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Mean-field reflected BSDEs with infinite horizon and applications

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Abstract

We establish the existence and uniqueness of solutions to a mean-field reflected backward stochastic differential equation with an infinite horizon under a Lipschitz condition on the coefficient. As an application, we prove the existence of an optimal strategy for the mean-field mixed stochastic control problem.

Keywords: Reflected BSDEs, Mean-field, Infinite horizon, mixed stochastic control problem, Hamiltonian
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1 Introduction

Let $(B_t)_{t\geq 0}$ be a standard d-dimensional Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{F} = \{\mathcal{F}_s,\ s \geq 0\}$ be the filtration generated by the process (B_s) augmented with the $\mathbb{P}$-null sets of $\mathcal{F}$, and $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. We denote $\mathcal{S}^2(0,\infty,\mathbb{R}) :=$ the space of continuous, $\mathcal{F}_t$-adapted processes φ such that

$$\mathbb{E} \sup_{0\leq t\leq \infty} |\varphi_t|^2 < +\infty.$$

$\mathcal{M}^2(0,\infty,\mathbb{R}^d) :=$ the space of $\mathcal{F}_t$-progressively measurable processes φ satisfying

$$\mathbb{E} \int_0^\infty |\varphi_t|^2 \, dt < +\infty.$$

$L^2 :=$ the space of $\mathcal{F}_\infty-$measurable random variable ξ s.t. satisfying $\mathbb{E}\,|\xi|^2 < +\infty$.

$\mathcal{T}_t :=$ the space of $\mathcal{F}_t$-stopping time v such that $\mathbb{P}-a.s.\ v \geq t$.

$\mathcal{P} :=$ the σ-algebra of progressively measurable subsets of $[0,\infty) \times \Omega$.

Let $\mathcal{B}^2$ be the Banach space of processes (Y,Z) with values in $\mathbb{R}^{1+d}$ such that $Y \in \mathcal{S}^2, Z \in \mathcal{M}^2$ and $\|(Y,Z)\|_{\mathcal{B}^2} := \left(\|Y\|^2_{\mathcal{S}^2} + \|Z\|^2_{\mathcal{M}^2}\right)^{1/2}$.

The space $\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}\right) := (\Omega\times\Omega, \mathcal{F}\otimes\mathcal{F}, \mathbb{P}\otimes\mathbb{P})$ is endowed with the filtration $\overline{\mathbb{F}}=\left\{\overline{\mathcal{F}}_t = \mathcal{F}\otimes\mathcal{F}_t,\ t\geq 0\right\}$.

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The random variable $\xi \in L^0(\Omega,\mathcal{F},\mathbb{P},\mathbb{R})$ is canonically extended to $\overline{\Omega}$ by putting $\xi'(\omega',\omega) := \xi(\omega)$ for $(\omega',\omega) \in \overline{\Omega}$. For any $\theta \in L^1\left(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbb{P}}\right)$, the variable $\theta(.,\omega) : \Omega \to \mathbb{R}$ belongs to $L^1(\Omega,\mathcal{F},\mathbb{P})$ $\mathbb{P}(d\omega) - a.s.$ We denote its expectation by

$$\mathbb{E}'[\theta(.,\omega)] = \int_\Omega \theta(\omega',\omega)\mathbb{P}(d\omega').$$

Note that $\mathbb{E}'[\theta] := \mathbb{E}'[\theta(.,\omega)] \in L^1(\Omega,\mathcal{F},\mathbb{P})$ and

$$\overline{\mathbb{E}}[\theta] = \int_\Omega \theta d\overline{\mathbb{P}} = \int_\Omega \mathbb{E}'[\theta(.,\omega)]\,\mathbb{P}(d\omega) = \mathbb{E}[\mathbb{E}'[\theta]].$$

We consider the following assumption:

(H1)–**(i)** f is a mapping from $\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}$ such that

(a) there exist two positive deterministic functions $u_1(t)$ and $u_2(t)$ such that for any (y'_i, z'_i, y_i, z_i) in $\mathbb{R}^{1+d+1+d}$, $(i = 1,2)$ and any $t \in [0,\ \infty]$, we have

$$|f(t,y'_1,z'_1,y_1,z_1) - f(t,y'_2,z'_2,y_2,z_2)| \le u_1(t)\left(|y'_1 - y'_2| + |y_1 - y_2|\right) + u_2(t)\left(|z'_1 - z'_2| + |z_1 - z_2|\right).$$

(b) $\mathbb{E}(\int_0^\infty |f(s,0,0,0,0)|\,ds)^2 < \infty$.

(c) $\int_0^\infty u_1(t)dt < \infty$, $\int_0^\infty u_2^2(t)dt < \infty$.

