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Closeness of Lindley distribution to an exponential distribution with the presence of outliers

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Abstract

The problem of distinguishing between distributions is always important. It becomes more complicated when data is contaminated by outliers. Here, we use two well-known Lindley and exponential distributions infected by outliers. The closeness of the Lindley distribution in comparison with the exponential distribution with outliers is discussed in this research. Three ways such as likelihood ratio, asymptotic likelihood ratio tests and minimum Kolmogorov distance are used to select the proper fitted model for a real data set. We perform Monte Carlo simulation to obtain the probability of correct selection for various values of sample sizes and parameters based on the best criteria in the distributions. In general, it has been seen that the Lindley distribution is closer to exponential distribution contaminated by outliers based on the likelihood ratio and Kolmogorov criteria. An actual example of real data is used to see the behaviour of the distributions.

Keywords: Lindley distribution, Exponential distribution, Outliers, Likelihood ratio test, Kolmogorov distance, Probability of correct selection, Monte Carlo simulation
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1 Introduction

A application of real-life of current numeral methods in various fields such as medical profession, investment, engineering in biology and statistics can be seen in Aslam and Kazmi [1] and Al-Mutairi et al. [2]. Recently, Nedjar and Zeghdoudi [11] proposed a modern distribution, titled as gamma Lindley distribution (GLD), according to combination of gamma distribution $(2, \delta)$ (GD) and one-parameter Lindley distribution (LD). Although, in the literature, it is the first time to use a mixture of GD $(2, \delta)$ and LD to generate GLD. Mazucheli and Achcar [9] showed that strength data is followed the LD as well as for modeling general lifetime data. Recently, many researchers have gone through this distribution and developed it. See for example, [3, 10, 13]. Statistical inference for LD in the form of entire and censored data is considered by various authors. For more details refer to [1, 2] and [7, 8].

For $\gamma > 0$ and $\delta > 0$, the two-parameter LD (power Lindley) has the following probability density function (PDF) and cumulative distribution function (CDF), respectively of,

$$h(z; \gamma, \delta) = \frac{\gamma\delta^2}{1+\delta} z^{\gamma-1} (1 + z^{\gamma}) e^{-\delta z^{\gamma}}, \; z > 0, \; \gamma > 0, \; \delta > 0, \tag{1.1}$$

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and

$$H(z;\gamma,\delta)=1-\left[1+\frac{\delta}{1+\delta}\right]e^{-\delta z^{\gamma}}, z>0,\ \gamma>0,\ \delta>0. \tag{1.2}$$

However, the LD may be shown as a combination of an exponential distribution and GD (see, [14]). Hence in the present paper, we consider closeness of the LD to exponential distribution contaminated by outliers (EOD) which is from GD.

Dixit, et al. [4] has considered the random variables (rvs) $Z_1, Z_2, \ldots, Z_s$ are such that k of them come from $w_2(z,\delta)$ and the other $(s-k)$ rvs are from $w_1(z,\delta)$. Therefore, the joint PDF of the rvs is

$$w(z_1,z_2,\ldots,z_s;\delta)=\frac{k!\,(s-k)!}{s!}\prod_{l=1}^{s}w_1(z_l,\delta)\sum_{*}\prod_{j=1}^{k}\frac{w_2(z_{B_j},\delta)}{w_1(z_{B_j},\delta)}, \tag{1.3}$$

where, $\sum_* = \sum_{B_1=1}^{s-k+1}\sum_{B_2=B_1+1}^{s-k+2}\cdots\sum_{B_k=B_{k-1}+1}^{s}$ (see [4]-[6]). One may note that the marginal PDF of Z is

$$w(z;\delta)=\frac{s-k}{s}w_1(z;\delta)+\frac{k}{s}w_2(z;\delta). \tag{1.4}$$

For, $w_1(z;\epsilon)=\epsilon e^{-\epsilon z},\ z>0,\ \epsilon>0$ and, $w_2(z;\epsilon)=\epsilon^2 z e^{-\epsilon z},\ z>0,\ \epsilon>0$, it has been seen that

$$w(z;\epsilon)=\frac{s-k}{s}\epsilon e^{-\epsilon z}+\frac{k}{s}\epsilon^2 z e^{-\epsilon z},\ z>0,\ \epsilon>0. \tag{1.5}$$

Also,

$$E(Z)=\frac{s-k}{s}\int_0^{\infty}\epsilon z e^{-\epsilon z}dz+\frac{k}{s}\int_0^{\infty}\epsilon^2 z^2 e^{-\epsilon z}dz=\frac{s+k}{s\epsilon}.$$

2 Methodology

In this section, the parameters of the LD and EOD are estimated.

