Int. J. Nonlinear Anal. Appl. 16 (2025) 9, 41–55
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2023.26401.4430

Non-resonant Nabla fractional boundary value problems

Jagan Mohan Jonnalagadda

Department of Mathematics, Birla Institute of Technology & Science Pilani, Hyderabad, Telangana, 500078, India

(Communicated by Abdolrahman Razani)

Abstract

We consider two simple non-resonant boundary value problems for a nabla fractional difference equation. First, we construct associated Green's functions and obtain some of their properties. Under suitable constraints on the nonlinear part of the nabla fractional difference equation, we deduce sufficient conditions for the existence of solutions to the considered problems through an appropriate fixed point theorem. We also provide two examples to demonstrate the applicability of the established results.

Keywords: Nabla fractional difference, boundary value problem, Green's function, resonance, memory property, fixed point, existence of a solution
2020 MSC: 39A12

1 Introduction

Nabla fractional calculus is an integrated theory of arbitrary order sums and differences in the backward sense. The concept of nabla fractional difference traces back to the works of many famous researchers in the last two decades. For a detailed introduction to the evolution of nabla fractional calculus, we refer to a recent monograph [7] and the references therein.

During the past decade, there has been an increasing interest in analyzing nabla fractional boundary value problems. To name a few notable works, we refer to [2, 3, 5, 6, 8, 9, 11, 12, 13]. In this line, we investigate two simple non-resonant nabla fractional boundary value problems. Specifically, we shall consider

$$\begin{cases} \big(\nabla^{\nu}_{\rho(a)}u\big)(t) + f(t,u(t)) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \\ \big(\nabla u\big)(a+1) = 0, \quad \big(\nabla u\big)(b) = 0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \big(\nabla^{\nu}_{\rho(a)}u\big)(t) = f(t,u(t)), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = 0, \quad \big(\nabla u\big)(a+1) = \big(\nabla u\big)(b), \end{cases} \tag{1.2}$$

where a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_3 = \{3, 4, 5, \cdots\}$, $\mathbb{N}^{b}_{a+2} = \{a+2, a+3, \cdots, b\}$, $f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}$, $1 < \nu < 2$ and $\nabla^{\nu}_{\rho(a)}u$ denotes the ν^{th}-order Riemann–Liouville nabla fractional difference of u based at $\rho(a) = a - 1$.

The present article is organized as follows: Section 2 contains preliminaries on discrete fractional calculus. We construct associated Green's functions and obtain some of their properties in Section 3. We deduce sufficient conditions

Email address: j.jaganmohan@hotmail.com (Jagan Mohan Jonnalagadda)

Received: May 2023 *Accepted:* October 2023

for the existence of solutions to (1.1) and (1.2) in Section 4 by employing the Leray–Schauder nonlinear alternative coupled with some suitable constraints on the nonlinear part of the nabla fractional difference equation. We also provide two examples to demonstrate the applicability of the established results in Section 5.

Remark 1.1. The boundary value problems (1.1) and (1.2) are not at resonance because the corresponding homogeneous problems

$$\begin{cases} \big(\nabla^{\nu}_{\rho(a)}u\big)(t) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ \big(\nabla u\big)(a+1) = 0, \quad \big(\nabla u\big)(b) = 0, \end{cases} \tag{1.3}$$

and

$$\begin{cases} \big(\nabla^{\nu}_{\rho(a)}u\big)(t) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ u(a) = 0, \quad \big(\nabla u\big)(a+1) = \big(\nabla u\big)(b), \end{cases} \tag{1.4}$$

have only trivial solutions. As $\nu \to 2$, the boundary value problems (1.1) and (1.2) reduce to the following second-order discrete boundary value problems, respectively.

$$\begin{cases} \big(\nabla^2 u\big)(t) + f(t, u(t)) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ \big(\nabla u\big)(a+1) = 0, \quad \big(\nabla u\big)(b) = 0, \end{cases} \tag{1.5}$$

and

$$\begin{cases} \big(\nabla^2 u\big)(t) = f(t, u(t)), \quad t \in \mathbb{N}^b_{a+2}, \\ u(a) = 0, \quad \big(\nabla u\big)(a+1) = \big(\nabla u\big)(b). \end{cases} \tag{1.6}$$

These problems are at resonance because the corresponding homogeneous problems

$$\begin{cases} \big(\nabla^2 u\big)(t) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ \big(\nabla u\big)(a+1) = 0, \quad \big(\nabla u\big)(b) = 0, \end{cases} \tag{1.7}$$

and

$$\begin{cases} \big(\nabla^2 u\big)(t) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ u(a) = 0, \quad \big(\nabla u\big)(a+1) = \big(\nabla u\big)(b), \end{cases} \tag{1.8}$$

have nontrivial solutions. This shows that the memory property of nabla fractional difference operators play an important role in describing the behaviour of solutions of the corresponding nabla fractional boundary value problems.

2 Preliminaries

We shall use the following fundamentals of discrete calculus [4] and discrete fractional calculus [7] throughout the article. Denote by $\mathbb{N}_c = \{c, c+1, c+2, \dots\}$ and $\mathbb{N}^d_c = \{c, c+1, c+2, \dots, d\}$, for any real numbers c and d such that $d - c \in \mathbb{N}_1$.