(ii) ξ is an $\mathcal{F}_\infty-$measurable random variable ξ s.t. satisfying $\mathbb{E}|\xi|^2 < +\infty$.

(iii) $S \in \mathcal{S}^2$ and $S_\infty \le \xi$ $a.s.$

Definition 1.1. A solution of a mean-field reflected backward stochastic differential equation (MFRBSDE), with the data (f,ξ,S), is an $(\mathcal{F}_t)$-adapted process $(Y,Z,K) := (Y_t,Z_t,K_t)_{t\ge 0}$ which satisfies the following equation

$$\begin{cases} \textbf{(i)}\ Y \in \mathcal{S}^2, Z \in \mathcal{M}^2, K_\infty \in L^2, \\ \textbf{(ii)}\ Y_t = \xi + \int_t^{+\infty} \mathbb{E}'[f(s,Y'_s,Z'_s,Y_s,Z_s)]\,ds + K_\infty - K_t - \int_t^{+\infty} Z_s dB_s, \qquad t \ge 0, \\ \textbf{(iii)}\ \forall t \ge 0, S_t \le Y_t. \\ \textbf{(iv)}\ K_t \text{ is continuous and increasing, } K_0 = 0 \text{ and } \int_0^{+\infty}(Y_t - S_t)dK_t = 0. \end{cases} \tag{1.1}$$

where

$$\begin{aligned}\mathbb{E}'[f(s,Y'_s,Z'_s,Y_s,Z_s)](\omega) &= \mathbb{E}'[f(s,Y'_s,Z'_s,Y_s(\omega),Z_s(\omega))] \\ &= \int_\Omega f(\omega',\omega,s,Y_s(\omega'),Z_s(\omega'),Y_s(\omega),Z_s(\omega))\mathbb{P}(d\omega')\end{aligned}$$

ξ is called the terminal value, f the generator (or coefficient) and S the barrier (or obstacle). In the sequel the previous equation will be labeled $eq(f,\xi,S)$ or $bsde(f,\xi,S)$.

The one barrier reflected backward stochastic differential equations (RBSDEs in short) were first studied by El Karoui et al. [4]. For these RBSDEs, that is for the equations of the form

$$\begin{cases} \textbf{(i)}\ Y \in \mathcal{S}^2, Z \in \mathcal{M}^2, K_T \in L^2. \\ \textbf{(ii)}\ Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \qquad 0 \le t \le T, \\ \textbf{(iii)}\ \forall t \in [0,T],\ S_t \le Y_t. \\ \textbf{(iv)}\ K_t \text{ is continuous and increasing, } K_0 = 0 \text{ and } \int_0^{+\infty}(Y_t - S_t)dK_t = 0. \end{cases} \tag{1.2}$$

In [4] the existence and uniqueness of solutions to (1.2) is proved under a square integrability of the terminal value ξ and a Lipschitz continuous condition on the generator f while the barrier S is assumed continuous and satisfies $\mathbb{E}\left[\sup(S_t^+)^2\right] < +\infty$. In Hamadène et al. [8], the existence and uniqueness of solutions to the one barrier RBSDEs with infinite horizon has been proved under the same assumptions in [4]. The authors of [12] have proved the existence and uniqueness of a class of BSDEs with mean-field type of the form

$$Y_t = \xi + \int_t^{+\infty} \mathbb{E}'[f(s,Y'_s,Z'_s,Y_s,Z_s)]\,ds - \int_t^{+\infty} Z_s dB_s, \qquad t \ge 0. \tag{1.3}$$

The mean-field BSDEs (1.3) were introduced in Buckdahn et al. [2] in the context of a finite horizon. In finite horizon, the existence of solutions to mean-field BSDEs (1.2) has been proved in Li and Luo [10] in the case where the terminal value ξ is square integrable, the generator f is uniformly Lipschitz in y, z. They also show the existence and uniqueness of solution when the data f, ξ and S satisfy assumption (**H1**). Our work distinguishes itself from previous studies by addressing reflected backward stochastic differential equations of mean-field type with double barriers and an infinite horizon.

The main result of this work consists of proving the existence and uniqueness of solutions for backward stochastic differential equations of mean-field type in infinite horizon. This result is an extension of the results stated in [8, 10]. As an application, we establish the existence of optimal strategy for the mean-field mixed control problem in infinite horizon.