2.1 Estimation Parameters of LD

Assume that a random sample $Z_1, Z_2, \ldots, Z_s$ is selected from the LD ie. $Z \sim LD(\gamma,\delta)$. By using observation, the profile likelihood function is $L_{LD}(\gamma,\delta;z)$

$$L_{LD}(\gamma,\delta;z)=\left(\frac{\gamma\delta^2}{1+\delta}\right)^{s}\left(\prod_{l=1}^{s}z_l^{\gamma-1}\left(1+z_l^{\gamma}\right)\right)e^{-\delta\sum_{l=1}^{s}z_l^{\gamma}}. \tag{2.1}$$

Let $l_{LD}=\ln(L_{LD}(\gamma,\delta;z))$, so

$$l_{LD}=s\ln(\gamma)+s\ln\left(\frac{\delta^2}{1+\delta}\right)+(\gamma-1)\sum_{l=1}^{s}\ln(z_l)+\sum_{l=1}^{s}\ln(1+z_l^{\gamma})-\delta\sum_{l=1}^{s}z_l^{\gamma}. \tag{2.2}$$

Then

$$\frac{\partial l_{LD}}{\partial\delta}=\frac{2s}{\delta}-\frac{s}{1+\delta}-\sum_{l=1}^{s}z_l^{\gamma}=0. \tag{2.3}$$

For given γ, maximum likelihood estimator (MLE) of δ is

$$\widehat{\delta}=\frac{s-\sum_{l=1}^{s}z_l^{\gamma}+\sqrt{\left(\sum_{l=1}^{s}z_l^{\gamma}-s\right)^2+8s\sum_{l=1}^{s}z_l^{\gamma}}}{2\sum_{l=1}^{s}z_l^{\gamma}}. \tag{2.4}$$

Now, to obtain the MLE of γ, we need to solve $\frac{\partial l_{LD}}{\partial \gamma} = 0$ with respect to γ. So

$$\frac{s}{\gamma} + \sum_{l=1}^{s} \frac{z_l^{\gamma} \ln(z_l)}{1+z_l^{\gamma}} - \delta \sum_{l=1}^{s} z_l^{\gamma} \ln(z_l) = 0, \tag{2.5}$$

and

$$s + \gamma \sum_{l=1}^{s} \frac{z_l^{\gamma} \ln(z_l)}{1+z_l^{\gamma}} - \gamma\delta \sum_{l=1}^{s} z_l^{\gamma} \ln(z_l) = 0. \tag{2.6}$$

Using Newton-Raphson method, the solution is as the following form.

$$\gamma_{l+1} = \gamma_l - \frac{\nabla(\gamma_l)}{\nabla'(\gamma_l)'}, \tag{2.7}$$

where,

$$\nabla(\gamma) = s + \gamma \sum_{l=1}^{s} \ln(z_l) + \gamma \sum_{l=1}^{s} \frac{z_l^{\gamma} \ln(z_l)}{1+z_l^{\gamma}} - \gamma\widehat{\delta} \sum_{l=1}^{s} z_l^{\gamma} \ln(z_l) = 0, \tag{2.8}$$

and

$$\begin{aligned} \nabla'(\gamma) &= \sum_{l=1}^{s} \ln(z_l) + \sum_{l=1}^{s} \frac{z_l^{\gamma} \ln(z_l)}{1+z_l^{\gamma}} + \gamma \sum_{l=1}^{s} \frac{z_l^{\gamma} (\ln(z_l))^2 (1+z_l^{\gamma}) - (z_l^{\gamma} \ln(z_l))^2}{(1+z_l^{\gamma})^2} - \widehat{\delta} \sum_{l=1}^{s} z_l^{\gamma} \ln(z_l) \\ &- \gamma \sum_{l=1}^{s} z_l^{\gamma} (\ln(z_l))^2 = 0. \end{aligned} \tag{2.9}$$