Definition 2.1. [4] The backward jump operator $\rho : \mathbb{N}_{c+1} \to \mathbb{N}_c$ is defined by

$$\rho(t) = t - 1, \quad t \in \mathbb{N}_{c+1}.$$

Definition 2.2. [7] For $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ and $\mu \in \mathbb{R}$ such that $(t + \mu) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, the generalized rising function is defined by

$$t^{\overline{\mu}} = \frac{\Gamma(t+\mu)}{\Gamma(t)}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Also, if $t \in \{\dots, -2, -1, 0\}$ and $\mu \in \mathbb{R}$ such that $(t + \mu) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, then we use the convention that $t^{\overline{\mu}} = 0$.

Definition 2.3. [7] The μ^{th}-order nabla fractional Taylor monomial is defined by

$$H_\mu(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)}, \quad \mu \in \mathbb{R} \setminus \{\dots, -2, -1\},$$

provided the right-hand side exists.

Definition 2.4. [4] Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order nabla difference of u is defined by

$$\big(\nabla u\big)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1}u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 2.5. [7] Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-\nu} u\big)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-\nu} u\big)(a) = 0$.

Definition 2.6. [7] Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \leq N$. The ν^{th}-order *Riemann–Liouville* nabla difference of u based at a is given by

$$\big(\nabla_a^{\nu} u\big)(t) = \Big(\nabla^N\big(\nabla_a^{-(N-\nu)}u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

In the subsequent lemmas, we collect a few properties of nabla fractional Taylor monomials.

Lemma 2.7. [7] The following properties hold, provided the expressions in this lemma are well defined.

1. $H_\mu(a, a) = 0$;
2. $H_0(t, a) = 1$;
3. $H_\mu(t, a) = 0$ for $t \in \mathbb{N}_a$ and $\mu \in \{\dots, -2, -1\}$;
4. $H_\mu(t, \rho(a)) = H_\mu(t + 1, a)$;
5. $H_\mu(t, a + 1) = H_\mu(t - 1, a)$;
6. $\nabla H_\mu(t, a) = H_{\mu-1}(t, a)$;
7. $\nabla H_\mu(b, t) = -H_{\mu-1}(b + 1, t)$;
8. $H_\mu(t, a) - H_{\mu-1}(t, a) = H_\mu(t, a + 1)$;
9. If $0 < \mu < t \leq r$, then $r^{\overline{-\mu}} \leq t^{\overline{-\mu}}$.

Lemma 2.8. [9] Let $s \in \mathbb{N}_a$ and $\mu > -1$. Then, the following properties hold:

(a) If $t \in \mathbb{N}_{\rho(s)}$, then $H_\mu(t, \rho(s)) \geq 0$;
(b) If $t \in \mathbb{N}_s$, then $H_\mu(t, \rho(s)) > 0$;
(c) If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geq 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of s;
(d) If $t \in \mathbb{N}_s$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is an increasing function of s;
(e) If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geq 0$, then $H_\mu(t, \rho(s))$ is a nondecreasing function of t;
(f) If $t \in \mathbb{N}_s$ and $\mu \geq 0$, then $H_\mu(t, \rho(s))$ is an increasing function of t;
(g) If $t \in \mathbb{N}_{s+1}$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of t.

We require the following composition rule of nabla fractional sum in the next section.

Lemma 2.9. [7] Let $k \in \mathbb{N}_0$, $\mu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \mu \leq N$. Then,

$$\Big(\nabla^k\big(\nabla_a^{-\mu}u\big)\Big)(t) = \big(\nabla_a^{k-\mu}u\big)(t), \quad t \in \mathbb{N}_{a+k}.$$

3 Green's Functions & Their Properties

In this section, we construct associated Green's functions for the following linear boundary value problems and deduce their properties.

$$\begin{cases} \big(\nabla^{\nu}_{\rho(a)} u\big)(t) + h(t) = 0, & t \in \mathbb{N}^{b}_{a+2}, \\ \big(\nabla u\big)(a+1) = 0, & \big(\nabla u\big)(b) = 0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \big(\nabla^{\nu}_{\rho(a)} u\big)(t) = h(t), & t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = 0, & \big(\nabla u\big)(a+1) = \big(\nabla u\big)(b), \end{cases} \tag{3.2}$$

where $h : \mathbb{N}^{b}_{a+2} \to \mathbb{R}$. We begin with the following lemma and remark.

Lemma 3.1. Let $0 < \mu < 1$. Denote by

$$G_{\mu}(t) = H_{\mu}(t, \rho(a)) - \mu H_{\mu}(t, a), \quad t \in \mathbb{N}_{a}.$$

Then, the following properties hold:

(I) $G_{\mu}(a) = 1$;
(II) $G_{\mu}(t) > 0$ for $t \in \mathbb{N}_{a}$;
(III) $G_{\mu-1}(t) > 0$ for $t \in \mathbb{N}_{a}$;
(IV) $G_{\mu}(t)$ is a nondecreasing function of t;
(V) $0 \leq \big(\nabla G_{\mu}\big)(t) < G_{\mu}(t)$ for $t \in \mathbb{N}_{a+1}$.

