} + h_n\nu) - \psi(\hat{x})}{h_n}$, for some sequence $\{h_n\} \downarrow 0$. This is equivalent to

$$\begin{aligned} y(t) &= \left(\lim_{n\to\infty} \frac{\psi(\hat{x} + h_n\nu) - \psi(\hat{x})}{h_n}\right)(t) = \lim_{n\to\infty} \frac{\psi(\hat{x} + h_n\nu)(t) - \psi(\hat{x})(t)}{h_n} \\ &= \lim_{n\to\infty} \frac{\psi_t(\hat{x} + h_n\nu) - \psi_t(\hat{x})}{h_n} \in \mathfrak{D}\psi_t(\hat{x},\nu), \end{aligned}$$

as required. □

Theorem 3.9. Let $\hat{x} \in S$ be a local weakly efficient solution for (VP) and ψ be Lipschitz around $\hat{x}$. Then, for each $\nu \in \Gamma_C(S, \hat{x})$ we can find some $t \in T$ such that $\psi_t^D(\hat{x}; \nu) \geq 0$. Equivalently,

$$\{\nu \in \mathcal{X} \mid \psi_t^D(\hat{x}; \nu) < 0, \ \forall t \in T\} \cap \Gamma_C(S, \hat{x}) = \emptyset.$$

Proof . Suppose that $\nu \in \Gamma_C(S, \hat{x})$ is given. Owing to Theorem 3.4, we deduce that

$$\mathfrak{D}\psi(\hat{x}; \nu) \subseteq \mathcal{Y} \setminus (-int(\mathcal{Y}^+)) = \{y \in \mathcal{Y} \mid \exists t \in T, \ y(t) \geq 0\}.$$

This means that for each $y \in \mathfrak{D}\psi(\hat{x}; \nu)$ there exist some $t \in T$ with $y(t) \geq 0$. From this and Theorem 3.8 we deduce that $\mathfrak{D}\psi_t(\hat{x}; \nu) \cap [0, +\infty) \neq \emptyset$, for some $t \in T$. From this and (3.3) we conclude that $\psi_t^D(\hat{x}; \nu) \geq 0$ for some $t \in T$, and the proof is complete. □

The following Theorem is a generalization of [7, Theorem 3.4, Step 1] to infinite T.

Theorem 3.10. Let $\hat{x}$ be a local weakly efficient solution for (VP) and ψ be Lipschitz around $\hat{x}$. Then, $\psi_t : \mathcal{X} \to \mathbb{R}$ is Lipschitz around $\hat{x}$ for all $t \in T$, and

$$\Big(\bigcup_{t \in T} \partial_C \psi_t(\hat{x})\Big)^{\prec} \cap \Gamma_C(S, \hat{x}) = \emptyset.$$

Proof . Since ψ is Lipschitz around $\hat{x}$, there exist some neighborhood U of $\hat{x}$ and some constant $L_U > 0$ such that

$$\|\psi(y) - \psi(x)\| \leq L_U \|x - y\|, \quad \forall x, y \in U.$$

This inequality and the definition of $\|\psi(y) - \psi(x)\|$ imply

$$\max\Big\{\big|\overbrace{\psi(y)(t)}^{\psi_t(y)} - \overbrace{\psi(x)(t)}^{\psi_t(x)}\big| \mid t \in T\Big\} \leq L_U \|x - y\|, \quad \forall x, y \in U.$$

Hence, for all $t \in T$ we have

$$|\psi_t(y) - \psi_t(x)| \leq L_U \|x - y\|, \quad \forall x, y \in U,$$

which concludes that ψ_t is Lipschitz around $\hat{x}$. Since $\hat{x}$ is a local weakly efficient solution of (VP), by Theorem 3.9 we have

$$\{\nu \in \mathcal{X} \mid \psi_t^D(\hat{x}; \nu) < 0, \ \forall t \in T\} \cap \Gamma_C(S, \hat{x}) = \emptyset.$$