2.2 Parameters Estimation of EOD

Let the rvs $Z_1, Z_2, \ldots, Z_s$ are such that k out of s are with PDF

$$w_2(z; \epsilon) = \epsilon^2 z e^{-\epsilon z}, \ z > 0, \ \epsilon > 0, \tag{2.10}$$

and the other $(s-k)$ rvs are distrusted with

$$w_1(z; \epsilon) = \epsilon e^{-\epsilon z}, \ z > 0, \ \epsilon > 0. \tag{2.11}$$

Therefore, the joint PDF of the rvs is given by

$$w(z_1, \ldots, z_s; \epsilon) = \frac{k!\,(s-k)!}{s!} \epsilon^s e^{-\epsilon \sum_{l=1}^{s} z_l} \sum_{*} \prod_{j=1}^{k} \frac{\epsilon^2 z_{Bj} e^{-\epsilon z_{Bj}}}{\epsilon e^{-\epsilon z_{Bj}}} = \frac{k!\,(s-k)!}{s!} \epsilon^{s+k} e^{-s\epsilon\overline{z}} \sum_{*} \prod_{j=1}^{k} z_{Bj}. \tag{2.12}$$

To estimate ϵ, let, $l(z, \epsilon) = \ln(w(z_1, \ldots, z_s; \epsilon))$, so

$$l(z, \epsilon) = (s+k)\ln(\epsilon) - s\epsilon\overline{z} + \ln\left(\frac{k!\,(s-k)!}{s!} \sum_{*} \prod_{j=1}^{k} z_{Bj} \right). \tag{2.13}$$

From $\frac{\partial l(z,\epsilon)}{\partial \epsilon} = 0$, it easy to see that $\frac{s+k}{\epsilon} - s\overline{z} = 0$, so

$$\widehat{\epsilon} = \frac{s+k}{s\overline{z}}. \tag{2.14}$$

2.3 Estimation of k

In this subsection, in the case of unknown k, it is estimated by obtaining the likelihood respect to k and selecting the one that maximizes the likelihood. Form the equation (2-12), for k_1 and k_2, we consider the ratio P as

$$P = \frac{\frac{k_1!(s-k_1)!}{s!}\epsilon^{s+k_1}e^{-s\epsilon\bar{z}}\sum_{*k_1}\prod_{j=1}^{k_1} z_{Bj}}{\frac{k_2!(s-k_2)!}{s!}\epsilon^{s+k_2}e^{-s\epsilon\bar{z}}\sum_{*k_2}\prod_{j=1}^{k_2} z_{Bj}}, \tag{2.15}$$

where, $\sum_{*k_i} = \sum_{B_1=1}^{s-k_i+1}\sum_{B_2=B_1+1}^{s-k_i+2}\cdots\sum_{B_{k_i}=B_{k_{i-1}+1}}^{s}$, for $i = 1, 2$. If $P > 1$, then $k = k_1$, otherwise $k = k_2$.

3 Likelihood Ratio Test (LRT)

Here, the LRT is considered to select the appropriate fitting model between the LD and EOD. Let a random of size s is selected from either $L_{LD}(\gamma, \delta)$ or $Exp(\epsilon)$. The log-LR statistic Λ is derived from the logarithm of ratio maximum likelihood (RML functions).