Proof . The proof of (I) follows from Definition 2.3 and Lemma 2.7 (1). To prove (II), for $t \in \mathbb{N}_{a}$, consider

$$\begin{aligned} G_{\mu}(t) &= H_{\mu}(t, \rho(a)) - \mu H_{\mu}(t, a) \\ &= H_{\mu}(t+1, a) - \mu H_{\mu}(t, a) \quad (\text{By} \quad \text{Lemma 2.7 (4)}) \\ &> H_{\mu}(t+1, a) - H_{\mu}(t, a) \\ &> 0. \quad (\text{By} \quad \text{Lemma 2.8 (f)}) \end{aligned}$$

The proof of (II) is complete. To prove (III), for $t \in \mathbb{N}_{a}$, consider

$$\begin{aligned} G_{\mu-1}(t) &= H_{\mu-1}(t, \rho(a)) - (\mu - 1) H_{\mu-1}(t, a) \\ &> 0. \quad (\text{By} \quad \text{Lemma 2.8 (a, b)}) \end{aligned}$$

The proof of (III) is complete. To prove (IV), for $t \in \mathbb{N}_{a+1}$, consider

$$\begin{aligned} \big(\nabla G_{\mu}\big)(t) &= \nabla \left[H_{\mu}(t, \rho(a)) - \mu H_{\mu}(t, a)\right] \\ &= H_{\mu-1}(t, \rho(a)) - \mu H_{\mu-1}(t, a) \quad (\text{By} \quad \text{Lemma 2.7 (6)}) \\ &= \frac{1}{\Gamma(\mu)} \left[(t - \rho(a))^{\overline{\mu-1}}\right] - \frac{\mu}{\Gamma(\mu)} \left[(t - a)^{\overline{\mu-1}}\right] \quad (\text{By} \quad \text{Definition 2.3}) \\ &= \frac{1}{\Gamma(\mu)} \left[\frac{\Gamma(t - a + \mu)}{\Gamma(t - a + 1)} - \mu \, \frac{\Gamma(t - a + \mu - 1)}{\Gamma(t - a)}\right] \quad (\text{By} \quad \text{Definition 2.2}) \\ &= \frac{1}{\Gamma(\mu)} \frac{\Gamma(t - a + \mu - 1)}{\Gamma(t - a)} \left[\frac{(t - a + \mu - 1)}{(t - a)} - \mu\right] \\ &= \frac{1}{\Gamma(\mu)} \frac{\Gamma(t - a + \mu - 1)}{\Gamma(t - a)} \left[1 + \frac{(\mu - 1)}{(t - a)} - \mu\right] \\ &= \frac{(1 - \mu)}{\Gamma(\mu)} \frac{\Gamma(t - a + \mu - 1)}{\Gamma(t - a)} \left[1 - \frac{1}{(t - a)}\right]. \end{aligned}$$

Since $0 < \mu < 1$ and $t \in \mathbb{N}_{a+1}$, we have $(1 - \mu) > 0$, $\Gamma(\mu) > 0$, $\Gamma(t - a + \mu - 1) > 0$, $\Gamma(t - a) > 0$ and $1 - \frac{1}{(t-a)} \geq 0$, implying that $\big(\nabla G_{\mu}\big)(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$. Therefore, $G_{\mu}(t)$ is a nondecreasing function of t. The proof of (IV) is complete. To prove (V), for $t \in \mathbb{N}_{a+1}$, consider

$$\begin{aligned}\big(\nabla G_\mu\big)(t) - G_\mu(t) &= \nabla\left[H_\mu(t,\rho(a)) - \mu H_\mu(t,a)\right] - \left[H_\mu(t,\rho(a)) - \mu H_\mu(t,a)\right]\\ &= \left[H_{\mu-1}(t,\rho(a)) - \mu H_{\mu-1}(t,a)\right] - \left[H_\mu(t,\rho(a)) - \mu H_\mu(t,a)\right] \quad \text{(By Lemma 2.7 (6))}\\ &= -\left[H_\mu(t,\rho(a)) - H_{\mu-1}(t,\rho(a))\right] + \mu\left[H_\mu(t,a) - H_{\mu-1}(t,a)\right]\\ &= -H_\mu(t,a) + \mu H_\mu(t,a+1) \quad \text{(By Lemma 2.7 (8))}\\ &= -H_\mu(t,a) + \mu H_\mu(t-1,a) \quad \text{(By Lemma 2.7 (5))}\\ &< -H_\mu(t,a) + H_\mu(t-1,a)\\ &= -\left[H_\mu(t,a) - H_\mu(t-1,a)\right] < 0. \quad \text{(By Lemma 2.8 (f))}\end{aligned}$$

From (IV), we have that $\big(\nabla G_\mu\big)(t) \geq 0$ for $t \in \mathbb{N}_{a+1}$. The proof of (V) is complete. □

Remark 3.2. Since $(b-a) \in \mathbb{N}_3$ and $(2-\nu) < 3$, from Definition 2.3 and Lemma 2.7 (9), we have

$$H_{\nu-2}(b,a) = \frac{1}{\Gamma(\nu-1)}\left[(b-a)^{\overline{\nu-2}}\right] \leq \frac{1}{\Gamma(\nu-1)}\left[3^{\overline{\nu-2}}\right] = \frac{\nu(\nu-1)}{2},$$

implying that $H_{\nu-2}(b,a) < 1$.