Let us give a few explanations. Let ϕ and φ be bounded continuous functions, consider the following system

$$\left\{\begin{array}{l} dX_t = b(t, X_., \mathbb{E}^u\left[\phi(X_s)\right], u_t)dt + \sigma(t, X_.)dB_t^u. \\ X_0 = x \in \mathbb{R}^d, \qquad t \geq 0. \end{array}\right. \tag{1.4}$$

where $u \in U$ is an admissible control and

$$B_t^u = B_t - \int_0^t \sigma^{-1}(s, X_.)b(s, X_., \mathbb{E}^u\left[\phi(X_s)\right], u_s)ds, \qquad t \geq 0,$$

is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}^u)$. The probability measures $\mathbb{P}$ and $\mathbb{P}^u$ are equivalent on $(\Omega, \mathcal{F})$. The payoff associated to the system (1.4) and Mean-field RBSDE $eq(f, \xi, S)$ is defined by

$$J(u, \tau) = \mathbb{E}^u\left[\int_0^\tau h(s, X_., \mathbb{E}^u\left[\varphi(X_s)\right], u_s)ds + S_\tau 1_{\{\tau<+\infty\}} + \xi 1_{\{\tau=+\infty\}}\right], \tag{1.5}$$

where $\mathbb{E}^u$ denotes the expectation with respect to $\mathbb{P}^u$. The problem consists to find $(\widehat{u}, \widehat{\tau})$ such that

$$J(u, \tau) \leq J(\widehat{u}, \widehat{\tau}), \qquad \forall (u, \tau) \in U \times \mathcal{T} \tag{1.6}$$

where

$$\widehat{\tau} = \left\{\begin{array}{ll} \inf\{t \in [0, +\infty), Y_t^* \leq S_t\} & \\ +\infty, & \text{otherwise.} \end{array}\right.$$

Any $(\widehat{u}, \widehat{\tau}) \in U \times \mathcal{T}$ satisfying (1.6) is called optimal strategy or saddle-point strategy. Our work extends these of [8, 7] to the case of a mean equation.

The paper is organized as follows. In Section 2, we use the ideas developed in [8] and [10] to establish the existence and uniqueness of a solution (Y, Z, K) for an infinite horizon mean-field reflected backward stochastic differential equation. In Section 3, we prove the existence of an optimal strategy for the mean-field mixed control problem (1.5).

2 Mean-field reflected BSDE with infinite horizon

In this Section, we establish the existence of solutions (Y, Z, K) to the infinite horizon mean-field reflected BSDEs $eq(f, \xi, S)$ when assumption (H1) is satisfied. Let $(y, z) \in \mathcal{S}^2 \times \mathcal{M}^2$ and consider the mean-field reflected BSDEs

$$\left\{\begin{array}{l} (\mathbf{i})\ Y \in \mathcal{S}^2, Z \in \mathcal{M}^2, K_\infty \in L^2. \\ (\mathbf{ii})\ Y_t = \xi + \int_t^{+\infty} \mathbb{E}'\left[f(s, y_s', z_s', Y_s, Z_s)\right] ds + K_\infty - K_t - \int_t^{+\infty} Z_s dB_s, \qquad t \geq 0, \\ (\mathbf{iii})\ \forall t \geq 0, S_t \leq Y_t. \\ (\mathbf{iv})\ K_t \text{ is continuous and increasing, } K_0 = 0 \text{ and } \int_0^{+\infty}(Y_t - S_t)dK_t = 0. \end{array}\right. \tag{2.1}$$

where

$$\begin{aligned} \mathbb{E}'\left[f(s, y_s', z_s', Y_s, Z_s)\right](\omega) &= \mathbb{E}'\left[f(s, y_s', z_s', Y_s(\omega), Z_s(\omega))\right] \\ &= \int_\Omega f(\omega', \omega, s, y_s(\omega'), z_s(\omega'), Y_s(\omega), Z_s(\omega))\mathbb{P}(d\omega') \end{aligned}$$

Proposition 2.1. Assume that (H1) is satisfied. If $(y, z) \in \mathcal{S}^2 \times \mathcal{M}^2$, then there exists a unique solution (Y, Z, K) of the mean-field reflected BSDE (2.1) such that (Y, Z) belongs to $\mathcal{B}^2 \times \mathcal{B}^2$.

Proof . See [8] Theorem 3.2. □

The main result of this section is the following.

Theorem 2.2. Let assumption (H1) holds. Then, the mean-field RBSDEs $eq(f,\xi,S)$ with the data (f,ξ,S) has a unique solution (Y,Z,K).

