Three weak solutions for a (p,q)-Schrödinger-Kirchhoff type equation

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(Communicated by Abdolrahman Razani)

Abstract

In this manuscript, we investigate a (p,q)-Schrödinger-Kirchhoff equation involving a continuous positive potential that meets the del Pino-Felmer type conditions. Using Recceri's classical variational approach, we prove the existence of three weak solutions.

Keywords: Double phase problems, Recceri's vartianonel principle, Nonlocal Schrödinger-Kirchhoff type equation.
2020 MSC: Primary 35Q55, Secondary 35J60, 35D30, 35J20

1 Introduction

Our objective in this work is to discuss the existence of three weak solutions for Schrödinger double phase Kirchhoff-type elliptic equation involving potential nonlinearities. The approach is based on a technical strategy that employs the critical points theorem developed by B. Ricceri, which is documented in [35].

More precisely, we have studied the following equation:

$$K_1\Big(\int_{\mathbb{R}^N}\frac{1}{p}|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N}\mathcal{V}(x)|u|^p dx\Big)\Big(-\Delta_p u + \mathcal{V}(x)|u|^{p-2}u\Big)$$
$$+ K_2\Big(\int_{\mathbb{R}^N}\frac{1}{q}|\nabla u|^q dx + \frac{1}{q}\int_{\mathbb{R}^N}\mathcal{V}(x)|u|^q dx\Big)\Big(-\Delta_q u + \mathcal{V}(x)|u|^{q-2}u\Big)$$
$$= \alpha f_1(x,u) + \beta f_2(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $1 < p < q < N < 2q$, $\Delta_s u = \operatorname{div}(|\nabla u|^{s-2}\nabla u)$, with $s \in \{p,q\}$, is the s-Laplacian, α, β are two real parameters. We suppose that $\mathcal{V}: \mathbb{R}^N \longrightarrow \mathbb{R}$ be a continuous function that adheres to the conditions elucidated by del Pino and Felmer in their reference [11]]:

(V1)

$$\mathcal{V} \in \mathcal{C}(\mathbb{R}^N), \quad \inf_{\mathbb{R}^N}\mathcal{V}(x) \geq \mathcal{V}_0 > 0 \text{ and } \operatorname{meas}\Big\{x \in \mathbb{R}^N : \mathcal{V}(x) \leq M\Big\} < \infty, \text{ for all } M > 0,$$

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and let K_1, $K_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are a continuous function and satisfies the conditions :

(K1)

$$k_1 t^{\sigma_1 - 1} \leq K_1(t) \quad \text{for all} \quad t > 0,$$

where k_1 and σ_1 are positive constants such that $\sigma_1 \in \left(1, \frac{q^*}{q}\right)$.

(K2)

$$k_2 t^{\sigma_2 - 1} \leq K_2(t) \quad \text{for all} \quad t > 0,$$

where k_2 and σ_2 are positive constants such that $\sigma_2 \in \left(1, \frac{q^*}{q}\right)$.

Definition 1.1. Let $1 < z < \infty$ with $z < N$. For any Carathéodory function $f_1 : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ such that, we denote the class $\mathcal{A}$ as

$$\sup_{(x,t) \in \mathbb{R}^N \times \mathbb{R}} \frac{|f_1(x,t)|}{|t|^{s-1}} < \infty,$$

for all $s \in [p, q^*]$.

If $f_1 \in \mathcal{A}$, we assume that:

(F1)

$$\sup_{u \in W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)} \int_{\mathbb{R}^N} F_1(x,u)dx > 0,$$

(F2)

$$\limsup_{t \to 0} \frac{\sup_{x \in \mathbb{R}^N} F_1(x,t)}{|t|^{q \max(\sigma_1, \sigma_2)}} = 0,$$

(F3)

$$\limsup_{t \to \infty} \frac{\sup_{x \in \mathbb{R}^N} F_1(x,t)}{|t|^{p \min(\sigma_1, \sigma_2)}} = 0.$$

The operator defined in (1.1) is used in many branches of mathematics, including calculus of variations and PEDs. It has also been applied in a wide range of physical and engineering contexts, including fluid filtration in porous media, optimal control, image processing, elastoplasticity, financial mathematics, and elsewhere, for more details, see [40, 41, 42] and the references therein. The Kirchhoff-type problem was primarily introduced in [18] to generalise the classical D'Alembert wave equation for free vibrations of elastic strings. Some interesting research by variational methods can be found in [6, 10, 20, 23] for Kirchhoff-type problems. More precisely, Kirchhoff introduced a famous equation defined as

$$r\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

that it is related to the problem (1.1). In (1.2), L is the length of the string, h is the area of the cross-section, P_0 is the initial tension, r is the mass density, and E is the Young modulus of the material. See the paper [5, 15, 27] for more details.

In recent research, specifically when considering the functions $K_1(t) = 1 + at$ and $K_2(t) = 1 + bt$, alongside the simplifications $f_1(x, u) = f(u)$ and $f_2(x, u) \equiv 0$, where a and b are both strongly positive, W. Zhang, J. Zuo, and P. Zhao, as documented in [39], made significant advancements. They successfully demonstrated the existence of multiple positive solutions to the following problem:

$$-\left(1 + a\int_{\mathbb{R}^N} |\nabla u|^p dx\right)\Delta_p u - \left(1 + b\int_{\mathbb{R}^N} |\nabla u|^q dx\right)\Delta_q u + \mathcal{V}_\varepsilon(x)\left(|u|^{p-2}u + |u|^{q-2}u\right) = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

In a similar vein, V. Ambrosio and T. Isernia, as discussed in [3], also conducted a comprehensive study to establish the existence of multiple solutions for problem (1.3). These groundbreaking contributions have significantly enriched the understanding of this complex problem.

When $K_1 \equiv K_2 \equiv 1$, the expression in (1.1) transforms into a (p, q)-Schrödinger equation structured as

$$-\Delta_p u - \Delta_q u + \mathcal{V}(x)\Big(|u|^{p-2}u + |u|^{q-2}u\Big) = \alpha f_1(x, u) + \beta f_2(x, u) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

It is worth noting that recent years have seen several researchers establish a variety of results related to (1.4), including existence, multiplicity, and regularity. In [16], some regularity outcomes were established. Furthermore, [17] introduced an existence result by combining the concentration-compactness principle and the mountain pass theorem. Another work by [12] demonstrated the existence of a positive ground-state solution for a (p, q)-Schrödinger equation with critical growth. Multiple solutions and the concentration of nontrivial solutions for (1.4) were addressed in [2]. The authors in [22] employed refined variational methods grounded in critical point theory, along with Morse theory and truncation techniques, to establish multiplicity outcomes for a (p, q)-Laplacian problem within bounded domains. For additional insights, interested readers are encouraged to explore [4, 7, 9, 21, 28, 29, 30, 31], along with the references provided therein.

When $K_1 \equiv K_2 \equiv 1$ and $p = q$, (1.4) boils down a p-Schrödinger equation of the type

$$-\Delta_p u + \mathcal{V}(x)\Big(|u|^{p-2}u\Big) = \alpha f_1(x, u) + \beta f_2(x, u) \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

It is important to stress that the world of literature abounds with a variety of captivating and diverse findings. A comprehensive array of these results is available in sources such as [13]. Furthermore, it's essential to note that when $p = q = 2$. Equation (1.5) transforms into the widely studied Schrödinger equation, a subject that has garnered extensive attention over the past three decades. For a deeper exploration, readers are encouraged to examine works like [1, 14, 24, 25, 26].

It is important to note that Numerous studies have employed variational techniques and, by extension, Ricceri's principle to tackle a wide array of mathematical problems. These methodologies have been applied to Schrödinger double phase Kirchhoff-type elliptic equations with potential nonlinearities, among other equations, to ascertain the existence of weak solutions. The literature demonstrates that variational Ricceri's principle, along with its associated techniques, offers a robust framework for investigating critical points and establishing the existence of multiple solutions in a variety of mathematical settings. In this context and in our endeavor of establishing the presence of weak solutions for our central problem, we draw upon a theorem originally formulated by B. Ricceri in [35]. This theorem will serve as a key tool in our approach. For a more in-depth exploration of this theorem, we recommend referring to the following sources: [32, 33, 34, 37].