Theorem 3.3. The unique solution of the linear boundary value problem (3.1) is given by

$$u(t) = \sum_{s=a+2}^{b} \mathcal{G}(t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{3.3}$$

where

$$\mathcal{G}(t,s) = \begin{cases} \mathcal{G}_1(t,s), & t \in \mathbb{N}_a^{\rho(s)},\\ \mathcal{G}_2(t,s), & t \in \mathbb{N}_s^b. \end{cases} \tag{3.4}$$

Here

$$\mathcal{G}_1(t,s) = \frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} H_{\nu-2}(b,\rho(s)), \tag{3.5}$$

and

$$\mathcal{G}_2(t,s) = \mathcal{G}_1(t,s) - H_{\nu-1}(t,\rho(s)). \tag{3.6}$$

Proof . Applying $\nabla_{a+1}^{-\nu}$ on both sides of the nabla fractional difference equation in (3.1), we obtain

$$u(t) = C_1 H_{\nu-1}(t,\rho(a)) + C_2 H_{\nu-2}(t,\rho(a)) - \big(\nabla_{a+1}^{-\nu} h\big)(t), \quad t \in \mathbb{N}_a, \tag{3.7}$$

where C_1 and C_2 are arbitrary constants. Now, applying ∇ on both sides of (3.7), using Lemma 2.7 (6) and Lemma 2.9, we obtain

$$\big(\nabla u\big)(t) = C_1 H_{\nu-2}(t,\rho(a)) + C_2 H_{\nu-3}(t,\rho(a)) - \big(\nabla_{a+1}^{1-\nu} h\big)(t), \quad t \in \mathbb{N}_a. \tag{3.8}$$

Using the first boundary condition $\big(\nabla u\big)(a+1) = 0$ in (3.8), we get

$$(\nu-1)C_1 + (\nu-2)C_2 = 0. \tag{3.9}$$

Using the second boundary condition $\big(\nabla u\big)(b) = 0$ in (3.8), we get

$$C_1 H_{\nu-2}(b,\rho(a)) + C_2 H_{\nu-3}(b,\rho(a)) = \big(\nabla_{a+1}^{1-\nu} h\big)(b). \tag{3.10}$$

It follows from Lemma 2.7 (8) and (3.9) that

$$\begin{aligned} C_1 H_{\nu-2}(b,\rho(a)) + C_2 H_{\nu-3}(b,\rho(a)) &= C_1\left[H_{\nu-2}(b,\rho(a)) + \frac{(\nu-1)}{(2-\nu)} H_{\nu-3}(b,\rho(a))\right]\\ &= C_1\left[\frac{H_{\nu-2}(b,\rho(a)) - (\nu-1)H_{\nu-2}(b,a)}{(2-\nu)}\right]. \end{aligned} \tag{3.11}$$

Using (3.11) in (3.10) and rearranging the terms, we obtain

$$C_1 = \left[\frac{(2-\nu)}{H_{\nu-2}(b,\rho(a)) - (\nu-1)H_{\nu-2}(b,a)}\right] \sum_{s=a+2}^{b} H_{\nu-2}(b,\rho(s))h(s). \tag{3.12}$$

Then, from (3.9), we have

$$C_2 = \left[\frac{(\nu-1)}{H_{\nu-2}(b,\rho(a)) - (\nu-1)H_{\nu-2}(b,a)}\right] \sum_{s=a+2}^{b} H_{\nu-2}(b,\rho(s))h(s). \tag{3.13}$$

It follows from Lemma 2.7 (8) and (3.9) that

$$\begin{aligned}(2-\nu)H_{\nu-1}(t,\rho(a)) + (\nu-1)H_{\nu-2}(t,\rho(a)) &= H_{\nu-1}(t,\rho(a)) - (\nu-1)\left[H_{\nu-1}(t,\rho(a)) - H_{\nu-2}(t,\rho(a))\right] \\ &= H_{\nu-1}(t,\rho(a)) - (\nu-1)H_{\nu-1}(t,a). \end{aligned} \tag{3.14}$$

Substituting the expressions for C_1 and C_2 from (3.12) and (3.13) in (3.7) and using (3.14), we obtain (3.3). The proof is complete. □

Theorem 3.4. The unique solution of the linear boundary value problem (3.2) is given by

$$u(t) = \sum_{s=a+2}^{b} \mathcal{H}(t,s)h(s), \quad t \in \mathbb{N}_a^b, \tag{3.15}$$

where

$$\mathcal{H}(t,s) = \begin{cases} \mathcal{H}_1(t,s), & t \in \mathbb{N}_a^{\rho(s)}, \\ \mathcal{H}_2(t,s), & t \in \mathbb{N}_s^b. \end{cases} \tag{3.16}$$

Here

$$\mathcal{H}_1(t,s) = \frac{H_{\nu-1}(t,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)}, \tag{3.17}$$

and

$$\mathcal{H}_2(t,s) = \mathcal{H}_1(t,s) + H_{\nu-1}(t,\rho(s)). \tag{3.18}$$

Proof . Applying $\nabla_{a+1}^{-\nu}$ on both sides of the nabla fractional difference equation in (3.2), we obtain

$$u(t) = C_1 H_{\nu-1}(t,\rho(a)) + C_2 H_{\nu-2}(t,\rho(a)) + \left(\nabla_{a+1}^{-\nu}h\right)(t), \quad t \in \mathbb{N}_a, \tag{3.19}$$

where C_1 and C_2 are arbitrary constants. Now, applying ∇ on both sides of (3.19), using Lemma 2.7 (6) and Lemma 2.9, we obtain

$$\left(\nabla u\right)(t) = C_1 H_{\nu-2}(t,\rho(a)) + C_2 H_{\nu-3}(t,\rho(a)) + \left(\nabla_{a+1}^{1-\nu}h\right)(t), \quad t \in \mathbb{N}_a. \tag{3.20}$$

Using the first boundary condition $u(a) = 0$ in (3.19), we get

$$C_1 + C_2 = 0. \tag{3.21}$$

Using the second boundary condition $\left(\nabla u\right)(a+1) = \left(\nabla u\right)(b)$ in (3.20), we get

$$(\nu-1)C_1 + (\nu-2)C_2 = C_1 H_{\nu-2}(b,\rho(a)) + C_2 H_{\nu-3}(b,\rho(a)) + \left(\nabla_{a+1}^{1-\nu}h\right)(b). \tag{3.22}$$