$$\Lambda = \ln\left(\frac{L_{LD}\left(\widehat{\gamma},\widehat{\delta};z\right)}{w(\widehat{\epsilon};z)}\right) \tag{3.1}$$

$$\begin{aligned} &= \ln\left(\left(\frac{\widehat{\gamma}\widehat{\delta}^2}{1+\widehat{\delta}}\right)^s \frac{s!}{k!\,(s-k)!}\frac{\left(\prod_{l=1}^s z_l^{\widehat{\gamma}-1}\left(1+z_l^{\widehat{\gamma}}\right)\right)e^{-\widehat{\delta}\sum_{l=1}^s z_l^{\widehat{\gamma}}}}{\widehat{\epsilon}^{s+k}e^{-s\widehat{\epsilon}\bar{z}}\sum_*\prod_{j=1}^k z_{B_j}}\right) \\ &= s\ln\left(\frac{\widehat{\gamma}\widehat{\delta}^2}{1+\widehat{\delta}}\right) + \ln\left(\frac{s!}{k!\,(s-k)!}\right) - (s+k)\ln\left(\frac{s+k}{\sum_{l=1}^s z_l}\right) + (s+k) \\ &\quad + \sum_{l=1}^s\left[(\widehat{\gamma}-1)\ln(z_l) + \ln\left(1+z_l^{\widehat{\gamma}}\right) - \widehat{\delta}z_l^{\widehat{\gamma}}\right] - \ln\left(\sum_*\prod_{j=1}^k z_{B_j}\right), \end{aligned} \tag{3.2}$$

where $\widehat{\gamma}$ and $\widehat{\delta}$ are the MLE of γ and δ under the LD, respectively. For $k = 1$ and $k = 2$, the logarithm of RML is

$$\begin{aligned} \Lambda_1 &= s\ln\left(\frac{\widehat{\gamma}\widehat{\delta}^2}{1+\widehat{\delta}}\right) + \ln(s) - (s+1)\ln\left(\frac{s+1}{\sum_{l=1}^s z_l}\right) + (s+1) + \sum_{l=1}^s\left[(\widehat{\gamma}-1)\ln(z_l) + \ln\left(1+z_l^{\widehat{\gamma}}\right) - \widehat{\delta}z_l^{\widehat{\gamma}}\right] \\ &\quad -\ln\left(\sum_{B_1}^s z_{B_1}\right), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \Lambda_2 &= s\ln\left(\frac{\widehat{\gamma}\widehat{\delta}^2}{1+\widehat{\delta}}\right) + \ln\left(\frac{s(s-1)}{2}\right) - (s+2)\ln\left(\frac{s+2}{\sum_{l=1}^s z_l}\right) + (s+2) \\ &\quad + \sum_{l=1}^s\left[(\widehat{\gamma}-1)\ln(z_l) + \ln\left(1+z_l^{\widehat{\gamma}}\right) - \widehat{\delta}z_l^{\widehat{\gamma}}\right] - \ln\left(\sum_{B_1=1}^s\sum_{B_2=B_1+1}^{s-1} z_{B_1}z_{B_2}\right). \end{aligned} \tag{3.4}$$

This is the decision rule: we select the LD instead of the EOD when $\Lambda = \ln\left(\frac{L_{LD}(\widehat{\gamma},\widehat{\delta};z)}{w(\widehat{\epsilon};z)}\right) > 0$ and for $\Lambda \leq 0$ reject the LD versus of EOD.

3.1 Kolmogorov distance procedure

Usually, when one would like to evaluate similarity of two probability distributions, one of the methods is the Kolmogorov distance (KD) between two distributions. Assuming that $H_{LD}(\widehat{\gamma}, \widehat{\delta})$ and $W_{Exp}(\widehat{\epsilon})$ be the CDFs calculated by using the estimated parameters respectively of the LD and EOD. Further, consider $F_s(z)$ be the empirically observed CDF which is obtained from the real data. We define KDs respect to the two distributions such as $KD_{LD} = \sup_{-\infty<x<\infty} |H_{LD} - F_s(z)|$ and $KD_{Exp} = \sup_{-\infty<x<\infty} |W_{Exp} - F_s(z)|$.

Any two distributions which have the minimum value of the KD are selected as the appropriate model and it is the decision rule.

3.2 Asymptotic Results of the log-RML

In this sub section, asymptotic distribution of the log-RML statistics $L_{LD}(\gamma, \delta; z)$ and $w(\epsilon; z)$ under the null hypothesis in two approaches is derived. The outcomes are based on the Central Limit Theorem (CLT) and [15].