Motivated by the above research, we have successfully demonstrated the existence of three solutions to problem (1.1). What sets this study apart is our comprehensive approach to the Kirchhoff functions, K_1 and K_2, which extends beyond the scope of prior research, as exemplified in works like [3] and [39]. Furthermore, our approach yields entirely distinct results. This complexity is further amplified by the inherent lack of compactness in the embedding $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \subset L^s(\mathbb{R}^N)$ for $s \in [p, q^*]$ (as elucidated in section 2). These factors collectively present formidable challenges when applying Theorem 2.6, which stands as our primary analytical tool. In response to these challenges, we've embraced a fusion of classical and contemporary techniques. This amalgamation becomes especially apparent as we navigate the intricacies, as illustrated in Lemma 3.5, 3.6, and 3.8.

Our major result is the following theorems.

Theorem 1.2. Assuming that **(F1)**-**(F3)**, **(V1)**, **(K1)**, and **(K2)** are satisfied, thus, for any compact interval $\Lambda \subset (\Theta^*, \infty)$, there exists a positive number $\eta > 0$ with the following property: for any $\alpha \in \Lambda$ and any $f_2 \in \mathcal{A}$, there exists a positive constant $\beta^* > 0$ such that, for each $\beta \in [0, \beta^*]$, equation (1.1) admits at least three weak solutions with norms smaller than η.

The paper is organized as follows: In Section 2, we collect the main definitions and properties of Sobolev spaces. In section 3, we give the proof of the main results, and it also includes remarks and illustrative examples.

2 Some back results

Let z be a real such that $1 < z < N < \infty$. We denote by z' the dual exponent of z, i.e., $z' = \frac{z}{z-1}$, while z^* is the Sobolev dual in dimension N, namely

$$z^* := \frac{Nz}{N-z}.$$

The term $\mathcal{D}^{1,z}(\mathbb{R}^N)$ designates the set of functions obtained by taking the closure of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ functions, while considering the associated norm

$$|\nabla u|_z^z = \int_{\mathbb{R}^N} |\nabla u|^z dx.$$

By $W^{1,z}(\mathbb{R}^N)$, We refer to the Sobolev space, which is equipped with the following norm

$$\|u\|_{W^{1,z}(\mathbb{R}^N)} = (|u|_z^z + |\nabla u|_z^z)^{\frac{1}{z}}.$$

Theorem 2.1. (See [36]). Assume that $1 < z < N$. $W^{1,z}(\mathbb{R}^N)$ is continuously embedded in $L^t(\mathbb{R}^N)$ for any $t \in [z, z^*]$ and compactly in $L_{loc}^t(\mathbb{R}^N)$ for any $t \in [1, z^*)$.

Let $1 < p < q < N < 2q$ and set

$$W^{p,q}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),$$

endowed with the norm

$$\|u\|_{W^{p,q}(\mathbb{R}^N)} = \|u\|_{W^{1,p}(\mathbb{R}^N)} + \|u\|_{W^{1,q}(\mathbb{R}^N)}.$$

We introduce the space

$$W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) = \left\{ u \in W^{p,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \mathcal{V}(x)(|u|^p + |u|^q)dx < \infty \right\}.$$

Characterized by the norm

$$\|u\|_{W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)} = \|u\|_{\mathcal{V},p,q} = \|u\|_{\mathcal{V},p} + \|u\|_{\mathcal{V},q},$$

where

$$\|u\|_{\mathcal{V},r} = \left(|\nabla u|_r^r + \int_{\mathbb{R}^N} \mathcal{V}(x)|u|^r dx \right)^{\frac{1}{r}} \quad \text{for each} \quad r \in \{p, q\}.$$

Theorem 2.2. (See [36]). $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ is a separable reflexive Banach space.

Recall that the following embedding results.

Proposition 2.3. (See [39]). The space $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ is continuously embedded into $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. As a result $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ is continuously embedded in $L^s(\mathbb{R}^N)$ for any $s \in [p, q^*]$ and compactly embedded in $L_{loc}^s(\mathbb{R}^N)$ for any $s \in [1, q^*)$. Moreover, if $\mathcal{V}_\infty = \infty$, the embedding $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \subset L^s(\mathbb{R}^N)$ is compact for any $p \le s < q^*$.

Proposition 2.4. (Algebraic inequality)(See [19]). Let $\alpha \in (0,1)$, for any real numbers $a > 0, b > 0$, $c > 0$ and $d > 0$, we have

$$(a+b)^\alpha (c+d)^{1-\alpha} \ge a^\alpha c^{1-\alpha} + b^\alpha d^{1-\alpha}.$$

Definition 2.5. Consider a real Banach space X. We define $\mathcal{W}_X$ as the set of functionals $A : X \longrightarrow \mathbb{R}$ with the following characteristic: if $(u_n)_n$ is a sequence in X that converges weakly to $u \in X$, and $\liminf_{n\to\infty} A(u_n) \le A(u)$, then there exists a subsequence $(u_n)_n$ that converges strongly to u.

The following theorem plays an important role in this paper.

Theorem 2.6. (See [35]). Consider X as a separable and reflexive real Banach space, with the norm denoted by $\| \cdot \|$. Let $\Psi_1 : X \longrightarrow \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous $\mathcal{C}^1$ functional that falls under the class $\mathcal{W}_X$. Additionally, Ψ_1 is bounded on each bounded subset of X and has a derivative with a continuous inverse on X^*. Let $\Psi_2 : X \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1$ functional with a compact derivative.

Suppose that Ψ_1 possesses a strict local minimum at x_0, satisfying $\Psi_1(x_0) = \Psi_2(x_0) = 0$. Finally, we assume that:

$$\max \left\{ \limsup_{\|x\| \to +\infty} \frac{\Psi_2(x)}{\Psi_1(x)}, \limsup_{x \to x_0} \frac{\Psi_2(x)}{\Psi_1(x)} \right\} \leq 0,$$

and that

$$\sup_{x \in X} \min\{\Psi_1(x), \Psi_2(x)\} > 0.$$

Let

$$\Theta^* := \inf \left\{ \frac{\Psi_1(x)}{\Psi_2(x)} : x \in X, \min(\Psi_1(x), \Psi_2(x)) > 0 \right\}.$$

Hence, for every compact interval $\mathbf{\Lambda} \subset (\Theta^*, +\infty)$, there exists a positive number η such that the following holds: for any α within $\mathbf{\Lambda}$ and any $\mathcal{C}^1$ functional $\Psi_3 : X \longrightarrow \mathbb{R}$ with a compact derivative, there is a β^* greater than zero such that for each β in the interval $[0, \beta^*]$, the equation

$$\Psi_1'(x) = \alpha \Psi_2'(x) + \beta \Psi_3'(x), \tag{2.1}$$

has at least three solutions with norms smaller than η.

3 Proof of the main results

We introduce the functionals $\Psi_1, \Psi_2 : W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \longrightarrow \mathbb{R}$ by

$$\Psi_1(u) = \widehat{K}_1 \left(\varphi_{p,\mathcal{V}}(u) \right) + \widehat{K}_2 \left(\varphi_{q,\mathcal{V}}(u) \right), \tag{3.1}$$

and

$$\Psi_2(u) = \int_{\mathbb{R}^N} F_1(x, u) dx, \tag{3.2}$$

where

$$\varphi_{p,\mathcal{V}}(u) = \int_{\mathbb{R}^N} \frac{1}{p} \left(|\nabla u|^p + \mathcal{V}(x)|u|^p \right) dx, \quad \varphi_{q,\mathcal{V}}(u) = \int_{\mathbb{R}^N} \frac{1}{q} \left(|\nabla u|^q + \mathcal{V}(x)|u|^q \right) dx, \tag{3.3}$$

$$\widehat{K}_1(r) = \int_0^r K_1(s) ds, \quad \widehat{K}_2(r) = \int_0^r K_2(s) ds \quad \forall r \geq 0, \tag{3.4}$$

and

$$F_1(x, t) = \int_0^t f_1(x, s) ds \quad \text{for all } (x, t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.5}$$

Under the hypotheses **(F1)-(F3)**, we set

$$\Theta^* = \inf \left\{ \frac{\Psi_1(u)}{\Psi_2 u)} : u \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N), \int_{\mathbb{R}^N} F_1(x, u) dx > 0 \right\}.$$

Let us start by giving the definition of weak solution for the problem (1.1).