Solving (3.21) and (3.22) for C_1 and C_2, we obtain

$$C_1 = \frac{1}{1 - H_{\nu-2}(b,a)} \sum_{s=a+2}^{b} H_{\nu-2}(b,\rho(s))h(s), \tag{3.23}$$

and

$$C_2 = -\frac{1}{1 - H_{\nu-2}(b,a)} \sum_{s=a+2}^{b} H_{\nu-2}(b,\rho(s))h(s). \tag{3.24}$$

Substituting the expressions for C_1 and C_2 from (3.23) and (3.24) in (3.19), we obtain (3.15). The proof is complete. □

Lemma 3.5. The Green's function $\mathcal{G}(t,s)$ defined in (3.4) satisfies the following properties:

1. $\mathcal{G}_1(t,s) > 0$ for $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{t+1}^b$;
2. $\mathcal{G}_1(t,s)$ is an increasing function of s for a fixed $t \in \mathbb{N}_a^{b-1}$;
3. $\mathcal{G}_2(t,s)$ is an increasing function of s for a fixed $t \in \mathbb{N}_{a+2}^b$;
4. For $t \in \mathbb{N}_a^b$,

$$\max_{s\in\mathbb{N}_{a+2}^b} |\mathcal{G}(t,s)| = \tilde{G}(t),$$

where

$$\tilde{G}(t) = \begin{cases} \mathcal{G}_1(a,b), & t = a, \\ \mathcal{G}_1(a+1,b), & t = a+1, \\ \max\Big\{\mathcal{G}_1(t,b), |\mathcal{G}_2(t,a+2)|, |\mathcal{G}_2(t,b)|\Big\}, & t \in \mathbb{N}_{a+2}^{b-1}, \\ \max\Big\{ |\mathcal{G}_2(b,a+2)|, |\mathcal{G}_2(b,b)|\Big\}, & t = b. \end{cases} \tag{3.25}$$

Proof . Throughout the proof, ∇ denotes the first order backward difference operator with respect to s. To prove (1), for $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{t+1}^b$, consider

$$\mathcal{G}_1(t,s) = \frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} H_{\nu-2}(b,\rho(s)).$$

It follows from Lemma 2.8 (b) and Lemma 3.1 that $G_{\nu-1}(t) > 0$, $\big(\nabla G_{\nu-1}\big)(b) > 0$ and $H_{\nu-2}(b,\rho(s)) > 0$, implying that $\mathcal{G}_1(t,s) > 0$ for $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{t+1}^b$. The proof of (1) is complete. To prove (2), for a fixed $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{t+1}^b$, consider

$$\begin{aligned} \nabla\mathcal{G}_1(t,s) &= \nabla\left[\frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} H_{\nu-2}(b,\rho(s))\right] \\ &= \frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} \left[\nabla H_{\nu-2}(b,\rho(s))\right] \\ &= -\frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} H_{\nu-3}(b+1,\rho(s)) \quad (\text{By} \quad \text{Lemma 2.7 (7)}) \\ &= -\frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} \left[\frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+2)\Gamma(\nu-2)}\right] \\ &= (2-\nu)\frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} \left[\frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+2)\Gamma(\nu-1)}\right] \\ &= \frac{(2-\nu)}{(b-s+1)} \frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} H_{\nu-2}(b,\rho(s)). \end{aligned} \tag{3.26}$$

Clearly, $(2-\nu) > 0$ and $(b-s+1) > 0$ for $s \in \mathbb{N}_{t+1}^b$ and $t \in \mathbb{N}_a^{b-1}$. It follows from Lemma 2.8 (b) and Lemma 3.1 that $G_{\nu-1}(t) > 0$, $\big(\nabla G_{\nu-1}\big)(b) > 0$ and $H_{\nu-2}(b,\rho(s)) > 0$, implying that

$$\nabla\mathcal{G}_1(t,s) = \frac{(2-\nu)}{(b-s+1)} \frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} H_{\nu-2}(b,\rho(s)) > 0$$

for a fixed $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{t+1}^b$. Therefore, $\mathcal{G}_1(t,s)$ is an increasing function of s for a fixed $t \in \mathbb{N}_a^{b-1}$. The proof of (2) is complete. To prove (3), for a fixed $t \in \mathbb{N}_{a+2}^b$ and $s \in \mathbb{N}_{a+2}^t$, consider

$$\begin{aligned} \nabla\mathcal{G}_2(t,s) &= \nabla\left[\mathcal{G}_1(t,s) - H_{\nu-1}(t,\rho(s))\right] \\ &= \nabla\left[\frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} H_{\nu-2}(b,\rho(s)) - H_{\nu-1}(t,\rho(s))\right] \\ &= \frac{(2-\nu)}{(b-s+1)} \frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} H_{\nu-2}(b,\rho(s)) \quad (\text{By} \quad (3.26)) \\ &\quad + H_{\nu-2}(t+1,\rho(s)). \quad (\text{By} \quad \text{Lemma 2.7 (7)}) \end{aligned}$$