Proof . Thanks to Proposition 2.1, we can introduce the mapping $\Psi\left(y^i_\cdot,z^i_\cdot\right):=\left(Y^i_\cdot,Z^i_\cdot\right):\mathcal{B}^2\to\mathcal{B}^2$, through the following equation

$$Y^i_t=\xi+\int_t^{+\infty}\mathbb{E}'\left[f(s,y^{i\prime}_s,z^{i\prime}_s,Y^i_s,Z^i_s)\right]ds+K^i_\infty-K^i_t-\int_t^{+\infty}Z^i_s dB_s,\qquad t\geq 0. \tag{2.2}$$

The proof is divided into three steps. In step 1, under $\left(\int_0^\infty u(t)dt\right)^2+\int_0^\infty u^2(t)dt<\frac{1}{72}$ we prove that the mapping Ψ is a contraction. In step 2, we use assumption (H1)-(i)-(c) to show that there exists a sufficiently large constant $T_0>0$ such that $\left(\int_{T_0}^\infty u_1(s)ds\right)^2+\int_{T_0}^\infty u_2^2(s)ds<\frac{1}{72}$. This shows that the infinite horizon mean-field reflected BSDEs $eq(f,\xi,S)$ with the data (f,ξ,S) has a unique. In step 3, we prove the uniqueness of the solution of $eq(f,\xi,S)$.

Step 1. Contraction of the map $\Psi(y,z):=(Y,Z)$. Assume that

$$L:=\left(\int_0^\infty u(t)dt\right)^2+\int_0^\infty u^2(t)dt<\frac{1}{72}.$$

Let (y^i,z^i), $(i=1,2)$ be two elements of $\mathcal{B}^2$. We denote

$$\begin{array}{lll}(\widehat{y},\widehat{z}) & := & \left(y^1-y^2,z^1-z^2\right)\\ \left(\widehat{Y},\widehat{Z},\widehat{K}\right) & := & \left(Y^1-Y^2,Z^1-Z^2,K^1-K^2\right)\\ \widehat{f} & := & f(s,y^1_s,z^1_s,Y^1_s,Z^1_s)-f(s,y^2_s,z^2_s,Y^2_s,Z^2_s).\end{array}$$

The square-integrable solution Y^i_t of the MFRBSDE (2.2) can be be represented as follows:

$$Y^i_t=ess\sup_{v\in\mathcal{T}_t}\mathbb{E}\left[\int_t^v\mathbb{E}'\left[f(s,y^{i\prime}_s,z^{i\prime}_s,Y^i_s,Z^i_s)\right]ds+S_v 1_{\{v<\infty\}}+\xi 1_{\{v=\infty\}}\mid\mathcal{F}_t\right]. \tag{2.3}$$

Doob's inequality shows that

$$\begin{array}{lll}\left\|\widehat{Y}\right\|_{\mathcal{S}^2} & = & \mathbb{E}\left[\sup_{t\geq0}\left|\widehat{Y}_t\right|^2\right]\\ & \leq & \mathbb{E}\left[\sup_{t\geq0}\left(\mathbb{E}\left[\int_0^{+\infty}\mathbb{E}'\left[\widehat{f}(s)\right]ds\mid\mathcal{F}_t\right]\right)^2\right]\\ & \leq & 4\mathbb{E}\left[\left(\int_0^{+\infty}\mathbb{E}'\left[\widehat{f}(s)\right]ds\right)^2\right].\end{array}$$

Using the fact that $\int_t^{+\infty}\widehat{Y}_s d\widehat{K}_s\leq 0$, then Itô's formula applied to $\left|\widehat{Y}_t\right|^2$ shows that

$$\begin{array}{lll}\left|\widehat{Y}_t\right|^2+\int_t^{+\infty}\left|\widehat{Z}_s\right|^2ds & = & 2\int_t^{+\infty}\widehat{Y}_s\mathbb{E}'\left[\widehat{f}(s)\right]ds+2\int_t^{+\infty}\widehat{Y}_s d\widehat{K}_s-2\int_t^{+\infty}\widehat{Y}_s\widehat{Z}_s ds\\ & \leq & 2\int_t^{+\infty}\widehat{Y}_s\mathbb{E}'\left[\widehat{f}(s)\right]ds-2\int_t^{+\infty}\widehat{Y}_s\widehat{Z}_s dB_s.\end{array}$$

Hence,

$$\begin{aligned}\left\|\widehat{Z}\right\|^2_{\mathcal{M}^2} &= \mathbb{E}\left[\int_0^{+\infty}\left|\widehat{Z}_s\right|^2 ds\right]\\ &\leq 2\mathbb{E}\left[\int_0^{+\infty}\widehat{Y}_s\mathbb{E}'\left[\widehat{f}(s)\right]ds\right]\\ &\leq 5\mathbb{E}\left[\left(\int_0^{+\infty}\mathbb{E}'\left[\widehat{f}(s)\right]ds\right)^2\right].\end{aligned}$$

The Lipschitz condition allows to show that

$$\mathbb{E}\left[\left(\int_0^{+\infty}\mathbb{E}'\left[\widehat{f}(s)\right]ds\right)^2\right] \leq 4L\left[\|(\widehat{y},\widehat{z})\|^2_{\mathcal{B}} + \left\|(\widehat{Y},\widehat{Z})\right\|^2_{\mathcal{B}}\right].$$

