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A new generalized Laplace transform

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Abstract

A new generalized Laplace transform is introduced, which is a common generalization of the Laplace transform, Sumudu transform, ELzaki transform, and a transformation of X. J. Yang. The basic properties of this generalized Laplace transform are derived, so that these properties coincide with the properties of the known transformations mentioned above.

Keywords: Laplace transform, Sumudu transform, ELzaki transform, Convolution
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1 Introduction

Integral transforms are used to solve differential equations and integral equations. So, they are considered as an applicable tool to understand our nature. The foremost integral transforms are Laplace transform and Fourier transform. The main advantage of a Laplace transform is that it transforms real functions to real functions. For this purpose, there are many generalizations and variations of Laplace transform towards applications. This article proposes a common generalization of Lapalce transform, Sumudu transform, ELzaki transform and a transformation of X - J. Yang, etc.

The classical Laplace transform [1] is given by

$$\bar{f}(s) = L[f(t)] = \int_0^\infty e^{-st} f(t) dt,$$

for $Re(s) > 0$, in particular for $s > 0$. Sumudu [3] modified this transform in the following form

$$\bar{f}(p) = S[\phi(\tau)] = \frac{1}{p} \int_0^\infty e^{-\frac{\tau}{p}} \phi(\tau) d\tau.$$

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The ELzaki transform [2] is given by

$$\bar{f}(v) = E[f(t)] = v \int_0^\infty e^{-\frac{t}{v}} f(t)dt, \ t \geq 0, \ k_1 \leq v \leq k_2.$$

X-J. Yang [4] introduced the following integral transform

$$\bar{f}(w) = Y[f(t)] = \int_0^\infty e^{-\frac{t}{w}} f(t)dt, \ t > 0.$$

One may observe that all these transformations take the following common form $L_{\phi,\psi}[f(t)] = \phi(s) \int_0^\infty [e^{-\psi(s)t}] \ f(t) \ dt$, with $\psi(s) > 0$. This article considers a generalized integral transform in this form. With $\phi(s) = 1$ and $\psi(s) = s$; $\phi(s) = \frac{1}{s}$ and $\psi(s) = \frac{1}{s}$; $\phi(s) = s$ and $\psi(s) = \frac{1}{s}$; and $\phi(s) = 1$ and $\psi(s) = \frac{1}{s}$, it is understood that Laplace transform, Sumudu transform, ELzaki transform and a transform of X-J. Yang are generalized. The purpose of this article is to derive transforms of some standard functions and some basic properties of this generalized Laplace transform. Further, convolution product is introduced and a convolution theorem is obtained for this generalized integral transform.

2 New Generalized Laplace Transform

For given real valued functions ϕ and ψ defined on $(0, \infty)$, a new generalized Laplace transform of the function $f(t)$ is defined by the integral

$$L_{\phi,\psi}[f(t)] = \phi(s) \int_0^\infty [e^{-\psi(s)t}] \ f(t) \ dt = F_{\phi,\psi}(s), \ \text{say}, \ \text{where} \ \psi(s) > 0.$$

Here $f(t)$ is a real valued function on $(0, \infty)$. Later, it may also refer to a generalized function, the Dirac delta function. The subsequent sections provide basic properties of this transform. Necessary analytic assumptions can be assumed for exactness, and one can provide complete derivations of the following properties. These assumptions are not made explicitly.

3 Properties of a New Generalized Laplace Transform

Let us state properties of our transform with short derivations, because standard arguments are available in articles mentioned above.

Property 3.1 (Linearity Property). If $L_{\phi,\psi}[f(t)] = F_{\phi,\psi}(s)$ and $L_{\phi,\psi}[g(t)] = G_{\phi,\psi}(s)$, then

$$L_{\phi,\psi} \ [af(t) \pm bg(t)] = a \ F_{\phi,\psi}(s) \pm b \ G_{\phi,\psi}(s),$$

where a and b are arbitrary constants.