Now, we consider that data come from the LD with parameters γ and δ and computing distribution is an EOD with parameter ϵ. Now miss specified of ϵ, when data are coming from the LD, $\widetilde{\epsilon}$, is obtained when $\pi(\epsilon) = E_{LD}(\ln(w(z;\epsilon)))$ maximized. Similarly, $\widetilde{\epsilon}$ is the solution of

$$\frac{s-k}{\epsilon} - s\bar{z} = 0, \tag{3.5}$$

or

$$\widetilde{\epsilon} = \frac{s-k}{s\widehat{E_{LD}}(Z)}. \tag{3.6}$$

But from (1-1), $\widehat{E_{LD}}(Z) = \frac{1}{\widehat{\epsilon}}\frac{s+k}{s}$ and from (2-14) $\widehat{\epsilon} = \frac{s+k}{s\bar{z}}$. Hence

$$\widetilde{\epsilon} = \frac{s-k}{s\bar{z}}. \tag{3.7}$$

Consequently, for large value of s, $\Lambda = \ln\left(\frac{L_{LD}(\gamma,\delta;z)}{w(\widetilde{\epsilon};z)}\right)$ is asymptotically normal distributed with mean

$$\begin{aligned} E_{LD}(\Lambda) &= E_{LD}\left(\ln\left(L_{LD}(\gamma,\delta;z)\right) - \ln\left(w(\widetilde{\epsilon};z)\right)\right) \\ &= E_{LD}\left(\left(\frac{\gamma\delta^2}{1+\delta}\right)^s \left(\prod_{l=1}^{s} z_l^{\gamma-1}\left(1+z_l^{\gamma}\right)\right) e^{-\delta\sum_{l=1}^{s} z_l^{\gamma}} - \frac{k!\,(s-k)!}{s!}\widetilde{\epsilon}^{s+1} e^{-s\widetilde{\epsilon}\widetilde{z}} \sum_{*}\prod_{j=1}^{k} z_{B_j}\right), \end{aligned} \tag{3.8}$$

and variance

$$\begin{aligned} var_{LD}(\Lambda) &= var_{LD}\left(\ln\left(L_{LD}(\gamma,\delta;z)\right) - \ln\left(w(\widetilde{\epsilon};z)\right)\right) = var_{LD}\left(\left(\frac{\gamma\delta^2}{1+\delta}\right)^s \left(\prod_{l=1}^{s} z_l^{\gamma-1}\left(1+z_l^{\gamma}\right)\right) e^{-\delta\sum_{l=1}^{s} z_l^{\gamma}}\right. \\ &\left. - \frac{k!\,(s-k)!}{s!}\widetilde{\epsilon}^{s+1} e^{-s\widetilde{\epsilon}\widetilde{z}} \sum_{*}\prod_{j=1}^{k} z_{B_j}\right). \end{aligned} \tag{3.9}$$

When the true sample CDF was the LD and the computing CDF was EOD, the probability of correct selection (PCS) is asymptotically written by

$$\text{PCS}_{LD} = P(\Lambda > 0) \approx 1 - \Phi\left(\frac{-sE_{LD}(\Lambda)}{\sqrt{s\,var_{LD}(\Lambda)}}\right),$$

where, $\Phi(.)$ is used for standardized normal CDF.

4 Conclusion and Discussion

In this section, the simulation study and actual example are used to evaluate the results.

4.1 Simulation

Here, the Monte Carlo simulation is performed to evaluate which model between the LD and EOD more fitting on the data based on the assumed optimality criteria. Actually, the performance of the PCS of the LD and EOD according to LRT, KD and asymptotic LRT methods is considered. Here, the case such that $LD(1, \delta)$ is as the null distribution and the alternative is EOD is assumed. So, we consider that $s = 3\,(1)\,10(10)100$, δ=0.1,5 and k=1,2,3. By using the following algorithm, the Monte Carlo simulation is conducted.

Step 1: Simulate a random of size s from a LD.

Step 2: Obtain the MLE of the parameters for each certain value of δ. Also, compute the miss specified parameter $\widetilde{\epsilon}$ of EOD by using equations (3-6).