This equality and (2.5) deduce that

$$\{\nu \in \mathcal{X} \mid \psi_t^C(\hat{x}; \nu) < 0, \ \forall t \in T\} \cap \Gamma_C(S, \hat{x}) = \emptyset, \tag{3.4}$$

which by Theorem 2.5(vii) implies that

$$\{\nu \in \mathcal{X} \mid \langle \xi_t, \nu \rangle < 0, \ \forall \xi_t \in \partial_C \psi_t(\hat{x}), \forall t \in T\} \cap \Gamma_C(S, \hat{x}) = \emptyset.$$

This relation and the fact that

$$\Big\{\nu \in \mathcal{X} \mid \langle \xi_t, \nu \rangle < 0, \ \forall \xi_t \in \partial_C \psi_t(\hat{x}), \forall t \in T\Big\} = \Big\{\nu \in \mathcal{X} \mid \langle \xi, \nu \rangle < 0, \ \forall \xi \in \bigcup_{t \in T} \partial_C \psi_t(\hat{x})\Big\} = \Big(\bigcup_{t \in T} \partial_C \psi_t(\hat{x})\Big)^{\prec},$$

conclude that

$$\Big(\bigcup_{t \in T} \partial_C \psi_t(\hat{x})\Big)^{\prec} \cap \Gamma_C(S, \hat{x}) = \emptyset,$$

as required. □

The following definition is required for stating the KKT type necessary optimality condition for the problem (VP).

Definition 3.11. We say that the Abadie constraint qualification holds at $\hat{x} \in S$ if

$$\Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)^{\preceq} \subseteq \Gamma_C(S,\hat{x}),$$

where $J(\hat{x})$ denotes the set of active constraints at $\hat{x}$, i.e.,

$$J(\hat{x}) := \big\{j \in J \mid g_j(\hat{x}) = 0\big\}.$$

Note the above definition is an extension of Abadie constraint qualification, presented in [9] (resp. [4], [11, 12]) for linear (resp. convex, nonsmooth) multiobjective semi-infinite optimization problems. Now, we can present the KKT type necessary optimality condition at a local weakly efficient solution of (VP).

Theorem 3.12. Assume that $\hat{x} \in S$ is a local weakly efficient solution for (VP), ψ and g_j as $j \in J$ are Lipschitz around $\hat{x}$, and the Abadie constraint qualification holds at $\hat{x}$. Then, we can find some nonnegative scalars λ_t as $t \in T$ and μ_j as $j \in J(\hat{x})$, finite number of them are nonzero, such that $\sum_{t\in T} \lambda_t = 1$ and

$$0_{\mathcal{X}^*} \in cl^{w^*}\Big(\sum_{t\in T} \lambda_t \partial_C \psi_t(\hat{x}) + \sum_{j\in J(\hat{x})} \mu_j \partial_C g_j(\hat{x})\Big).$$

Proof . At the first, we claim that

$$h(\nu) \geq 0, \quad \forall \nu \in \Gamma_C(S,\hat{x}), \tag{3.5}$$

where the convex function $h : \mathcal{X} \to \mathbb{R}$ is defined as

$$h(\nu) := \max\{\psi_t^C(\hat{x};\nu) \mid t \in T\}.$$

Suppose, on the contrary, that there is a $\hat{\nu} \in \Gamma_C(S,\hat{x})$ with $h(\hat{\nu}) < 0$. So, $\psi_t^C(\hat{x};\hat{\nu}) < 0$ for all $t \in T$, and hence

$$\hat{\nu} \in \big\{\nu \in \mathcal{X} \mid \psi_t^C(\hat{x},\nu) < 0, \quad \forall t \in T\big\} \cap \Gamma_C(S,\hat{x}),$$

which contradicts (3.4). Thus, (3.5) is true, and the Abadie constraint qualification implies that

$$h(\nu) \geq 0, \qquad \forall \nu \in \Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)^{\preceq}.$$

The above inequality and $h(0_{\mathcal{X}}) = 0$ conclude that h attaints its minimum on the convex set $\Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)^{\preceq}$ at $\nu^* = 0_{\mathcal{X}}$, and by (2.3) and Theorem 2.5 we get

$$0_{\mathcal{X}^*} \in \partial h(0_{\mathcal{X}}) + N\bigg(\Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)^{\preceq}, 0_{\mathcal{X}}\bigg) = \partial h(0_{\mathcal{X}}) + cl^{w^*}\bigg(cone\Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)\bigg), \tag{3.6}$$

where the last equality holds by bipolar Theorem 2.3 and

$$N\bigg(\Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)^{\preceq}, 0_{\mathcal{X}}\bigg) = \bigg\{\nu \in \mathcal{X} \mid \langle \xi,\nu\rangle \leq 0,\ \forall \xi \in \Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)^{\preceq}\bigg\} = \bigg(\Big(\bigcup_{j\in J(\hat{x}} \partial_C g_j(\hat{x})\Big)^{\preceq}\bigg)^{\preceq}.$$