Proof .

$$\begin{aligned} L_{\phi,\psi}[af(t) \pm bg(t)] &= \phi(s) \int_0^\infty [e^{-\psi(s)t}] \ [af(t) \pm bg(t)] \ dt \\ &= a \ \phi(s) \int_0^\infty [e^{-\psi(s)t}] \ f(t) \ dt \pm b \ \phi(s) \int_0^\infty [e^{-\psi(s)t}] \ g(t) \ dt \\ &= a \ F_{\phi,\psi}(s) \pm b \ G_{\phi,\psi}(s). \end{aligned}$$

□

Property 3.2 (Change of Scale Property). If $L_{\phi,\psi}[f(t)] = F_{\phi,\psi}(s)$, then

$$L_{\phi,\psi}[f(at)] = \frac{1}{a}\ F_{\phi,\frac{\psi}{a}}(s),\ \text{ where } \psi(s) > 0 \ \text{ and } a > 0.$$

Proof .

$$\begin{aligned} L_{\phi,\psi}[f(at)] &= \phi(s) \int_0^\infty [e^{-\psi(s)t}]\ f(at)\ dt \\ &= \frac{1}{a}\ \phi(s) \int_0^\infty [e^{-\psi(s)(\frac{u}{a})}]\ f(u)\ du \\ &= \frac{1}{a}\ \phi(s) \int_0^\infty [e^{-(\frac{\psi(s)}{a})u}]\ f(u)\ du \\ &= \frac{1}{a}\ F_{\phi,\frac{\psi}{a}}(s). \end{aligned}$$

□

Property 3.3 (First Shifting Property). If $L_{\phi,\psi}[f(t)] = F_{\phi,\psi}(s)$, then

$$L_{\phi,\psi}[e^{-at} f(t)] = F_{\phi,\psi+a}(s) = [F_{\phi,\psi}(s)]_{\psi\to\psi+a}.$$

Proof .

$$\begin{aligned} L_{\phi,\psi}[e^{-at} f(t)] &= \phi(s) \int_0^\infty [e^{-\psi(s)t}]\ e^{-at}\ f(t)\ dt \\ &= \phi(s) \int_0^\infty e^{-(\psi(s)+a)t}\ f(t)\ dt \\ &= F_{\phi,\psi+a}(s) = [F_{\phi,\psi}(s)]_{\psi\to\psi+a} \end{aligned}$$

□

Property 3.4 (Second Shifting Property). If $L_{\phi,\psi}[f(t)] = F_{\phi,\psi}(s)$, then

$$L_{\phi,\psi}[f(t-a)H(t-a)] = e^{-a\psi(s)} F_{\phi,\psi}(s), \quad \text{where } a > 0.$$

(or) equivalently

$$L_{\phi,\psi}[f(t)H(t-a)] = e^{-a\psi(s)} L_{\phi,\psi}[f(t+a)],$$

where $H(t-a)$ is the Heaviside unit step function, $H(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a. \end{cases}$

Proof .

$$\begin{aligned} L_{\phi,\psi}[f(t-a)H(t-a)] &= \phi(s) \int_0^\infty [e^{-\psi(s)t}]\ f(t-a)\ H(t-a)\ dt \\ &= \phi(s) \int_a^\infty [e^{-\psi(s)t}]\ f(t-a)\ dt \\ &= e^{-a\psi(s)}\ [\phi(s) \int_0^\infty [e^{-\psi(s)u}]\ f(u)\ du] \\ &= e^{-a\psi(s)}\ F_{\phi,\psi}(s). \end{aligned}$$

□

4 Generalized Laplace Transform of Derivatives and Integrals

For solving differential and integral equations, there is a need for expressions for integral transforms of derivatives and integrals. They are stated in this section.

(I) If $L_{\phi,\psi}[f(t)] = F_{\phi,\psi}(s)$, then

1. $L_{\phi,\psi}[f'(t)] = \psi(s)F_{\phi,\psi}(s) - \phi(s)f(0)$.
2. $L_{\phi,\psi}[f''(t)] = [\psi(s)]^2F_{\phi,\psi}(s) - \psi(s)\phi(s)f(0) - \phi(s)f'(0)$.
3. $L_{\phi,\psi}[f^{(n)}(t)] = [\psi(s)]^nF_{\phi,\psi}(s) - [\psi(s)]^{n-1}\phi(s)f(0) - [\psi(s)]^{n-2}\phi(s)f'(0) - [\psi(s)]\phi(s)f^{(n-2)}(0) - \phi(s)f^{(n-1)}(0)$.