Definition 3.1. We say that $u \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ is a weak solution of problem (1.1), if

$$K_1 \left(\int_{\mathbb{R}^N} \frac{1}{p} \left(|\nabla u|^p + \mathcal{V}(x)|u|^p \right) dx \right) \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u \nabla v + \mathcal{V}(x)|u|^{p-2} uv \right) dx$$
$$+ K_2 \left(\int_{\mathbb{R}^N} \frac{1}{q} \left(|\nabla u|^q + \mathcal{V}(x)|u|^q \right) dx \right) \int_{\mathbb{R}^N} \left(|\nabla u|^{q-2} \nabla u \nabla v + \mathcal{V}(x)|u|^{q-2} uv \right) dx$$
$$= \alpha \int_\Omega f_1(x, u) v \, dx + \beta \int_\Omega f_2(x, u) v \, dx,$$

for all $v \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$.

The proof of Theorem 1.2 is methodically divided into three sequential steps.

Proof of Theorem 1.2

Step 1 : Technical lemmas

This subsection is dedicated to laying the groundwork by introducing essential technical lemmas that will find repeated application in this paper.

Lemma 3.2. Let $f_1 \in \mathcal{A}$, then the functional $\Psi_2 \in \mathcal{C}^1(W^{p,q}_{\mathcal{V}}(\mathbb{R}^N), \mathbb{R})$ with the derivative is given by

$$\langle \Psi'_2(u), v \rangle = \int_{\Omega} f_1(x, u)v \, dx, \tag{3.6}$$

for all $u, v \in W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$. In addition $\Psi'_2 : W^{p,q}_{\mathcal{V}}(\mathbb{R}^N) \longrightarrow \left(W^{p,q}_{\mathcal{V}}(\mathbb{R}^N) \right)^*$ is compact.

Proof . First, thanks to the definition of $\mathcal{A}$ and the embedding theorem, Ψ_2 is well dfined on $W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$. New we show that Ψ_2 is Gâteaux-differentiable on $W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$ with the derivative is given by (3.6). Actually, let $(u_n)_n \subset W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$ be a sequence converging strongly to $u \in W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$. Because $W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$ is compactly embedded in $L^s(\mathbb{R}^N)$, thus $(u_n)_n$ converges strongly to u in $L^s(\mathbb{R}^N)$. Consequently, there exists a subsequence of $(u_n)_n$, denoted as $(u_n)_n$, along with a function $\overline{u} \in L^s(\mathbb{R}^N)$. This subsequence converges to $\overline{u}$ almost everywhere in $\mathbb{R}^N$, satisfying $|u_n| \leq |\overline{u}|$ for every natural number n and nearly everywhere in $\mathbb{R}^N$. Given $f_1 \in \mathcal{A}$, for every measurable function $u : \mathbb{R}^N \longrightarrow \mathbb{R}$, the operator defined as $u \longmapsto f_1(\cdot, u(\cdot))$ maps $L^s(\mathbb{R}^N)$ into $L^{s'}(\mathbb{R}^N)$. Fix $v \in W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$ with $\|v\|_{W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)} \leq 1$, using Hölder's inequality and the continuous embedding of $W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$ into $L^s(\mathbb{R}^N)$, we have

$$
\begin{aligned}
\left| \langle \Psi'_2(u_n) - \Psi'_2(u), v \rangle \right| &= \left| \int_{\mathbb{R}^N} \Big(f_1(x, u_n(x)) - f_1(x, u(x)) \Big) v(x) dx \right| \\
&\leq \|f_1(x, u_n(x)) - f_1(x, u(x))\|_{L^{s'}(\mathbb{R}^N)} \|v\|_{L^s(\mathbb{R}^N)} \\
&\leq C_1 \|f_1(x, u_n(x)) - f_1(x, u(x))\|_{L^{s'}(\mathbb{R}^N)} \|v\|_{W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)},
\end{aligned}
$$

a certain positive constant C_1, we can pass to the supremum for $\|v\|_{W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)} \leq 1$, we get

$$\|\langle \Psi'_2(u_n) - \Psi'_2(u), v \rangle\|_{\left(W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)\right)^*} \leq \|f_1(x, u_n(x)) - f_1(x, u(x))\|_{L^{s'}(\mathbb{R}^N)}.$$

Since $f_1 \in \mathcal{A}$, we deduce

$$f_1(x, u_n(x)) - f_1(x, u(x)) \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

and

$$\left| f_1(x, u_n(x)) - f_1(x, u(x)) \right| \leq C_2 \Big(|\overline{u}(x)|^{s-1} + |u(x)|^{s-1} \Big).$$

Considering almost every $x \in \mathbb{R}^N$, it is worth noting that the majorant function in the previous relation lies in $L^{s'}(\mathbb{R}^N)$. Consequently, utilizing the dominated convergence theorem, we obtain that

$$\|f_1(x, u_n(x)) - f_1(x, u(x))\|_{s'} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

The continuity of Ψ'_2 is demonstrated. To establish the compactness of Ψ'_2, consider a sequence $(u_n)_n \subset W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$ that is bounded. Then there exists a subsequence of $(u_n)_n$ (still denoted by $(u_n)_n$) weakly convergent in $W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$ and strongly convergent in $L^s(\mathbb{R}^N)$. Using a similar argument as outlined above, we infer that the sequence $(\Psi'_2(u_n))_n$ exhibits strong convergence, and the compactness of the operator Ψ'_2 is established. $\square$

Lemma 3.3. Let **(V1)**, **(K1)** and **(K2)** hold true. Then the functional $\Psi_1 \in \mathcal{C}^1(W^{p,q}_{\mathcal{V}}(\mathbb{R}^N), \mathbb{R})$ and

$$
\begin{aligned}
\langle \Psi'_1(u), v \rangle &= K_1 \left(\int_{\mathbb{R}^N} \frac{1}{p} \Big(|\nabla u|^p + \mathcal{V}(x)|u|^p \Big) dx \right) \int_{\mathbb{R}^N} \Big(|\nabla u|^{p-2} \nabla u \nabla v + \mathcal{V}(x)|u|^{p-2} uv \Big) dx \\
&\quad + K_2 \left(\int_{\mathbb{R}^N} \frac{1}{q} \Big(|\nabla u|^q + \mathcal{V}(x)|u|^q \Big) dx \right) \int_{\mathbb{R}^N} \Big(|\nabla u|^{q-2} \nabla u \nabla v + \mathcal{V}(x)|u|^{q-2} uv \Big) dx,
\end{aligned} \tag{3.7}
$$

for each $u, v \in W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$. Morevoer, for all $u \in W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$, $\Psi'_1 \in \left(W^{p,q}_{\mathcal{V}}(\mathbb{R}^N) \right)^*$.

Proof . Initially, it is straightforward to observe that

$$\langle \Psi_1'(u), v \rangle = K_1 \left(\int_{\mathbb{R}^N} \frac{1}{p} \Big(|\nabla u|^p + \mathcal{V}(x)|u|^p \Big) dx \right) \int_{\mathbb{R}^N} \Big(|\nabla u|^{p-2} \nabla u \nabla v + \mathcal{V}(x)|u|^{p-2} uv \Big) dx$$

$$+ K_2 \left(\int_{\mathbb{R}^N} \frac{1}{q} \Big(|\nabla u|^q + \mathcal{V}(x)|u|^q \Big) dx \right) \int_{\mathbb{R}^N} \Big(|\nabla u|^{q-2} \nabla u \nabla v + \mathcal{V}(x)|u|^{q-2} uv \Big) dx,$$

for all $u, v \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$. It follows from (3.7) that for each $u \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$, $\Psi_1' \in \left(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \right)^*$.