Hence,

$$\begin{aligned}\left\|(\widehat{Y},\widehat{Z})\right\|^2_{\mathcal{B}^2} &= \left\|\widehat{Y}\right\|^2_{\mathcal{S}^2} + \left\|\widehat{Z}\right\|^2_{\mathcal{M}^2}\\ &\leq 9\mathbb{E}\left[\left(\int_0^{+\infty}\mathbb{E}'\left[\widehat{f}(s)\right]ds\right)^2\right]\\ &\leq 36L\left[\|(\widehat{y},\widehat{z})\|^2_{\mathcal{B}^2} + \left\|(\widehat{Y},\widehat{Z})\right\|^2_{\mathcal{B}^2}\right].\end{aligned}$$

Thus

$$\left\|(\widehat{Y},\widehat{Z})\right\|^2_{\mathcal{B}^2} \leq \Gamma^2\,\|(\widehat{y},\widehat{z})\|^2_{\mathcal{B}^2}$$

where

$$\Gamma^2 = \frac{36\left[\left(\int_0^\infty u_1(s)ds\right)^2 + \left(\int_0^\infty u_2^2(s)ds.\right)\right]}{1-36\left[\left(\int_0^\infty u_1(s)ds\right)^2 + \left(\int_0^\infty u_2^2(s)ds.\right)\right]}.$$

Therefore, using inequality $\left(\int_0^\infty u(t)dt\right)^2 + \int_0^\infty u^2(t)dt < \frac{1}{72}$, it follows that Ψ is a strict contraction. Hence, Ψ has a unique fixed point $(Y,Z)\in\mathcal{B}^2$ such that $(Y,Z)=\Phi(Y,Z)$. This shows the existence and uniqueness of (Y,Z). The existence and uniqueness of the process K are deduced from that of (Y,Z).

Step 2. For the general case.

It follows, from *Step 1*, that equation (2.1) has a unique solution $Y,Z,K)$ such that $(Y,Z)\in\mathcal{B}^2$. Using assumption (H1)-(i)-(c), one can show that there exists a sufficiently large constant $T_0>0$ such that,

$$\left(\int_{T_0}^\infty u_1(s)ds\right)^2 + \int_{T_0}^\infty u_2^2(s)ds < \frac{1}{72}.$$

Now, we respectively consider the following infinite horizon and finite horizon mean-field reflected BSDE,

$$\overline{Y}_t = \xi + \int_t^{+\infty} 1_{\{s\geq T_0\}}\mathbb{E}'\left[f(s,\overline{Y}'_s,\overline{Z}'_s,Y_s,Z_s)\right]ds + \overline{K}_\infty - \overline{K}_t - \int_t^{+\infty}\overline{Z}_s dB_s, \quad t\geq 0, \tag{2.4}$$

and

$$\widetilde{Y}_t = \overline{Y}_t + \int_t^{T_0}\mathbb{E}'\left[f(s,\widetilde{Y}'_s,\widetilde{Z}'_s,\widetilde{Y}_s,\widetilde{Z}_s)\right]ds + \widetilde{K}_{T_0} - \widetilde{K}_t - \int_t^{T_0}\widetilde{Z}_s dB_s, \quad t\in[0,T_0]. \tag{2.5}$$

According to *Step 1*, the infinite horizon mean-field reflected BSDE (2.4) has a unique solution $(\overline{Y}_t,\overline{Z}_t,\overline{K}_t)$. Moreover, from Li and Luo [10] Theorem 3.1, there exists a unique solution $(\widehat{Y}_t,\widehat{Z}_t,\widetilde{K}_t)$ satisfying the finite horizon mean-field reflected BSDE (2.5).

Thanks to *Step 1* and *Step 2*, the infinite horizon mean-field reflected BSDEs $eq(f,\xi,S)$ has a unique solution (Y,Z,K). From the uniqueness it follows that

$$Y_t = \begin{cases} \widetilde{Y}_t, & t \in [0,T_0] \\ \overline{Y}_t, & t \in [T_0,\infty] \end{cases} \quad ; \qquad Z_t = \begin{cases} \widetilde{Z}_t, & t \in [0,T_0] \\ \overline{Z}_t, & t \in [T_0,\infty] \end{cases}$$

$$K_t = \begin{cases} \widetilde{K}_t, & t \in [0,T_0] \\ \widetilde{K}_{T_0} - \overline{K}_t - \overline{K}_{T_0}, & t \in [T_0,\infty] \end{cases}$$