Clearly, $(2-\nu) > 0$ and $(b-s+1) > 0$ for $s \in \mathbb{N}_{a+2}^{t}$ and $t \in \mathbb{N}_{a+2}^{b}$. It follows from Lemma 2.8 (b) and Lemma 3.1 that $G_{\nu-1}(t) > 0$, $\big(\nabla G_{\nu-1}\big)(b) > 0$, $H_{\nu-2}(b,\rho(s)) > 0$ and $H_{\nu-2}(t+1,\rho(s)) > 0$, implying that

$$\nabla \mathcal{G}_2(t,s) = \frac{(2-\nu)}{(b-s+1)} \frac{G_{\nu-1}(t)}{\big(\nabla G_{\nu-1}\big)(b)} H_{\nu-2}(b,\rho(s)) + H_{\nu-2}(t+1,\rho(s)) > 0$$

for a fixed $t \in \mathbb{N}_{a+2}^{b}$ and $s \in \mathbb{N}_{a+2}^{t}$. Therefore, $\mathcal{G}_2(t,s)$ is an increasing function of s for a fixed $t \in \mathbb{N}_{a+2}^{b}$. Finally, we prove (4). It follows from (1) - (3) that

$$\max_{s\in\mathbb{N}_{t+1}^{b}} |\mathcal{G}_1(t,s)| = \max_{s\in\mathbb{N}_{t+1}^{b}} \mathcal{G}_1(t,s) = \mathcal{G}_1(t,b), \quad t \in \mathbb{N}_{a}^{b-1},$$

and

$$\max_{s\in\mathbb{N}_{a+2}^{t}} |\mathcal{G}_2(t,s)| = \max\Big\{ |\mathcal{G}_2(t,a+2)|, |\mathcal{G}_2(t,t)| \Big\}, \quad t \in \mathbb{N}_{a+2}^{b},$$

implying that

$$\max_{s\in\mathbb{N}_{a+2}^{b}} |\mathcal{G}(t,s)| = \max\Big\{\mathcal{G}_1(t,b), |\mathcal{G}_2(t,a+2)|, |\mathcal{G}_2(t,t)| \Big\}, \quad t \in \mathbb{N}_{a+2}^{b-1}. \tag{3.27}$$

We also have that

$$\max_{s\in\mathbb{N}_{a+1}^{b}} |\mathcal{G}_1(a,s)| = \max_{s\in\mathbb{N}_{a+1}^{b}} \mathcal{G}_1(a,s) = \mathcal{G}_1(a,b), \tag{3.28}$$

$$\max_{s\in\mathbb{N}_{a+1}^{b}} |\mathcal{G}_1(a+2,s)| = \max_{s\in\mathbb{N}_{a+1}^{b}} \mathcal{G}_1(a+2,s) = \mathcal{G}_1(a+2,b), \tag{3.29}$$

and

$$\max_{s\in\mathbb{N}_{a+2}^{b}} |\mathcal{G}(b,s)| = \max_{s\in\mathbb{N}_{a+2}^{b}} |\mathcal{G}_2(b,s)| = \max\Big\{ |\mathcal{G}_2(b,a+2)|, |\mathcal{G}_2(b,b)| \Big\}. \tag{3.30}$$

Then, from (3.27) - (3.30), we obtain (5). The proof is complete. □

Remark 3.6. Denote by

$$\Lambda = \max_{t\in\mathbb{N}_{a}^{b}} \tilde{G}(t).$$

Consequently,

$$\max_{(t,s)\in\mathbb{N}_{a}^{b}\times\mathbb{N}_{a+2}^{b}} |\mathcal{G}(t,s)| = \Lambda.$$

Lemma 3.7. The Green's function $\mathcal{H}(t,s)$ defined in (3.16) satisfies the following properties:

1. $\mathcal{H}_1(t,s) \geq 0$ for $t \in \mathbb{N}_{a}^{\rho(s)}$ and $s \in \mathbb{N}_{a+2}^{b}$;
2. $\mathcal{H}_2(t,s) > 0$ for $t \in \mathbb{N}_{s}^{b}$ and $s \in \mathbb{N}_{a+2}^{b}$;
3. $\mathcal{H}_1(t,s)$ is an increasing function of t for a fixed $s \in \mathbb{N}_{a+2}^{b}$;
4. $\mathcal{H}_2(t,s)$ is an increasing function of t for a fixed $s \in \mathbb{N}_{a+2}^{b}$;
5. For $s \in \mathbb{N}_{a+2}^{b}$,
$$\max_{t\in\mathbb{N}_{a}^{b}} \mathcal{H}(t,s) = \mathcal{H}(b,s).$$
6. For $s \in \mathbb{N}_{a+2}^{b}$,
$$\min_{t\in\mathbb{N}_{a+1}^{b}} \mathcal{H}(t,s) = \mathcal{H}(a+1,s).$$

Proof . Throughout the proof, ∇ denotes the first order backward difference operator with respect to t. To prove (1), for $t \in \mathbb{N}_{a}^{\rho(s)}$ and $s \in \mathbb{N}_{a+2}^{b}$, consider

$$\mathcal{H}_1(t,s) = \frac{H_{\nu-1}(t,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)}.$$

It follows from Lemma 2.8 (a, b) and Remark 3.2 that $H_{\nu-1}(t,a) \geq 0$, $H_{\nu-2}(b,\rho(s)) > 0$ and $H_{\nu-2}(b,a) < 1$, implying that $\mathcal{H}_1(t,s) \geq 0$ for $t \in \mathbb{N}_a^{\rho(s)}$ and $s \in \mathbb{N}_{a+2}^b$. The proof of (1) is complete. To prove (2), for $t \in \mathbb{N}_s^b$ and $s \in \mathbb{N}_{a+2}^b$, consider

$$\mathcal{H}_2(t,s) = \mathcal{H}_1(t,s) + H_{\nu-1}(t,\rho(s)) = \frac{H_{\nu-1}(t,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)} + H_{\nu-1}(t,\rho(s)).$$