Next, we prove that $\Psi_1' : W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \longrightarrow \left(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \right)^*$ is continuous. Let $(u_n)_n \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ with $u_n \longrightarrow u$ strongly in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$. We can assume, without sacrificing generality, that u_n converges to u almost everywhere in $\mathbb{R}^N$. Consequently, the sequences

$$(|\nabla u_n|^{p-2} \nabla u_n)_n \text{ is bounded in } L^{p'}(\mathbb{R}^N \times \mathbb{R}^N), \tag{3.8}$$

and

$$(|\nabla u_n|^{q-2} \nabla u_n)_n \text{ is bounded in } L^{q'}(\mathbb{R}^N \times \mathbb{R}^N), \tag{3.9}$$

as well as a.e. in $\mathbb{R}^N \times \mathbb{R}^N$,

$$(|\nabla u_n|^{p-2} \nabla u_n) \longrightarrow (|\nabla u|^{p-2} \nabla u),$$

and

$$(|\nabla u_n|^{q-2} \nabla u_n) \longrightarrow (|\nabla u|^{q-2} \nabla u).$$

Thus, the Brezis-Lieb lemma (see [3]) implies

$$\int_{\mathbb{R}^N} \lim_{n \to +\infty} \left| (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \right|^{p'} dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^p - |\nabla u|^p) dx,$$

and

$$\int_{\mathbb{R}^N} \lim_{n \to +\infty} \left| (|\nabla u_n|^{q-2} \nabla u_n - |\nabla u|^{q-2} \nabla u) \right|^{q'} dx = \lim_{n \to +\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^q - |\nabla u|^q) dx.$$

The strong convergence $u_n \longrightarrow u$ in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ implies that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^p - |\nabla u|^p) dx = 0, \tag{3.10}$$

and

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^q - |\nabla u|^q) dx = 0. \tag{3.11}$$

In a similar fashion,

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} \mathcal{V}(x) \left| |u_n(x)|^{p-2} u_n(x) - |u(x)|^{p-2} u(x) \right|^{p'} dx = 0, \tag{3.12}$$

and

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} \mathcal{V}(x) \left| |u_n(x)|^{q-2} u_n(x) - |u(x)|^{q-2} u(x) \right|^{q'} dx = 0. \tag{3.13}$$

Moreover, the continuity of K_1 and K_2 implies that

$$\lim_{n \to +\infty} K_1 \left(\int_{\mathbb{R}^N} \frac{1}{p} (|\nabla u_n|^p + \mathcal{V}(x)|u_n|^p) dx \right) \longrightarrow K_1 \left(\int_{\mathbb{R}^N} \frac{1}{p} (|\nabla u|^p + \mathcal{V}(x)|u|^p) dx \right), \tag{3.14}$$

and

$$\lim_{n \to +\infty} K_2 \left(\int_{\mathbb{R}^N} \frac{1}{q} (|\nabla u_n|^q + \mathcal{V}(x)|u_n|^q) dx \right) \longrightarrow K_2 \left(\int_{\mathbb{R}^N} \frac{1}{q} (|\nabla u|^q + \mathcal{V}(x)|u|^q) dx \right). \tag{3.15}$$

Utilizing Hölder's inequality along with the combination of (3.10)-(3.15), we get

$$\|\Psi_1(u_n) - \Psi_1(u)\|_{(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N))^*} = \sup_{u \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N), \|v\|_{\mathcal{V},p,q} \le 1} |\langle \Psi_1'(u_n) - \Psi_1'(u), v \rangle| \longrightarrow 0.$$

This completes the proof that $\Psi_1' : W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \longrightarrow \left(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \right)^*$ is continuous, and therefore $\Psi_1 \in \mathcal{C}^1(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N), \mathbb{R})$.
$\square$

Lemma 3.4. The functional Ψ_1 is sequentially weakly lower semicontinuous.

Proof . Consider a sequence $(u_n)_n$ such that $u_n \rightharpoonup u$ in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$. Due to the convexity of $\varphi_{p,\mathcal{V}}$ and $\varphi_{q,\mathcal{V}}$, for every n, it follows that

$$\varphi_{p,\mathcal{V}}(u) \leq \varphi_{p,\mathcal{V}}(u_n) + \langle \varphi'_{p,\mathcal{V}}(u), u - u_n \rangle,$$

and

$$\varphi_{q,\mathcal{V}}(u) \leq \varphi_{q,\mathcal{V}}(u_n) + \langle \varphi'_{q,\mathcal{V}}(u), u - u_n \rangle.$$

Taking the limit in the inequality above as n tends to infinity, we observe that $\varphi_{p,\mathcal{V}}$ and $\varphi_{q,\mathcal{V}}$ are sequentially weakly lower semi-continuous. Then we have

$$\int_{\mathbb{R}^N} \frac{1}{p} |\nabla u|^p dx \leq \liminf_{n\to+\infty} \int_{\Omega} \frac{1}{p} |\nabla u_n|^p dx, \tag{3.16}$$

$$\int_{\mathbb{R}^N} \frac{\mathcal{V}(x)}{p} |u|^p dx \leq \liminf_{n\to+\infty} \int_{\Omega} \frac{\mathcal{V}(x)}{p} |u_n|^p dx, \tag{3.17}$$

$$\int_{\mathbb{R}^N} \frac{1}{q} |\nabla u|^q dx \leq \liminf_{n\to+\infty} \int_{\Omega} \frac{1}{q} |\nabla u_n|^q dx, \tag{3.18}$$

and

$$\int_{\mathbb{R}^N} \frac{\mathcal{V}(x)}{q} |u|^q dx \leq \liminf_{n\to+\infty} \int_{\Omega} \frac{\mathcal{V}(x)}{q} |u_n|^q dx. \tag{3.19}$$

By (3.16)-(3.19) and since $\widehat{K}_1$ and $\widehat{K}_2$ are continuous and monotone, we have

$$
\begin{aligned}
\liminf_{n\to+\infty} \Psi_1(u_n) &= \liminf_{n\to+\infty} \widehat{K}_1 \left(\int_{\mathbb{R}^N} \frac{1}{p} \big(|\nabla u_n|^p + \mathcal{V}(x)|u_n|^p \big) dx \right) + \liminf_{n\to+\infty} \widehat{K}_2 \left(\int_{\mathbb{R}^N} \frac{1}{q} \big(|\nabla u_n|^q + \mathcal{V}(x)|u_n|^q \big) dx \right) \\
&\geq \widehat{K}_1 \left(\liminf_{n\to+\infty} \int_{\mathbb{R}^N} \frac{1}{p} \big(|\nabla u_n|^p + \mathcal{V}(x)|u_n|^p \big) dx \right) + \widehat{K}_2 \left(\liminf_{n\to+\infty} \int_{\mathbb{R}^N} \frac{1}{q} \big(|\nabla u_n|^q + \mathcal{V}(x)|u_n|^q \big) dx \right) \\
&\geq \widehat{K}_1 \left(\liminf_{n\to+\infty} \int_{\mathbb{R}^N} \frac{1}{p} |\nabla u_n|^p dx + \liminf_{n\to+\infty} \int_{\mathbb{R}^N} \frac{\mathcal{V}(x)}{p} |u_n|^p dx \right) \\
&\quad + \widehat{K}_2 \left(\liminf_{n\to+\infty} \int_{\mathbb{R}^N} \frac{1}{q} |\nabla u_n|^q dx + \liminf_{n\to+\infty} \int_{\mathbb{R}^N} \frac{\mathcal{V}(x)}{q} |u_n|^q dx \right) \\
&\geq \widehat{K}_1 \left(\int_{\mathbb{R}^N} \frac{1}{p} |\nabla u|^p dx + \int_{\mathbb{R}^N} \frac{\mathcal{V}(x)}{p} |u|^p dx \right) + \widehat{K}_2 \left(\int_{\mathbb{R}^N} \frac{1}{q} |\nabla u|^q dx + \int_{\mathbb{R}^N} \frac{\mathcal{V}(x)}{q} |u|^q dx \right) \\
&= \widehat{K}_1 \left(\int_{\mathbb{R}^N} \frac{1}{p} \big(|\nabla u|^p + \mathcal{V}(x)|u|^p \big) dx \right) + \widehat{K}_2 \left(\int_{\mathbb{R}^N} \frac{1}{q} \big(|\nabla u|^q + \mathcal{V}(x)|u|^q \big) dx \right), \\
&= \widehat{K}_1 \left(\varphi_{p,\mathcal{V}}(u) \right) + \widehat{K}_2 \left(\varphi_{q,\mathcal{V}}(u) \right) \\
&= \Psi_1(u), \tag{3.20}
\end{aligned}
$$

namely, Ψ_1 is sequentially weakly lower semi-continuous. $\square$

In the following lemma, we will demonstrate that the functional Ψ_1 belongs to the class $\mathcal{W}_X$, where $X = W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ as defined in Definition 2.5.