On the other hand, the well-known Pshenichnyi-Levin-Valadire Theorem ([6, pp. 267] and Theorem 2.5(vi) conclude that

$$\partial h(0_{\mathcal{X}}) \subseteq cl^{w^*}\bigg(conv\Big(\bigcup_{t\in T_*} \partial \psi_t^C(\hat{x};\cdot)(0_{\mathcal{X}})\Big)\bigg) = cl^{w^*}\bigg(conv\Big(\bigcup_{t\in T_*} \partial_C \psi_t(\hat{x})\Big)\bigg), \tag{3.7}$$

where T_* is defined as

$$T_* := \{t \in T \mid h(0_{\mathcal{X}}) = \psi_t^C(\hat{x};0_{\mathcal{X}}) = 0\}.$$

Owing to (3.6) and (3.7), we obtain that

$$0_{\mathcal{X}^*} \in cl^{w^*}\Bigg(conv\Big(\bigcup_{t\in T_*} \partial_C\psi_t(\hat{x})\Big)\Bigg) + cl^{w^*}\Bigg(cone\Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)\Bigg).$$

On the other hand, $conv\Big(\bigcup_{t\in T_*} \partial_C\psi_t(\hat{x})\Big)$ is norm bounded by [16, Lemma 4.1]. Thus, the above inclusion and Theorem 2.2 conclude that

$$0_{\mathcal{X}^*} \in cl^{w^*}\Bigg(conv\Big(\bigcup_{t\in T_*} \partial_C\psi_t(\hat{x})\Big) + cone\Big(\bigcup_{j\in J(x)} \partial_C g_j(\hat{x})\Big)\Bigg).$$

Now, according to Theorem 2.1, we can find some nonnegative scalars λ_t as $t \in T$ and μ_j as $j \in J(\hat{x})$, finite number of them are nonzero, such that $\sum_{t\in T} \lambda_t = 1$ and

$$0_{\mathcal{X}^*} \in cl^{w^*}\Big(\sum_{t\in T} \lambda_t\partial_C\psi_t(\hat{x}) + \sum_{j\in J(\hat{x})} \mu_j\partial_C g_j(\hat{x})\Big),$$

and the proof is complete. □

It should be noted that Theorem 3.12 is a generalization of the KKT necessary condition, presented in [1, 4, 8, 9, 10, 11, 12]. Note that the equality $\sum_{t\in T} \lambda_t = 1$ in Theorem 3.12 implies $\lambda_t > 0$ for some $t \in T$. It is noteworthy that if we have $\lambda_t > 0$ for all $t \in T$, the sum $\sum_{t\in T} \lambda_t\partial_C\psi_t(\hat{x})$ becomes meaningless, except when $|T| < \infty$. To get the result $\lambda_t > 0$ for all $t \in T := \{1, \ldots, |T|\}$, we need the following definition.

Definition 3.13. We say that the refined Abadie constraint qualification holds at $\hat{x} \in S$ if

$$\Big(\bigcup_{t\in T\setminus\{t_0\}} \partial_C\psi_t(\hat{x})\Big)^{\preceq} \cap \Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)^{\preceq} \subseteq \Gamma_C(Q_{t_0}, \hat{x}), \quad \forall t_0 \in T,$$

where

$$Q_{t_0} := \Big\{x \in \mathcal{X} \mid g_j(x) \le 0,\ \psi_t(x) \le 0,\ \forall j \in J,\ \forall t \in T \setminus \{t_0\}\Big\}.$$

Now, we can present the SKKT type necessary optimality condition at a local efficient solution of (VP).