It follows from Lemma 2.8 (b) and Remark 3.2 that $H_{\nu-1}(t,a) > 0$, $H_{\nu-2}(b,\rho(s)) > 0$, $H_{\nu-1}(t,\rho(s)) > 0$ and $H_{\nu-2}(b,a) < 1$, implying that $\mathcal{H}_2(t,s) > 0$ for $t \in \mathbb{N}_s^b$ and $s \in \mathbb{N}_{a+2}^b$. The proof of (2) is complete. To prove (3), for a fixed $s \in \mathbb{N}_{a+2}^b$ and $t \in \mathbb{N}_{a+1}^{\rho(s)}$, consider

$$\begin{aligned}\nabla\mathcal{H}_1(t,s) &= \nabla\left[\frac{H_{\nu-1}(t,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)}\right]\\ &= \frac{H_{\nu-2}(t,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)}. \quad \text{(By Lemma 2.7 (6))}\end{aligned}$$

It follows from Lemma 2.8 (b) and Remark 3.2 that $H_{\nu-2}(t,a) > 0$, $H_{\nu-2}(b,\rho(s)) > 0$ and $H_{\nu-2}(b,a) < 1$, implying that

$$\nabla\mathcal{H}_1(t,s) = \frac{H_{\nu-2}(t,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)} > 0$$

for a fixed $s \in \mathbb{N}_{a+2}^b$ and $t \in \mathbb{N}_{a+1}^{\rho(s)}$. Therefore, $\mathcal{H}_1(t,s)$ is an increasing function of t for a fixed $s \in \mathbb{N}_{a+2}^b$. The proof of (3) is complete. To prove (4), for a fixed $s \in \mathbb{N}_{a+2}^b$ and $t \in \mathbb{N}_s^b$, consider

$$\begin{aligned}\nabla\mathcal{H}_2(t,s) &= \nabla\left[\mathcal{H}_1(t,s) + H_{\nu-1}(t,\rho(s))\right]\\ &= \nabla\left[\frac{H_{\nu-1}(t,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)} + H_{\nu-1}(t,\rho(s))\right]\\ &= \frac{H_{\nu-2}(t,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)} + H_{\nu-2}(t,\rho(s)) \quad \text{(By Lemma 2.7 (6))}.\end{aligned}$$

It follows from Lemma 2.8 (b) and Remark 3.2 that $H_{\nu-2}(t,a) > 0$, $H_{\nu-2}(b,\rho(s)) > 0$, $H_{\nu-2}(t,\rho(s)) > 0$ and $H_{\nu-2}(b,a) < 1$, implying that

$$\nabla\mathcal{H}_2(t,s) = \frac{H_{\nu-2}(t,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)} + H_{\nu-2}(t,\rho(s)) > 0$$

for a fixed $s \in \mathbb{N}_{a+2}^b$ and $t \in \mathbb{N}_s^b$. Therefore, $\mathcal{H}_2(t,s)$ is an increasing function of t for a fixed $s \in \mathbb{N}_{a+2}^b$. The proof of (4) is complete. From (1) - (4), we have

$$0 \leq \mathcal{H}_1(a,s) \leq \mathcal{H}_1(t,s) \leq \mathcal{H}_1(\rho(s),s), \quad s \in \mathbb{N}_{a+2}^b,$$

and

$$0 < \mathcal{H}_2(s,s) \leq \mathcal{H}_2(t,s) \leq \mathcal{H}_2(b,s), \quad s \in \mathbb{N}_{a+2}^b.$$

Therefore,

$$\max_{t\in\mathbb{N}_a^b} \mathcal{H}(t,s) = \max\left\{\mathcal{H}_1(\rho(s),s), \mathcal{H}_2(b,s)\right\}, \quad s \in \mathbb{N}_{a+2}^b, \tag{3.31}$$

and

$$\min_{t\in\mathbb{N}_{a+1}^b} \mathcal{H}(t,s) = \min\left\{\mathcal{H}_1(a+1,s), \mathcal{H}_2(s,s)\right\}, \quad s \in \mathbb{N}_{a+2}^b. \tag{3.32}$$

We have

$$\mathcal{H}_1(\rho(s),s) = \frac{H_{\nu-1}(s-1,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)},$$

and

$$\mathcal{H}_2(b,s) = \mathcal{H}_1(b,s) + H_{\nu-1}(b,\rho(s)) = \frac{H_{\nu-1}(b,a)H_{\nu-2}(b,\rho(s))}{1 - H_{\nu-2}(b,a)} + H_{\nu-1}(b,\rho(s)).$$

Clearly, from Lemma 2.8 (b, f), we have

$$H_{\nu-1}(s-1,a) < H_{\nu-1}(b,a), \quad H_{\nu-1}(b,\rho(s)) > 0, \quad s \in \mathbb{N}_{a+2}^{b},$$

implying that

$$\mathcal{H}_1(\rho(s),s) < \mathcal{H}_2(b,s), \quad s \in \mathbb{N}_{a+2}^{b}.$$