(II) If $L_{\phi,\psi}[f(t)] = F_{\phi,\psi}(s)$, then

$$L_{\phi,\psi}[\int_0^t f(\tau)d\tau] = \frac{1}{\psi(s)}\ F_{\phi,\psi}(s).$$

Proof . (I) (1)

$$\begin{aligned} L_{\phi,\psi}[f'(t)] &= \phi(s)\int_0^\infty [e^{-\psi(s)t}]\ f'(t)\ dt \\ &= \phi(s)[e^{-\psi(s)t}f(t)]_0^\infty + \phi(s)\int_0^\infty \psi(s)[e^{-\psi(s)t}]\ f(t)\ dt. \end{aligned}$$

Assuming that $e^{-\psi(s)t}f(t) \to 0$, as $t \to \infty$ implies that

$$\begin{aligned} L_{\phi,\psi}[f'(t)] &= \psi(s)\ [\phi(s)\int_0^\infty [e^{-\psi(s)t}]\ f(t)\ dt] - \phi(s)f(0) \\ &= \psi(s)\ F_{\phi,\psi}(s) - \phi(s)f(0). \end{aligned}$$

(2)

$$\begin{aligned} L_{\phi,\psi}[f''(t)] &= \psi(s)L_{\phi,\psi}[f'(t)] - \phi(s)f'(0),\ by\ (1) \\ &= \psi(s)[\psi(s)F_{\phi,\psi}(s) - \phi(s)f(0)] - \phi(s)f'(0) \\ &= (\psi(s))^2F_{\phi,\psi}(s) - \psi(s)\phi(s)f(0) - \phi(s)f'(0), \end{aligned}$$

where we have assumed $e^{-\psi(s)t}f'(t) \to 0$, as $t \to \infty$. A similar procedure can be used to prove the result (3).

(II) Let $g(t) = \int_0^t f(\tau)\ d\tau$, so that $g(0) = 0$ and $g'(t) = f(t)$. By transform of derivatives,

$$L_{\phi,\psi}\,[g'(t)] = \psi(s)\ L_{\phi,\psi}\,[g(t)] - \phi(s)\ g(0).$$

Therefore,

$$L_{\phi,\psi}\,[f(t)] = \psi(s)\ \left[L_{\phi,\psi}\left[\int_0^t f(\tau)d\tau\right]\right] - 0.$$

This implies that

$$L_{\phi,\psi}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{\psi(s)}\ F_{\phi,\psi}(s).$$

□

5 Generalized Laplace Transform of some Basic Functions

Unless formulas for integral transforms of some standard functions are known, these transforms cannot be applied to solve equations. This section provides generalized Laplace transforms of some basic functions.

In the following $\delta(t-a)$ is the Dirac delta function about the point a. That is, $\delta(t-a) = 1$ if $t = a$ and zero otherwise, when it is considered as a generalized function.

S.No.	Function $f(t)$, $t \geq 0$	$L_{\phi,\psi}[f(t)] = F_{\phi,\psi}(s)$
1	k, a constant	k $\frac{\phi(s)}{\psi(S)}$
2	1	$\frac{\phi(s)}{\psi(S)}$
3	t	$\frac{\phi(s)}{[\psi(s)]^2}$
4	t^n, n is not a positive integer	$\frac{\phi(s)}{[\psi(s)]^{n+1}}\Gamma(n+1)$
5	t^n, n is a positive integer	$\frac{\phi(s)}{[\psi(s)]^{n+1}}n!$
6	e^{at}, a is a constant	$\frac{\phi(s)}{[\psi(s)-a]}$, (when $\psi(s) > a$)
7	e^{-at}, a is a constant	$\frac{\phi(s)}{[\psi(s)+a]}$, (when $\psi(s) > -a$)
8	sin at, a is real	$\frac{a\phi(s)}{[\psi(s)]^2+a^2}$, (when $\lvert\psi(s)\rvert > a$)
9	cos at, a is real	$\frac{\phi(s)\phi(s)}{[\psi(s)]^2+a^2}$, (when $\lvert\psi(s)\rvert > a$)
10	sinh at, a is real	$\frac{\phi(s)}{[\psi(s)]^2-a^2}$, (when $\lvert\psi(s)\rvert > a$)
11	cosh at, a is real	$\frac{\phi(s)\psi(s)}{[\psi(s)]^2-a^2}$, (when $\lvert\psi(s)\rvert > a$)
12	$H(t-a)$	$e^{-a\psi(s)}\,\frac{\phi(s)}{\psi(s)}$
13	$\delta(t-a)$	$\phi(s)e^{-a\psi(s)}$