Step 3: Calculate the LRT statistics δ based on Step 2.

Step 4: Calculate Monte Carlo samples Λ_i by T times repetition of Steps 1-3, where $i = 1, 2, \ldots, T$.

Step 5: Asymptotic PCS based on the LRT and the KD is respectively explained based on the true distribution is the LD.

$$\mathrm{PCS}_{hw}^{LRT} \approx \frac{\#\text{ of value in step } 3 > 0}{T},$$

and

$$\mathrm{PCS}_{hw}^{KD} \approx \frac{\# \, of \text{ time } KD_{LD} \text{ is minimum with respect to } KD_{Exp}}{T}.$$

Also, PCS_{LD} is the approximate PCS according to the normal distribution of Λ as $s \to \infty$ (by using equations 3-7 and 3-8). The results by using **R** statistical software are computed from Monte Carlo simulation So, the LRT, asymptotic approximation of LRT (ALRT) and the KD results are shown in Tables 1 & 2. for different values of the sample size as well as the different values of the number of outliers k.

Table 1: The PCS based on the LRT, KD and ALRT methods for the actual model of LD and computing model of EOD when $\gamma = 1$ and $\delta = 0.1$.

n	$k=1$			$k=2$			$k=3$		
	PCS_{hw}^{LRT}	PCS_{hw}^{KD}	PCS_{LD}	PCS_{hw}^{LRT}	PCS_{hw}^{KD}	PCS_{LD}	PCS_{hw}^{LRT}	PCS_{hw}^{KD}	PCS_{LD}
3	0.7881	0.3544	0.99958	–	–	–	–	–	–
4	0.7932	0.4068	0.99996	0.7452	0.3902	0.99969	–	–	–
5	0.7973	0.4180	1.00000	0.7675	0.4321	0.99998	0.7184	0.7669	0.99982
6	0.8033	0.4200	1.00000	0.7711	0.4857	1.00000	0.7383	0.5111	0.99998
7	0.8156	0.4134	1.00000	0.7960	0.5385	1.00000	0.7613	0.4860	1.00000
8	0.8158	0.3960	1.00000	0.7942	0.5569	1.00000	0.7763	0.5433	1.00000
9	0.8222	0.3878	1.00000	0.8127	0.5524	1.00000	0.7827	0.5877	1.00000
10	0.8299	0.3671	1.00000	0.8222	0.5597	1.00000	0.8016	0.6075	1.00000
20	0.8752	0.3024	1.00000	0.8761	0.5148	1.00000	0.8693	0.6452	1.00000
30	0.9116	0.2559	1.00000	0.9103	0.4672	1.00000	0.9050	0.5974	1.00000
40	0.9317	0.2331	1.00000	0.9337	0.4192	1.00000	0.9303	0.5574	1.00000
50	0.9502	0.2118	1.00000	0.9528	0.3825	1.00000	0.9514	0.5266	1.00000
60	0.9618	0.1862	1.00000	0.9612	0.3605	1.00000	0.9612	0.4985	1.00000
70	0.9706	0.1807	1.00000	0.9748	0.3433	1.00000	0.9713	0.4786	1.00000
80	0.9762	0.1574	1.00000	0.9784	0.3128	1.00000	0.9796	0.4545	1.00000
90	0.9821	0.1586	1.00000	0.9838	0.3069	1.00000	0.9842	0.4226	1.00000
100	0.9871	0.1467	1.00000	0.9870	0.2896	1.00000	0.9871	0.4050	1.00000

Table 2: The PCS based on the LRT, KD and ALRT methods when the true model is EOD and computing model is the LD for $\gamma = 1$ and $\delta = 5$.