Theorem 2.2 is proved. □

3 Mean-field mixed control problem with infinite horizon

In this section, we study the Mean-field mixed control problem with infinite horizon using mean-field reflected BSDEs $eq(f,\xi,S)$. We shall give a description of the problem. In the sequel $\Omega = \mathcal{C}([0,\infty[,\mathbb{R}^d)$ is the space of continuous functions from $[0,\infty[$ to $\mathbb{R}^d$. Let $X_0 = x \in \mathbb{R}^d$ and $X = (X_t)_{t\geq 0}$ be the solution of the following standard functional differential equation

$$\begin{cases} dX_t = \sigma(t,X_.)dB_t \\ X_0 = x \in \mathbb{R}^d, \quad t \geq 0. \end{cases} \tag{3.1}$$

We consider the following assumptions:

(**H2**) The coefficient $\sigma : (t,\zeta) \in [0,\infty[\times\Omega \to \sigma(t,\zeta) \in \mathbb{R}^d \otimes \mathbb{R}^d$ satisfy

(i) σ is $\mathcal{P}$-measurable.

(ii) For any $t \in [0,+\infty[$ and $\zeta \in \Omega$, $\sigma(t,\zeta)$ is invertible and $\sigma^{-1}(t,\zeta)$ is bounded.

(iii) There exist $K > 0$ such that $|\sigma(t,\zeta) - \sigma(t,\zeta')| \leq K\left\|\zeta - \zeta'\right\|_t$ and $|\sigma(t,\zeta)| \leq K(1 + \|\zeta\|_t)$, where for any $(\zeta,\zeta') \in \Omega$ and any $t \geq 0$, $\|\zeta\|_t = \sup_{s\leq t}|\zeta_s|$.

Remark 3.1. Let assumption (**H2**). holds. Then, according to Revuz and Yor [11] pp. 375, the standard functional differential equation (3.1) has a unique solution.

Let $\mathcal{A}$ be a compact metric space. Let $\mathcal{U}$ be the set of all $\mathcal{P}$-measurable processes $u := (u_t)_{t\geq 0}$ with values in $\mathcal{A}$. Let b be a function from $[0,+\infty[\times\Omega\times\mathbb{R}^d \times \mathcal{A}$ into $\mathbb{R}^d$ satisfy the following conditions:

(**H3**)

(i) b is a Borel measurable functions.

(ii) For any $t \in [0,+\infty)$ and $(\zeta,x) \in \Omega\times\mathbb{R}^d$, $b(t,\zeta,x,.)$ is continuous on $\mathcal{A}$.

(iii) There exists a deterministic function $C(t)$ satisfying $\int_0^{+\infty} C^2(t)dt < +\infty$ such that

$$|b(t,\zeta,x,u)| \leq C(t), \quad \text{for } a.s.\omega \text{ and any } \ t \geq 0, \ (\zeta,x) \in \Omega\times\mathbb{R}^d, \ u \in \mathcal{A}.$$

Let ϕ be a bounded continuous function from $\mathbb{R}^d$ onto $\mathbb{R}^d$. For each $u \in \mathcal{U}$, we define a probability $\mathbb{P}^u$ on $(\Omega,\mathcal{F})$ by

$$\frac{d\mathbb{P}^u}{d\mathbb{P}} = \exp\left\{\int_0^{+\infty} \sigma^{-1}(s,X_.)b(s,X_.,\mathbb{E}^u\left[\phi(X_s)\right],u_s)dB_s - \frac{1}{2}\int_0^{+\infty} \left|\sigma^{-1}(s,X_.)b(s,X_.,\mathbb{E}^u\left[\phi(X_s)\right],u_s)\right|^2 ds\right\},$$

where $\mathbb{E}^u$ denotes the expectation with respect to $\mathbb{P}^u$. Using assumptions (**H2**), (**H3**) and Girsanov's theorem (see [9], [11]), we see that the process

$$B_t^u = B_t - \int_0^t \sigma^{-1}(s,X_.)b(s,X_.,\mathbb{E}^u\left[\phi(X_s)\right],u_s)ds, \quad t \geq 0,$$

is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}^u)$ and X is a weak solution of the following mean-field SDE

$$\begin{cases} dX_t = b(t, X_., \mathbb{E}^u[\phi(X_s)], u_t)dt + \sigma(t, X_.)dB_t^u, \\ X_0 = x \in \mathbb{R}^d, \quad t \geq 0. \end{cases} \tag{3.2}$$

In the present section, suppose that we have a system (3.2), whose evolution is described by process X, which has an effect on the wealth of a controller. On the other hand the controller has no influence on the system (3.2). For instance, the process X may represent the price of an asset on the market and the controller a small share holder or a small investor. Here $\mathcal{U}$ is the set of admissible controls, the controller acts to protect his advantages by means of $u \in \mathcal{U}$ via the probability $\mathbb{P}^u$. On the other hand he has also the possibility at any time $\tau \in \mathcal{T}$ to stop controlling. The control is not free.