6 Convolution

Integral transform of a convolution product of two functions is the product of integral transforms of these individual functions. This required property holds even for our generalized Laplace transform, with a slight modification.

Definition 6.1. The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and is defined by the integral

$$f(t) * g(t) = \int_0^t f(t-u)\ g(u)\ du.$$

Theorem 6.2 (Convolution Theorem). If $L_{\phi,\psi}[f(t)] = F_{\phi,\psi}(s)$ and $L_{\phi,\psi}[g(t)] = G_{\phi,\psi}(s)$, then

$$L_{\phi,\psi}[f(t) * g(t)] = \frac{1}{\phi(s)}\ [F_{\phi,\psi}(s).G_{\phi,\psi}(s)].$$

Proof .

$$
\begin{aligned}
L_{\phi,\psi}[f(t)*g(t)] &= \phi(s)\int_0^\infty \left[e^{-\psi(s)t}\right]\ [f(t)*g(t)]\ dt\\
&= \phi(s)\int_0^\infty \left[e^{-\psi(s)t}\right]\ dt\ \int_0^t f(t-u)\ g(u)\ du\\
&= \left[\int_0^\infty g(u)\ du\right]\ \left[\phi(s)\int_u^\infty \left[e^{-\psi(s)t}\right]\ f(t-u)\ dt\right]\\
&= \left[\int_0^\infty e^{-\psi(s)u}\ g(u)\ du\right]\ \left[\phi(s)\int_0^\infty \left[e^{-\psi(s)x}\right]\ f(x)\ dx\right]\\
&= \frac{1}{\phi(s)}\left[\phi(s)\int_0^\infty e^{-\psi(s)u}\ g(u)\ du\right]\ \left[\phi(s)\int_0^\infty \left[e^{-\psi(s)x}\right]\ f(x)\ dx\right]\\
&= \frac{1}{\phi(s)}\ \left[F_{\phi,\psi}(s).G_{\phi,\psi}(s)\right].
\end{aligned}
$$

□

7 Conclusion

The functions $\phi(s)$ and $\psi(s)$ have been fixed in earlier articles for application purpose. The same thing should be done when our generalized Laplace transform is used for application. The advantage of our generalized Laplace transform is the possibility to change the fixed functions $\phi(s)$ and $\psi(s)$, so that they are suitable for solving equations chosen. For example, let us consider the case $\phi(s) = \psi(s) = s$. Let us consider the equations

$$\frac{d^2x}{dt^2} - x = 0;\ x(0) = 1, \frac{dx}{dt}(0) = -1,$$

and let us apply the corresponding generalized Laplace transform on the first equation to obtain

$$
\begin{aligned}
s^2F_{\phi,\psi}(s) - s^2x(0) - s\frac{dx}{dt}(0) - F_{\phi,\psi}(s) = 0,\\
(s^2-1)F_{\phi,\psi}(s) - s^2 + s = 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
F_{\phi,\psi}(s) &= \frac{s^2-s}{s^2-1} = \frac{s}{s+1}\\
&= \frac{\phi(s)}{\psi(s)+1}.
\end{aligned}
$$

So, $x(t) = e^{-t}$, when the corresponding inverse transform is applied. Thus, $x = e^{-t}$ is the solution of the previous initial value problem.

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