Theorem 3.14. Assume that $\hat{x} \in S$ is a local efficient solution for (VP), ψ is Lipschitz around $\hat{x}$, $|T| < \infty$, and the refined Abadie constraint qualification holds at $\hat{x}$. Then, there exist some positive numbers $\lambda_t > 0$ as $t \in T$ and some negative scalars μ_j as $j \in J(\hat{x})$, finite number of them are nonzero, such that

$$0_{\mathcal{X}_*} \in cl^{w^*}\Big(\sum_{t\in T} \lambda_t\partial_C\psi_t(\hat{x}) + \sum_{j\in J(\hat{x})} \mu_j\partial_C g_j(\hat{x})\Big).$$

Proof . We claim that for all $t_0 \in T$, $\hat{x}$ is a local minimizer of the following problem:

$$\begin{array}{llll} (P_{t_0}): & \min & \psi_{t_0}(x) & \\ & s.t. & g_j(x) \le 0, & j \in J, \\ & & \psi_t(x) \le 0, & t \in T \setminus \{t_0\}. \end{array}$$

On the contrary, suppose that there exist some $t_0 \in T$ such that $\hat{x}$ is not minimizer of (P_{t_0}). Thus, there are some neighborhood U of $\hat{x}$ and some $x^* \in U$ such that

$$\begin{cases} \psi_{t_0}(x^*) < \psi_{t_0}(\hat{x}), & \\ \psi_t(x^*) \le \psi_t(\hat{x}), & t \in T \setminus \{t_0\} \\ g_j(x^*) \le 0, & j \in J \end{cases} \implies \begin{cases} x^* \in S \cap U, & \\ \psi_t(x^*) \le \psi_t(\hat{x}), & \forall t \in T, \\ \psi_{t_0}(x^*) < \psi_{t_0}(\hat{x}), & \exists t_0 \in T, \end{cases}$$

which contradicts Lemma 3.2. Thus, our claim is proved. Since the refined Abadie constraint qualification holds at $\hat{x}$, we have

$$\Big(\bigcup_{t\in T\setminus\{t_0\}} \partial_C\psi_t(\hat{x})\Big)^{\preceq} \cap \Big(\bigcup_{j\in J(\hat{x})} \partial_C g_j(\hat{x})\Big)^{\preceq} \subseteq \Gamma_C(Q_{t_0},\hat{x}).$$

Since the feasible set of (P_{t_0}) is Q_{t_0}, the above inclusion concludes that the Abadie constraint qualification holds at $\hat{x}$ for (P_{t_0}), and thus, Theorem 3.12 implies that there exist some nonnegative $\lambda_t^{(t_0)}$ as $t \in T\setminus\{t_0\}$ and $\mu_j^{(t_0)}$ as $j \in J(\hat{x})$ such that

$$0_{\mathcal{X}_*} \in cl^{w^*}\Big(\partial_C\psi_{t_0}(\hat{x}) + \sum_{t\in T\setminus\{t_0\}} \lambda_t^{(t_0)}\partial_C\psi_t(\hat{x}) + \sum_{j\in J(\hat{x})} \mu_j^{(t_0)}\partial_C g_j(\hat{x})\Big).$$

By repeating the above inclusion for each $t_0 \in T = \{1,\cdots,|T|\}$ and summing them, and considering [16, Lemma 4.1] and Theorem 2.2, we obtain that

$$\begin{aligned} 0_{\mathcal{X}_*} &\in cl^{w^*}\Big(\sum_{t_0\in T}\Big(\partial_C\psi_{t_0}(\hat{x}) + \sum_{t\in T\setminus\{t_0\}} \lambda_t^{(t_0)}\partial_C\psi_t(\hat{x}) + \sum_{j\in J(\hat{x})} \mu_j^{(t_0)}\partial_C g_j(\hat{x})\Big)\Big) \\ &= cl^{w^*}\Big(\sum_{t\in T}\lambda_t\partial_C\psi_t(\hat{x}) + \sum_{j\in J(\hat{x})}\mu_j\partial_C g_j(\hat{x})\Big), \end{aligned}$$

where

$$\lambda_k := 1 + \sum_{t\in T\setminus\{k\}} \lambda_k^{(t)} > 0, \quad \forall k \in T \qquad \textit{and} \qquad \mu_r := \sum_{t\in T}\mu_r^{(t)} \geq 0, \quad \forall j \in J.$$

The result is proved. □

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