Lemma 3.5. The functional Ψ_1 belongs to the class $\mathcal{W}_{W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)}$.

Proof . Since $\widehat{K}_1$ and $\widehat{K}_2$ are continuous and strictly increasing, it suffices to show that $\varphi_{p,\mathcal{V}}, \varphi_{q,\mathcal{V}} \in \mathcal{W}_{W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)}$. Consider a sequence $(u_n)_n$ that converges weakly to u within the space $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ and let $\liminf_{n\to+\infty} \varphi_{p,\mathcal{V}}(u_n) \leq \varphi_{p,\mathcal{V}}(u)$ and $\liminf_{n\to+\infty} \varphi_{q,\mathcal{V}}(u_n) \leq \varphi_{q,\mathcal{V}}(u)$. Since the functional $\varphi_{p,\mathcal{V}}, \varphi_{q,\mathcal{V}}$ is sequentially weakly lower semicontinuous, then there exists a subsequence of $(u_n)_n$, bearing the same notation $(u_n)_n$ such that $\liminf_{n\to+\infty} \varphi_{p,\mathcal{V}}(u_n) \geq \varphi_{p,\mathcal{V}}(u)$ and $\liminf_{n\to+\infty} \varphi_{q,\mathcal{V}}(u_n) \geq \varphi_{q,\mathcal{V}}(u)$, so:

$$\liminf_{n\to+\infty} \varphi_{p,\mathcal{V}}(u_n) = \varphi_{p,\mathcal{V}}(u), \tag{3.21}$$

and

$$\liminf_{n\to+\infty} \varphi_{q,\mathcal{V}}(u_n) = \varphi_{q,\mathcal{V}}(u). \tag{3.22}$$

According to (3.21)-(3.22), we obtain

$$\lim_{n \to \infty} \Psi_1(u_n) = \Psi_1(u),$$

that is,

$$\lim_{n \to \infty} \|u_n\|_{\mathcal{V},p,q} = \|u\|_{\mathcal{V},p,q}.$$

Since $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ is uniformly convex of the space, we can deduce the strong convergence of u_n to u in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$. □

Lemma 3.6. (a) Let $u, v \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$, this implies that

$$\langle \varphi'_{p,\mathcal{V}}(u) - \varphi'_{p,\mathcal{V}}(u)'(v), u - v \rangle \geq \left(\|u\|_{\mathcal{V},p}^{p-1} - \|u\|_{\mathcal{V},p}^{p-1} \right) \left(\|u\|_{\mathcal{V},p} - \|u\|_{\mathcal{V},p} \right). \tag{3.23}$$

$$\langle \varphi'_{q,\mathcal{V}}(u) - \varphi'_{q,\mathcal{V}}(u)'(v), u - v \rangle \geq \left(\|u\|_{\mathcal{V},q}^{q-1} - \|u\|_{\mathcal{V},q}^{q-1} \right) \left(\|u\|_{\mathcal{V},q} - \|u\|_{\mathcal{V},q} \right). \tag{3.24}$$

(b) Suppose the sequence $(u_n)_n$ weakly converges to u in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$, and

$$\limsup_{n \to \infty} \langle \varphi'_{p,\mathcal{V}}(u_n), u_n - u \rangle \leq 0. \tag{3.25}$$

$$\limsup_{n \to \infty} \langle \varphi'_{q,\mathcal{V}}(u_n), u_n - u \rangle \leq 0. \tag{3.26}$$

As a result, the sequence $(u_n)_n$ strongly converges to u in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$.

Proof .

(a) For $u, v \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$, we have

$$\langle \varphi'_{p,\mathcal{V}}(u) - \varphi'_{p,\mathcal{V}}(u)'(v), u - v \rangle$$

$$= \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2} \nabla u + \mathcal{V}(x)|u|^{p-2}u \right) (u - v) \, dx - \int_{\mathbb{R}^N} \left(|\nabla v|^{p-2} \nabla v + \mathcal{V}(x)|v|^{p-2}v \right) (u - v) \, dx$$

$$= \|u\|_{\mathcal{V},p}^p + \|u\|_{\mathcal{V},p}^p - \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u.v \, dx - \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v.u \, dx - \int_{\mathbb{R}^N} \mathcal{V}(x)|u|^{p-2}uv \, dx - \int_{\mathbb{R}^N} \mathcal{V}(x)|v|^{p-2}vu \, dx.$$

The algebraic inequality Proposition 2.4, with $\alpha = \frac{p-1}{p}$ combining with Hölder inequality, allows us to deduce

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u.v \, dx + \int_{\mathbb{R}^N} \mathcal{V}(x)|u|^{p-2}uv \, dx \leq \|u\|_{\mathcal{V},p}^{p-1} \|v\|_{\mathcal{V},p},$$

and

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v.u \, dx + \int_{\mathbb{R}^N} \mathcal{V}(x)|v|^{p-2}v.u \, dx \leq \|v\|_{\mathcal{V},p}^{p-1} \|u\|_{\mathcal{V},p}.$$

Then we get (3.23). Following a similar line of thought, we get (3.24).

(b) Let $(u_n)_n$ converges weakly to u in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ such that

$$\limsup_{n \to \infty} \langle \varphi'_{p,\mathcal{V}}(u_n), u_n - u \rangle \leq 0, \text{ and } \limsup_{n \to \infty} \langle \varphi'_{q,\mathcal{V}}(u_n), u_n - u \rangle \leq 0. \tag{3.27}$$

Since $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ is reflexive Banach space, its isomorphic to a locally uniformly convex space. Then, all you need to prove the norm convergence, which is attainable by using the assertion (a) above with (3.27).

□

In this lemma, we demonstrate the invertibility and continuity of the energy Ψ'_1.

Lemma 3.7. Let **(V1)**, **(K1)** and **(K2)** hold true. Then the operator $\Psi'_1 : W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \longrightarrow \left(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \right)^*$ is, in fact, invertible, and $\Psi'_1{}^{-1}$ is notably continuous. continuous.

Proof . To begin, we make the initial assumption that the operator $\Psi_1' : W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \longrightarrow \left(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)\right)^*$ possesses invertibility within the space $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$. As stipulated by the Minty-Browder Theorem (refer to [3, 3]), our task is to demonstrate that Ψ_1' fulfills the criteria of being strictly monotone, hemicontinuous, and coercive in the context of monotone operators. With this in mind, consider u and v as elements of $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$, where $u \neq v$, and let μ_1 and μ_2 be in the interval $[0,1]$, with $\mu_1 + \mu_2 = 1$. Using (3.23) and (3.24) , we have $\varphi_{p,\mathcal{V}}', \varphi_{q,\mathcal{V}}'(u) : W_{\mathcal{V}}^{p,q}(\mathbb{R}^N) \longrightarrow \left(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)\right)^*$ are strictly monotone, so by [3], f is strictly convex. Moreover, since K_1 and K_2 are nondecreasing, the functions $\widehat{K}_1$ and $\widehat{K}_2$ are convex in $\mathbb{R}^+$, thus we have

$$
\begin{aligned}
\Psi_1(\mu_1 u + \mu_2 v) &= \widehat{K}_1\left(\varphi_{p,\mathcal{V}}(\mu_1 u + \mu_2 v)\right) + \widehat{K}_2\left(\varphi_{q,\mathcal{V}}(\mu_1 u + \mu_2 v)\right) \\
&< \widehat{K}_1\left(\mu_1 \varphi_{p,\mathcal{V}}(u) + \mu_2 \varphi_{p,\mathcal{V}}(v)\right) + \widehat{K}_2\left(\mu_1 \varphi_{q,\mathcal{V}}(u) + \mu_2 \varphi_{q,\mathcal{V}}(v)\right) \\
&\leq \mu_1 \widehat{K}_1\left(\varphi_{p,\mathcal{V}}(u)\right) + \mu_2 \widehat{K}_1\left(\varphi_{p,\mathcal{V}}(v)\right) + \mu_1 \widehat{K}_2\left(\varphi_{q,\mathcal{V}}(u)\right) + \mu_2 \widehat{K}_2\left(\varphi_{q,\mathcal{V}}(v)\right) \\
&\leq \mu_1\left(\widehat{K}_1\left(\varphi_{p,\mathcal{V}}(u)\right) + \widehat{K}_2\left(\varphi_{q,\mathcal{V}}(u)\right)\right) + \mu_2\left(\widehat{K}_1\left(\varphi_{p,\mathcal{V}}(v)\right) + \widehat{K}_2\left(\varphi_{q,\mathcal{V}}(v)\right)\right) \\
&\leq \mu_1 \Psi_1(u) + \mu_2 \Psi_1(v).
\end{aligned}
\tag{3.28}
$$