n	$k=1$			$k=2$			$k=3$		
	PCS_{hw}^{LRT}	PCS_{hw}^{KD}	PCS_{LD}	PCS_{hw}^{LRT}	PCS_{hw}^{KD}	PCS_{LD}	PCS_{hw}^{LRT}	PCS_{hw}^{KD}	PCS_{LD}
3	0.0027	0.3565	0.53590	–	–	–	–	–	–
4	0.0151	0.4007	0.59779	0.1272	0.4504	0.77741	–	–	–
5	0.0428	0.4196	0.68182	0.0869	0.4239	0.75484	0.2429	0.8117	0.89733
6	0.0801	0.4166	0.76509	0.0610	0.4924	0.73378	0.2120	0.5808	0.89804
7	0.1353	0.4103	0.85234	0.0417	0.5338	0.70948	0.1768	0.4916	0.88991
8	0.1908	0.3990	0.91518	0.0269	0.5443	0.68091	0.1565	0.5511	0.88844
9	0.2556	0.3838	0.96061	0.0206	0.5601	0.66824	0.1318	0.5879	0.87876
10	0.3065	0.3856	0.98223	0.0248	0.5686	0.69296	0.1054	0.6214	0.86112
20	0.5402	0.3113	1.00000	0.3636	0.5284	0.99964	0.1008	0.6584	0.93284
30	0.5746	0.2614	1.00000	0.5042	0.4695	1.00000	0.3941	0.6165	0.99999
40	0.5986	0.2394	1.00000	0.5524	0.4361	1.00000	0.4983	0.5730	1.00000
50	0.5965	0.2217	1.00000	0.5600	0.3951	1.00000	0.5367	0.5361	1.00000
60	0.5906	0.1933	1.00000	0.5805	0.3726	1.00000	0.5476	0.5148	1.00000
70	0.6027	0.1771	1.00000	0.5890	0.3529	1.00000	0.5695	0.4823	1.00000
80	0.6012	0.1675	1.00000	0.5809	0.3206	1.00000	0.5751	0.4605	1.00000
90	0.5886	0.1557	1.00000	0.6030	0.3101	1.00000	0.5843	0.4532	1.00000
100	0.6005	0.1522	1.00000	0.5889	0.2917	1.00000	0.5804	0.4279	1.00000

4.2 Conclusion

A random sample is simulated from the LD and based on these values; the unknown parameters of two distributions are estimated. Then, the PCSs based on the LRT, KD and ALRT methods are obtained and tabulated in Tables 1&2 for various values of s and k. According to the Tables, PCSs of LRT and ALRT are increasing respect to n, but PCSs of KD are decreased when the sample size increased. It means that increasing the sample size leads to increase the chances of choosing the LD instead of the EOD. But, it is reverse using PCS based on KD. Also, the PCSs are almost decreasing respect to the number of outliers (k). Further, when the sample size excesses than 10, the PCSs based on the ALRT are become 1.

4.3 Data analysis

In 1982, Nelson [12] has shown that the breakdown times of insulated fluid for the voltages are follows ED. Data are as follows.

0.19 0.78 0.96 1.31 2.78 3.16 4.15 4.67 4.85 6.50

7.35 8.01 8.27 12.06 31.75 32.52 33.91 36.71 72.89

In this example, the main objective is that to select the proper model to fit on the data among the LD and EOD. To estimate the parameters of the LD and EOD, ML method is used. For the EOD, we should estimate k. So, the likelihood function respect to k is tabulated in Table 3.

Table 3. Likelihood function of EOD for different values of k.

k	1	2	3
Likelihood function	5.947149e-31	5.832186e-31	5.457728e-31

It is observed that k=1 maximizes the likelihood. So, the number of outliers is one and the MLE of the parameters of EOD are shown in Table 4. Also, MLE of the parameters of the LD is presented in Table 4.