(**H4**) The function $h : [0, \infty[\times \Omega \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}$ satisfies assumption (**H3**).

The payoff functional $J(u, \tau)$, $(u, \tau) \in \mathcal{U} \times \mathcal{T}$, associated with the controlled mean-field SDE (3.2) is

$$J(u, \tau) = \mathbb{E}^u\left[\int_0^\tau h(s, X_., \mathbb{E}^u[\varphi(X_s)], u_s)ds + S_\tau 1_{\{\tau < +\infty\}} + \xi 1_{\{\tau = +\infty\}}\right] \tag{3.3}$$

where φ be a bounded continuous function from $\mathbb{R}^d$ to $\mathbb{R}^d$, S and ξ are the same as in assumption (**H1**).

In this controller, the coefficients $h(t, X_., \mathbb{E}^u[\varphi(X_t)], u)$ is the instantaneous reward, S and ξ are respectively the rewards if he decides to stop before or until infinite time. The problem is to look for an optimal strategy for the controller i.e. a strategy $(\widehat{u}, \widehat{\tau})$ such that

$$J(u, \tau) \leq J(\widehat{u}, \widehat{\tau}), \quad \forall (u, \tau) \in \mathcal{U} \times \mathcal{T}$$

We define the usual Hamiltonian function associated with this mean-field mixed stochastic control problem $H : [0, +\infty[\times \Omega \times \mathbb{M}_1(\mathbb{R}^d) \times \mathcal{A} \to \mathbb{R}$, by

$$H(t, X_., \mu, p, u) := pb\left(t, X_., \int \phi d\mu, p, u\right) + h\left(t, X_., \int \varphi d\mu, p, u\right), \tag{3.4}$$

where $\mathbb{M}_1(\mathbb{R}^d)$ denotes the space of probability measures in $\mathbb{R}^d$ and μ_t is the marginal probability distribution of X_t under the probability measure $\mathbb{P}^u$. To ease the notation, we use the following notation for the Hamiltonian

$$H(t, X_., p, u) := pb(t, X_., \mathbb{E}^u[\phi(X_t)], p, u) + h(t, X_., \mathbb{E}^u[\varphi(X_t)], p, u)$$

The following lemma on the existence $u^* = u^*(t, X_., \mu, p)$ which maximize the Hamiltonian H, has been obtained by Benes [1] (see also [3]).

Lemma 3.2. There exists a Borel measurable function $u^* : [0, +\infty[\times \Omega \times \mathbb{M}_1(\mathbb{R}^d) \to \mathcal{A}$ such that

$$H(t, X_., p, u^*) = \max_{u \in U}\left(pb(t, X_., \mathbb{E}^u[\phi(X_t)], p, u) + h(t, X_., \mathbb{E}^u[\varphi(X_t)], p, u)\right)$$

Remark 3.3. Under assumption (**H3**) − (**iii**), $H(t, X_., p, u)$ satisfies the Lipschitz condition in p. Then, the function $H(t, X_., p, u^*)$ also satisfies the Lipschitz assumption (**H3**) − (**iii**) in p.

The main result of this section is given by the following theorem.

Theorem 3.4. Assume that assumption (**H1**) − (**H4**) are satisfied. Let (Y^*, Z^*, K^*) be the solution of the infinite horizon mean-field reflected BSDEs, with the data $(H(t, X_., Z^*, u^*), \xi, S)$, let

$$\widehat{\tau} = \begin{cases} \inf\{t \in [0, +\infty), Y_t^* \leq S_t\} \\ +\infty, \quad \text{otherwise}. \end{cases} \tag{3.5}$$

Then $Y_0^* = J(u^*, \widehat{\tau})$ and $(u^*, \widehat{\tau})$ is an optimal strategy for the mean-field mixed stochastic game problem (3.3).

Proof . According to Theorem 2.2, it is clear that under assumption (**H1**) – (**H4**), the infinite horizon mean-field reflected BSDEs, with the data $(H(t,X_{.},Z^*,u^*),\xi,S)$ has a unique solution (Y^*,Z^*,K^*). Then for any $t\geq 0$ we have,

$$Y_t^* = \xi + \int_t^{+\infty} H(t,X_{.},Z^*,u^*)ds + K_\infty^* - K_t^* - \int_t^{+\infty} Z_s^* dB_s, \qquad t\geq 0, \tag{3.6}$$