This shows that Ψ_1 is strictly convex and is already said, that Ψ_1' is strictly monotone. Let $\|u\|_{\mathcal{V},p} > 1$ and $\|u\|_{\mathcal{V},q} > 1$, by **(K1)** and **(K2)**, we have

$$
\begin{aligned}
\langle \Psi_1'(u), u \rangle &= K_1\left(\int_{\mathbb{R}^N} \frac{1}{p}\left(|\nabla u|^p + \mathcal{V}(x)|u|^p\right)dx\right)\int_{\mathbb{R}^N}\left(|\nabla u|^{p-2}\nabla u \nabla u + \mathcal{V}(x)|u|^{p-2}uu\right)dx \\
&\quad + K_2\left(\int_{\mathbb{R}^N} \frac{1}{q}\left(|\nabla u|^q + \mathcal{V}(x)|u|^q\right)dx\right)\int_{\mathbb{R}^N}\left(|\nabla u|^{q-2}\nabla u \nabla u + \mathcal{V}(x)|u|^{q-2}uu\right)dx \\
&\geq k_1\left(\int_{\mathbb{R}^N} \frac{1}{p}\left(|\nabla u|^p + \mathcal{V}(x)|u|^p\right)dx\right)^{\sigma_1-1}\|u\|_{\mathcal{V},p} + k_2\left(\int_{\mathbb{R}^N} \frac{1}{q}\left(|\nabla u|^q + \mathcal{V}(x)|u|^q\right)dx\right)^{\sigma_2-1}\|u\|_{\mathcal{V},q} \\
&\geq \frac{k_1}{p}\|u\|_{\mathcal{V},p}^{p\sigma_1} + \frac{k_2}{q}\|u\|_{\mathcal{V},q}^{q\sigma_2} \\
&\geq \frac{\min(k_1,k_2)}{q}\left(\|u\|_{\mathcal{V},p}^{p\min(\sigma_1,\sigma_2)} + \|u\|_{\mathcal{V},q}^{p\min(\sigma_1,\sigma_2)}\right) \\
&\geq \frac{\min(k_1,k_2)}{q2^{p\min(\sigma_1,\sigma_2)-1}}\|u\|_{\mathcal{V},p,q}^{p\min(\sigma_1,\sigma_2)-1}.
\end{aligned}
\tag{3.29}
$$

Hence, Ψ_1' is coercive. Using Lemma 3.2, we can establish that Ψ_1 belongs to $C^1(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N), \mathbb{R})$. Consequently, Ψ_1' exhibits hemicontinuity. Therefore, in accordance with the Minty-Browder Theorem, we deduce the existence of $\Psi_1'^{-1} : \left(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)\right)^* \longrightarrow W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$, and it is subject to boundedness.

To establish the continuity of $\Psi_1'^{-1}$, we aim to demonstrate its sequential continuity. Consider a sequence $(u_n)_n$ in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)^*$ that strongly converges to $u \in \left(W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)\right)^*$. Consider $v_n = \Psi_1'^{-1}(u_n)$, and $v = \Psi_1'^{-1}(u)$. We can establish that the sequence $(v_n)_n$ is bounded within $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$, so, we may make the assumption that it converges weakly to a certain $v_0 \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$. As a consequence of the strong convergence of u_n to u, we can establish:

$$
\lim_{n\to\infty}\langle \Psi_1'(v_n), v_n - v_0 \rangle = \langle u_n, v_n - v_0 \rangle = 0,
$$

that is,

$$
\lim_{n\to\infty} K_1\left(\frac{\|v_n\|_{\mathcal{V},p}^p}{p}\right)\langle \varphi_{p,\mathcal{V}}'(v_n), v_n - v_0 \rangle = 0,
\tag{3.30}
$$

and

$$
\lim_{n\to\infty} K_2\left(\frac{\|v_n\|_{\mathcal{V},q}^q}{q}\right)\langle \varphi_{q,\mathcal{V}}'(v_n), v_n - v_0 \rangle = 0.
\tag{3.31}
$$

Because v_n is bounded in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$, we can deduce that $\varphi_{p,\mathcal{V}}(v_n)$ and $\varphi_{q,\mathcal{V}}(v_n)$ are also bounded, therefore

$$
\|u\|_{\mathcal{V},p} \to t_0 \geq 0, \quad \|u\|_{\mathcal{V},q} \to t_1 \geq 0, \text{ as } n \to \infty.
$$

If $t_0 = t_1 = 0$, hence, we establish that $(v_n)_n$ exhibits strong convergence to v_0 within the function space $W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$. Thanks to the continuity and injectivity of $\Psi_1'^{-1}$. As a result, we obtain the desired conclusion. If $t_0 > 0$ (respectively $t_1 > 0$), it follows from the continuity of the function K_1 (respectively K_2) that

$$K_1\left(\frac{\|v_n\|^p_{\mathcal{V},p}}{p}\right) \to K_1\left(\frac{t_0}{p}\right), \text{ respectively } K_2\left(\frac{\|v_n\|^q_{\mathcal{V},q}}{q}\right) \to K_2\left(\frac{t_1}{q}\right) \text{ as } n \to \infty.$$

Thus, by **(K1)** (respectively **(K2)**), when n is taken to be sufficiently large, the result becomes apparent:

$$K_1\left(\frac{\|v_n\|^p_{\mathcal{V},p}}{p}\right) \geq \tau_0 > 0, \text{ respectively } K_2\left(\frac{\|v_n\|^q_{\mathcal{V},q}}{q}\right) \geq \tau_1 > 0. \tag{3.32}$$

As a result of (3.30) and (3.32), the following is obtained:

$$\lim_{n\to\infty} \langle \Psi_1'(v_n), v_n - v_0 \rangle = 0. \tag{3.33}$$

Utilizing (3.33) and considering the weak convergence of v_n to v_0 in $W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$, we can employ Lemma 3.6 to conclude that v_n undergoes a strong convergence to v_0 within the same function space $W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$. $\square$

Step 2 : Coercivity of Ψ_1

Lemma 3.8. Under the hypothesis **(V1)**, **(K1)** and **(K2)** the functional Ψ_1 is coercive.