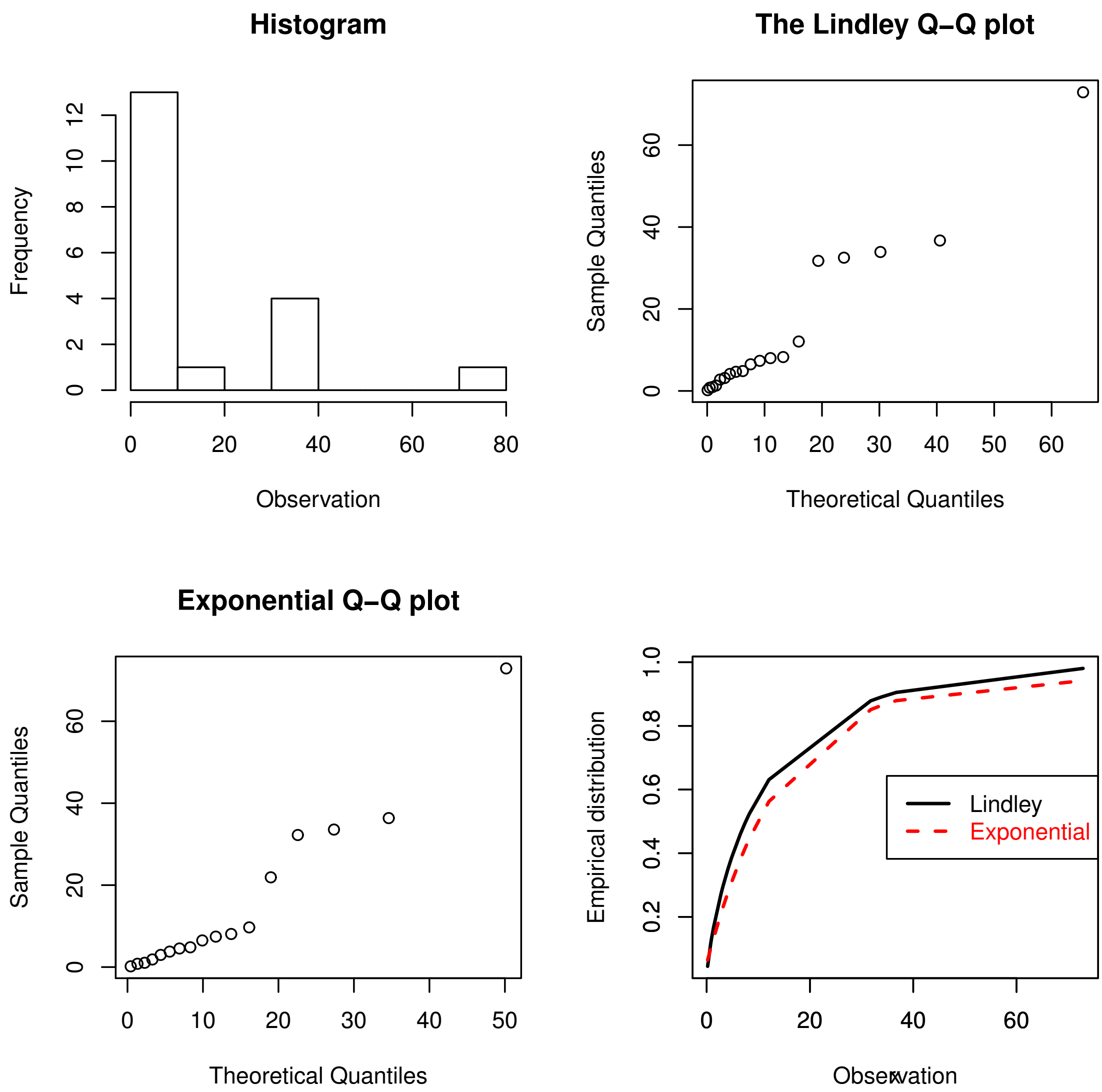

Figure 1. Histogram, Q-Q plot and the estimated distribution function of two distributions.

Table 4. MLE of the parameters of the LD and EOD.

$\widehat{\gamma}$	$\widehat{\delta}$	$\widehat{\epsilon}$
0.61406	0.39788	0.07331

Histogram and the Q-Q plot of two distributions are shown in Figure 1. to evaluate the goodness of fit. To make the comparison purpose, we have estimated the distribution functions of two distributions for data set and plotted in Figure 1. Fitting probability model shows that the two fitted models are very similar. So, it is necessary to obtain a tool which is discriminated among them.

Finally, it has been seen that the log-likelihood values related to the LD and EOD is -68.53093 and -69.59723, respectively. Thus, Λ=1.066293 and we conclude that the LD is preferred.

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