Since Y_0^* is deterministic, then we can show that

$$\begin{aligned} Y_0^* &= \mathbb{E}^{u^*}\left[Y_0^*\right] \\ &= \mathbb{E}^{u^*}\left[\xi + \int_0^{+\infty} H(t,X_{.},Z_s^*,u^*)ds + K_\infty^* - \int_0^{+\infty} Z_s^* dB_s\right] \\ &= \mathbb{E}^{u^*}\left[Y_{\widehat{\tau}}^* + \int_0^{\widehat{\tau}} H(t,X_{.},Z_s^*,u^*)ds + K_{\widehat{\tau}}^* - \int_0^{\widehat{\tau}} Z_s^* dB_s\right] \\ &= \mathbb{E}^{u^*}\left[Y_{\widehat{\tau}}^* + \int_0^{\widehat{\tau}} h(s,X_{.},\mathbb{E}^{u^*}\left[\varphi(X_s)\right],u^*)ds + K_{\widehat{\tau}}^* - \int_0^{\widehat{\tau}} Z_s^* dB_s^{u^*}\right] \end{aligned}$$

Assumption (**H3**) and Burkholder Davis-Gundy's inequality allows to show that

$$\mathbb{E}^{u^*}\left[\sup_{t\geq 0}\left|\int_0^t Z_s^* dB_s^{u^*}\right|\right] \leq C\left(\mathbb{E}\left[\left(\frac{d\mathbb{P}^u}{d\mathbb{P}}\right)^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left(\int_0^\infty Z_s^* ds\right)^2\right]\right)^{\frac{1}{2}}$$

Since $\mathbb{E}\left[\left(\frac{d\mathbb{P}^u}{d\mathbb{P}}\right)^2\right] < \infty$, $\mathbb{E}^{u^*}\left[\sup_{t\geq 0}\left|\int_0^t Z_s^* dB_s^{u^*}\right|\right] < \infty$. Therefore $(\int_0^t Z_t^* dB_t^{u^*}, t\in[0,+\infty])$ is a $\mathbb{P}^{u^*}$-martingale. Moreover, from (3.5) and the properties of reflected BSDEs, it is easy to see that the process $K_{\widehat{\tau}}^*$ does not increase between 0 and $\widehat{\tau}$ and hence $K_{\widehat{\tau}}^* = 0$. It follows that

$$Y_0^* = \mathbb{E}^{u^*}\left[\int_0^{\widehat{\tau}} h(s,X_{.},\mathbb{E}^{u^*}\left[\varphi(X_s)\right],u^*)ds + Y_{\widehat{\tau}}^*\right].$$

Since $Y_{\widehat{\tau}}^* = S_{\widehat{\tau}}1_{\{\widehat{\tau}<+\infty\}} + \xi 1_{\{\widehat{\tau}=+\infty\}}$ $\mathbb{P}^{u^*}$-$a.s$, it holds that

$$Y_0^* = J(u^*,\widehat{\tau}).$$

On the other hand, let u be an admissible control and τ be a stopping time. Since $\mathbb{P}$ and $\mathbb{P}^u$ are equivalent probabilities on $(\Omega,\mathcal{F})$, we have

$$\begin{aligned} Y_0^* &= \mathbb{E}^{u}\left[Y_0^*\right] \\ &= \mathbb{E}^{u}\left[\xi + \int_0^{+\infty} H(t,X_{.},Z_s^*,u^*)ds + K_\infty^* - \int_0^{+\infty} Z_s^* dB_s\right] \\ &= \mathbb{E}^{u}\left[Y_\tau^* + \int_0^{\tau} H(t,X_{.},Z_s^*,u^*)ds + K_\tau^* - \int_0^{\tau} Z_s^* dB_s\right] \\ &= \mathbb{E}^{u}\left[Y_\tau^* + \int_0^{\tau} h(s,X_{.},\mathbb{E}^{u}\left[\varphi(X_s)\right],u)ds + K_\tau^* - \int_0^{\tau} Z_s^* dB_s^u + \int_0^\tau H(t,X_{.},Z^*,u^*) - H(t,X_{.},Z^*,u)ds\right]. \end{aligned}$$

Since the terms $Y_\tau^* \geq S_\tau 1_{\{\tau<+\infty\}} + \xi 1_{\{\tau=+\infty\}}$, K_τ^* and $\int_0^\tau H(t,X_{.},Z^*,u^*) - H(t,X_{.},Z,u)$ are positive $\mathbb{P}^u$-$a.s.$, $(\int_0^t Z_t^* dB_t^u, t\in[0,+\infty])$ is a $\mathbb{P}^u$-martingale, the following result holds

$$J(u^*,\widehat{\tau}) = Y_0^* \geq \mathbb{E}^u\left[\int_0^\tau h(s,X_{.},\mathbb{E}^u\left[\varphi(X_s)\right],u)ds + K_\tau^* + Y_\tau^* - \int_0^\tau Z_s^* dB_s^u\right] = J(u,\tau).$$

It follows that the control $(u^*,\widehat{\tau})$ is an optimal strategy for the mean-field mixed stochastic game problem (3.3).
□

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