Proof . In cases where $\|u\|_{\mathcal{V},p,q}$ is exceptionally large, it allows for the choice of $\|u\|_{\mathcal{V},p} \geq 1$ and $\|u\|_{\mathcal{V},q} \geq 1$. Then, by **(K1)** and **(K2)** we have

$$
\begin{aligned}
\Psi_1(u) &= \widehat{K}_1\left(\varphi_{p,\mathcal{V}}(u)\right) + \widehat{K}_2\left(\varphi_{q,\mathcal{V}}(u)\right) \\
&\geq \frac{k_1}{p\sigma_1}\|u\|^{p\sigma_1}_{\mathcal{V},p} + \frac{k_2}{q\sigma_2}\|u\|^{q\sigma_2}_{\mathcal{V},q} \\
&\geq \frac{\min(k_1,k_2)}{q\max(\sigma_1,\sigma_2)}\left(\|u\|^{p\min(\sigma_1,\sigma_2)}_{\mathcal{V},p} + \|u\|^{p\min(\sigma_1,\sigma_2)}_{\mathcal{V},q}\right) \\
&\geq \frac{\min(k_1,k_2)}{q\max(\sigma_1,\sigma_2)2^{p\min(\sigma_1,\sigma_2)-1}}\|u\|^{p\min(\sigma_1,\sigma_2)}_{\mathcal{V},p,q}.
\end{aligned}
\tag{3.34}
$$

Which shows that Ψ_1 is coercive. $\square$

Step 3 : Application of Theorem 2.6

Taking $X = W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$, Ψ_1 and Ψ_2 are in accordance with the definitions set earlier, thanks to Lemma 3.2 Ψ_2 is C^1-functional with compact derivative. Additionally, Lemma 3.4 highlights the sequential weak lower continuity of Ψ_1 and by Lemma 3.5 Ψ_1 is C^1-functional belonging to $\mathcal{W}_{W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)}$. Moreover, as implied by Lemma 3.7, it's essential to note that the operator Ψ_1' possesses a continuous inverse on $\left(W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)\right)^*$, and by Lemma 3.8 Ψ_1 is coercive.

It is evident that $u_0 = 0$ is global minimum of Ψ_1 and that $\Psi_1(u_0) = \Psi_2(u_0) = 0$. Furthermore, it is essential to note that Ψ_1 remains bounded when restricted to any bounded subset of $W^{p,q}_{\mathcal{V}}(\mathbb{R}^N)$. In fact, if $\|u\|_{\mathcal{V},p,q} \leq c$, we have:

$$
\begin{aligned}
\Psi_1(u) &= \widehat{K}_1\left(\varphi_{p,\mathcal{V}}(u)\right) + \widehat{K}_2\left(\varphi_{q,\mathcal{V}}(u)\right) \\
&\leq \widehat{K}_1\left(\frac{\|u\|^p_{\mathcal{V},p}}{p}\right) + \widehat{K}_2\left(\frac{\|u\|^q_{\mathcal{V},q}}{q}\right) \\
&\leq \widehat{K}_1\left(\frac{c^p}{p}\right) + \widehat{K}_2\left(\frac{c^q}{q}\right).
\end{aligned}
\tag{3.35}
$$

Now, in accordance with the assumptions mentioned in **(F2)**, for any $\varepsilon > 0$, there exists a positive value ζ_1 satisfying:

$$|F_1(x,t)| \leq \varepsilon |t|^{q\max(\sigma_1,\sigma_2)}, \tag{3.36}$$

for every $x \in \mathbb{R}^N$ and $t \in [-\zeta_1, \zeta_1]$. Given that $f_1 \in \mathcal{A}$, for $r \in [q\max(\sigma_1, \sigma_2), q^*]$, we can choose $c_1 > 0$ so that

$$|F_1(x,t)| \le c_1|t|^r, \tag{3.37}$$

for all $(x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus [-\zeta_1, \zeta_1])$. Combining (3.36) and (3.37), we get

$$|F_1(x,t)| \le \varepsilon|t|^{q\max(\sigma_1, \sigma_2)} + c_1|t|^r, \tag{3.38}$$

for each $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. So by Lemma 2.3, the embedding $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ in $L^{q\max(\sigma_1, \sigma_2)}(\mathbb{R}^N)$ and $L^r(\mathbb{R}^N)$ is compact. Then for some positive constants c_2, and c_3, one has for all $u \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$

$$
\begin{aligned}
\Psi_2(u) &\le \varepsilon \|u\|_{L^{\max(p\sigma_1, q\sigma_2)}(\mathbb{R}^N)}^{q\max(\sigma_1, \sigma_2)} + c_1 \|u\|_{L^r(\mathbb{R}^N)}^r \\
&\le \varepsilon c_2 \|u\|_{\mathcal{V},p,q}^{q\max(\sigma_1, \sigma_2)} + c_1 c_3 \|u\|_{\mathcal{V},p,q}^r.
\end{aligned}
$$

Or by **(K1)** and **(K2)**, if $\|u\|_{\mathcal{V},p,q} \le 1$, it permits the selection of values for $\|u\|_{\mathcal{V},p} < 1$ and $\|u\|_{\mathcal{V},p} < 1$. Then, as implied by **(K1)** and **(K2)**, we have:

$$
\begin{aligned}
\Psi_1(u) &\ge \frac{\min(k_1, k_2)}{q\max(\sigma_1, \sigma_2)} \left(\|u\|_{\mathcal{V},p}^{p\max(\sigma_1, \sigma_2)} + \|u\|_{\mathcal{V},q}^{q\max(\sigma_1, \sigma_2)} \right) \\
&\ge \frac{\min(k_1, k_2)}{q\max(\sigma_1, \sigma_2)2^{q\max(\sigma_1, \sigma_2)-1}} \|u\|_{\mathcal{V},p,q}^{q\max(\sigma_1, \sigma_2)}.
\end{aligned} \tag{3.39}
$$

Hance

$$\|u\|_{\mathcal{V},p,q}^{q\max(\sigma_1, \sigma_2)} \le \frac{q\max(\sigma_1, \sigma_2)2^{q\max(\sigma_1, \sigma_2)-1}}{\min(k_1, k_2)} \Psi_1(u),$$

then

$$\Psi_2(u) \le \varepsilon c_2 \frac{q\max(\sigma_1, \sigma_2)2^{q\max(\sigma_1, \sigma_2)-1}}{\min(k_1, k_2)} \Psi_1(u) + c_1 c_3 \left(\frac{q\max(\sigma_1, \sigma_2)2^{q\max(\sigma_1, \sigma_2)-1}}{\min(k_1, k_2)} \Psi_1(u) \right)^{\frac{r}{q\max(\sigma_1, \sigma_2)}}.$$

Consequently, since $r > q\max(\sigma_1, \sigma_2)$ and given that $\Psi_1(u) \to 0$ as $u \to 0$, we can deduce that:

$$\limsup_{u \to 0} \frac{\Psi_2(u)}{\Psi_1(u)} \le \varepsilon c_2 \frac{q\max(\sigma_1, \sigma_2)2^{q\max(\sigma_1, \sigma_2)-1}}{\min(k_1, k_2)}. \tag{3.40}$$

On the flip side, as implied by **(F3)**, it follows that, for all $\varepsilon > 0$, there exists a $\zeta_2 > 0$ satisfying:

$$|F_1(x,t)| \le \varepsilon|t|^{p\min(\sigma_1, \sigma_2)}, \tag{3.41}$$

for any $x \in \mathbb{R}^N$ and $|t| > \zeta_2$, and given that $f \in \mathcal{A}$, there exists a constant $c_4 > 0$ for $r \in [p, p\min(\sigma_1, \sigma_2)]$, satisfying:

$$|F_1(x,t)| \le c_4|t|^r, \tag{3.42}$$

for each $x \in \mathbb{R}^N$ and $|t| \le \zeta_2$. If $\|u\|_{\mathcal{V},p,q} > 1$, we can apply Lemma 3.8, resulting in:

$$
\begin{aligned}
\frac{\Psi_2(u)}{\Psi_1(u)} &= \frac{\displaystyle\int_\Omega F_1(x,u)dx}{\widehat{K}_1\left(\varphi_{p,\mathcal{V}}(u)\right) + \widehat{K}_2\left(\varphi_{q,\mathcal{V}}(u)\right)} \\
&\le \frac{\displaystyle\int_{\{x \in \mathbb{R}^N : |u(x)| \le \zeta_2\}} F_1(x,u)dx}{\frac{q\max(\sigma_1, \sigma_2)2^{p\min(\sigma_1, \sigma_2)-1}}{\min(k_1, k_2)} \|u\|_{\mathcal{V},p,q}^{p\min(\sigma_1, \sigma_2)}} + \frac{\displaystyle\int_{\{x \in \mathbb{R}^N : |u(x)| > \zeta_2\}} F_1(x,u)dx}{\frac{q\max(\sigma_1, \sigma_2)2^{p\min(\sigma_1, \sigma_2)-1}}{\min(k_1, k_2)} \|u\|_{\mathcal{V},p,q}^{p\min(\sigma_1, \sigma_2)}} \\
&\le \frac{c_4 \|u\|_{L^r(\mathbb{R}^N)}^r}{\frac{q\max(\sigma_1, \sigma_2)2^{p\min(\sigma_1, \sigma_2)-1}}{\min(k_1, k_2)} \|u\|_{\mathcal{V},p,q}^{p\min(\sigma_1, \sigma_2)}} + \frac{\varepsilon \|u\|_{\mathcal{V},p,q}^{p\min(\sigma_1, \sigma_2)}}{\frac{q\max(\sigma_1, \sigma_2)2^{p\min(\sigma_1, \sigma_2)-1}}{\min(k_1, k_2)} \|u\|_{\mathcal{V},p,q}^{p\min(\sigma_1, \sigma_2)}} \\
&\le \frac{c_4 \|u\|_{L^r(\mathbb{R}^N)}^r}{\frac{q\max(\sigma_1, \sigma_2)2^{p\min(\sigma_1, \sigma_2)-1}}{\min(k_1, k_2)} \|u\|_{\mathcal{V},p,q}^{p\min(\sigma_1, \sigma_2)}} + \varepsilon c_5.
\end{aligned} \tag{3.43}
$$

Since $p \min(\sigma_1, \sigma_2) > r$,

$$\limsup_{\|u\|_{\mathcal{V},p,q} \to \infty} \frac{\Psi_2(u)}{\Psi_1(u)} \leq \varepsilon c_5. \tag{3.44}$$

As $\varepsilon > 0$ is arbitrary, the expressions (3.40) and (3.44) signify that:

$$\max \left\{ \limsup_{\|u\|_{\mathcal{V},p,q} \to \infty} \frac{\Psi_2(u)}{\Psi_1(u)}, \; \limsup_{\|u\|_{\mathcal{V},p,q} \to 0} \frac{\Psi_2(u)}{\Psi_1(u)} \right\} \leq 0.$$

Thus, all the conditions of Theorem 2.6 are verified. Hence, for every compact interval $\mathbf{\Lambda} \subset (\Theta^*, +\infty)$, there exists a positive η that meets the criteria outlined in the conclusion of Theorem 2.6. Let $\alpha \in \mathbf{\Lambda}$ and $f_2 \in \mathcal{A}$. We can set:

$$\Psi_3(u) = \int_{\mathbb{R}^N} F_2(x, u) dx, \quad F_2(x, t) = \int_0^t f_2(x, s) ds,$$

for every $u \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$, we have Ψ_3 as a C^1 functional on $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ with a derivative that is compact. As a result, there exists a positive β^* such that, for any β within the interval $[0, \beta^*]$, the equation:

$$\Psi_1'(u) = \alpha \Psi_2'(u) + \beta \Psi_3'(u),$$

consists of at least three solutions, with the norm of each being smaller than η. Importantly, the solutions in $W_{\mathcal{V}}^{p,q}(\mathbb{R}^N)$ for the equation above are precisely the weak solutions of problem (1.1). Thus, we have completed the proof of Theorem 1.2. $\qquad\square$

Here, we offer an illustrative example to expound upon the conclusions laid out in Theorem 1.2.

Corollary 3.9. Let $p \in [2, N[$, $q \in [p+1, N[$ and $\xi > \max\left(2, \max(\sigma_1 q, \sigma_2 q)\right)$ such that $p < q < \xi$, where $1 < \sigma_1 < \frac{N}{N-q}$ and $1 < \sigma_2 < \frac{N}{N-q}$.

We consider

$$K_1(t) = (1+t)^{\sigma_1 - 1}, \quad K_2(t) = (1+t)^{\sigma_2 - 1} \text{ for all } t \geq 0. \tag{3.45}$$

and

$$f_1(t) = \xi \sin(t) \cos(t) |\sin(t)|^{\xi - 2} \text{ for every } t \in \mathbb{R}. \tag{3.46}$$

Therefore, by referencing both (3.45) and (3.46), we obtain the following:

$$\widehat{K}_1(t) = \frac{(1+t)^{\sigma_1}}{\sigma_1}, \quad \widehat{K}_2(t) = \frac{(1+t)^{\sigma_2}}{\sigma_2} \text{ and } F_1(x, t) = F_1(t) = |\sin(t)|^{\xi}.$$

Then, the following nonlinear perturbed (p,q)-Schrödinger-Kirchhoff problem

$$\left(1 + \left(\int_{\mathbb{R}^N} \frac{1}{p} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} \mathcal{V}(x) |u|^p dx\right)^{\sigma_1 - 1}\right) \left(-\Delta_p u + \mathcal{V}(x) |u|^{p-2} u\right)$$

$$+ \left(1 + \left(\int_{\mathbb{R}^N} \frac{1}{q} |\nabla u|^q dx + \frac{1}{q} \int_{\mathbb{R}^N} \mathcal{V}(x) |u|^q dx\right)^{\sigma_2 - 1}\right) \left(-\Delta_q u + \mathcal{V}(x) |u|^{q-2} u\right)$$

$$= \alpha \xi \sin(u) \cos(u) |\sin(u)|^{\xi - 2} + \beta f_2(x, u) \quad \text{in } \mathbb{R}^N. \tag{3.47}$$

Possesses a minimum of three weak solutions, all with norms below η.

Proof . For any $t \in \mathbb{R}$, we can assert that f_1 is a member of $\mathcal{A}$. This assertion is supported by the following reasoning:

$$\sup_{t \in \mathbb{R}} \frac{|f_1(t)|}{|t|^{s-1}} < \infty,$$

is true for every $p < q < s < \xi$. Conversely, the following holds:

$$\limsup_{t \to 0} \frac{|\sin(t)|^\xi}{|t|^{q \max(\sigma_1, \sigma_2)}} = 0 \quad \text{and} \quad \limsup_{t \to \infty} \frac{|\sin(t)|^\xi}{|t|^{p \min(\sigma_1, \sigma_2)}} = 0.$$

Choose a positively measured compact set $\mathcal{U}$ in $\mathbb{R}^N$ and select $v \in W_{\mathcal{U}}^{p,q}(\mathbb{R}^N)$ so that it satisfies $v(x) = \frac{\pi}{2}$ within $\mathcal{U}$ and $0 \le v(x) \le \frac{\pi}{2}$ outside of $\mathbb{R}^N \backslash \mathcal{U}$. This results in:

$$\int_{\mathbb{R}^N} |\sin(v(x))|^\xi dx = \text{meas}(\mathcal{U}) + \int_{\mathbb{R}^N \backslash \mathcal{U}} |\sin(v(x))|^\xi dx > 0,$$

signifying that **(F1)**, **(F2)** and **(F3)** are verified. Also, for $k_1 = 1$ and $k_2 = 1$ the condition **(K1)** and **(K2)** is verified, we set

$$\Theta^* = \inf \left\{ \frac{\frac{\left(1 + \int_{\mathbb{R}^N} \frac{1}{p}(|\nabla v|^p + \mathcal{V}(x)|v|^p) dx\right)^{\sigma_1}}{\sigma_1} + \frac{\left(1 + \int_{\mathbb{R}^N} \frac{1}{q}(|\nabla v|^q + \mathcal{V}(x)|v|^q) dx\right)^{\sigma_2}}{\sigma_2}}{\int_{\mathbb{R}^N} |\sin(v(x))|^\xi dx} : v \in W_{\mathcal{V}}^{p,q}(\mathbb{R}^N), \int_{\mathbb{R}^N} |\sin(v(x))|^\xi dx > 0 \right\}.$$

Then, it follows from Theorem 1.2, we deduce that, within any compact interval $\mathbf{\Lambda} \subset (\Theta^*, \infty)$, there exist specific values $\eta > 0$ and $\beta^* > 0$, such that for each $\alpha \in \mathbf{\Lambda}$, and every $\beta \in [0, \beta^*]$, and for any $f_2 \in \mathcal{A}$, there exist a minimum of three weak solutions to problem (3.47), each having norms smaller than η. $\square$

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