The statement of (5) follows from (3.31). Also, we have

$$\mathcal{H}_1(a+1,s) = \frac{H_{\nu-1}(a+1,a)H_{\nu-2}(b,\rho(s))}{1-H_{\nu-2}(b,a)} = \frac{H_{\nu-2}(b,\rho(s))}{1-H_{\nu-2}(b,a)},$$

and

$$\mathcal{H}_2(s,s) = \mathcal{H}_1(s,s) + H_{\nu-1}(s,\rho(s)) = \frac{H_{\nu-1}(s,a)H_{\nu-2}(b,\rho(s))}{1-H_{\nu-2}(b,a)} + 1.$$

Clearly, from Lemma 2.8 (b, f), we have

$$H_{\nu-1}(s,a) > 1, \quad s \in \mathbb{N}_{a+2}^{b},$$

implying that

$$\mathcal{H}_1(a+1,s) < \mathcal{H}_2(s,s), \quad s \in \mathbb{N}_{a+2}^{b}.$$

The statement of (6) follows from (3.32). □

Remark 3.8. Denote by

$$\Theta = \max_{s \in \mathbb{N}_{a+2}^{b}} \mathcal{H}(b,s).$$

Consequently,

$$\max_{(t,s) \in \mathbb{N}_{a}^{b} \times \mathbb{N}_{a+2}^{b}} \mathcal{H}(t,s) = \Theta.$$

4 Main Results

In this section, we state and prove the existence results for the boundary value problems (1.1) and (1.2). By Theorem 3.3, we observe that u is a solution of (1.1) if and only if u is a solution of the summation equation

$$u(t) = \sum_{s=a+2}^{b} \mathcal{G}(t,s) f(s,u(s)), \quad t \in \mathbb{N}_{a}^{b}. \tag{4.1}$$

By Theorem 3.4, we observe that u is a solution of (1.2) if and only if u is a solution of the summation equation

$$u(t) = \sum_{s=a+2}^{b} \mathcal{H}(t,s) f(s,u(s)), \quad t \in \mathbb{N}_{a}^{b}. \tag{4.2}$$

Any solution $u : \mathbb{N}_{a}^{b} \to \mathbb{R}$ of (1.1) (or (1.2)) can be viewed as a real $(b-a+1)$-tuple vector. Let

$$E = \Big\{ u : u : \mathbb{N}_{a}^{b} \to \mathbb{R} \ \Big| \ \big(\nabla u\big)(a+1) = 0, \quad \big(\nabla u\big)(b) = 0 \Big\},$$

and

$$F = \Big\{ u : u : \mathbb{N}_{a}^{b} \to \mathbb{R} \ \Big| \ u(a) = 0, \quad \big(\nabla u\big)(a+1) = \big(\nabla u\big)(b) \Big\}.$$

Then, E and F are Banach spaces equipped with the maximum norm defined by

$$\|u\| = \max_{t \in \mathbb{N}_{a}^{b}} |u(t)|.$$

Define the operators $T : E \to E$ and $S : F \to F$ by

$$\big(Tu\big)(t) = \sum_{s=a+2}^{b} \mathcal{G}(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b, \tag{4.3}$$

and

$$\big(Su\big)(t) = \sum_{s=a+2}^{b} \mathcal{H}(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{4.4}$$

Clearly, u is a fixed point of T if and only if u is a solution of (1.1), and u is a fixed point of S if and only if u is a solution of (1.2). Now, we search for nontrivial fixed points of T and S in order to identify nontrivial solutions of (1.1) and (1.2), respectively. The following fixed point theorem will be essential in this direction.

Theorem 4.1. [1] (Leray–Schauder Nonlinear Alternative) Let $(E, \|\cdot\|)$ be a Banach space, K be a closed and convex subset of E, U be a relatively open subset of K such that $0 \in U$, and $T : \bar{U} \to K$ be completely continuous. Then, either

(i) $u = Tu$ has a solution in $\bar{U}$; or
(ii) There exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda T u$.

Theorem 4.2. Assume that

(A 1) There exist $\phi : \mathbb{N}_{a+2}^b \to [0,\infty)$ and a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$|f(t,r)| \le \phi(t)\psi\left(|r|\right), \quad (t,r) \in \mathbb{N}_{a+2}^b \times \mathbb{R}.$$

(A 2) There exists $L > 0$ such that

$$\frac{L}{\Lambda\Omega(b-a-1)\psi\left(L\right)} > 1,$$

where

$$\Omega = \max_{t \in \mathbb{N}_{a+2}^b} \phi(t).$$

Then, the boundary value problem (1.1) has a solution defined on $\mathbb{N}_a^b$.

Proof . We first show that T maps bounded sets into bounded sets. For this purpose, for $r > 0$, let

$$B_r = \{u \in E : \|u\| \le r\},$$

be a bounded subset of E. Then, by (A 1), for $t \in \mathbb{N}_a^b$ and $u \in B_r$,

$$\begin{aligned}
\left|\big(Tu\big)(t)\right| &\le \sum_{s=a+2}^{b} |\mathcal{G}(t,s)|\,|f(s,u(s))| \\
&\le \Lambda \sum_{s=a+2}^{b} \phi(s)\psi\left(|u(s)|\right) \\
&\le \Lambda\psi\left(\|u\|\right) \sum_{s=a+2}^{b} \phi(s) \\
&\le \Lambda\Omega(b-a-1)\psi\left(r\right),
\end{aligned}$$

implying that

$$\left|\big(Tu\big)(t)\right| \le \Lambda\Omega(b-a-1)\psi\left(r\right).$$

Thus, T maps B_r into a bounded set. Since $\mathbb{N}_a^b$ is a discrete set, it follows immediately that T maps B_r into an equicontinuous set. Therefore, by the Arzela–Ascoli theorem, T is completely continuous. Next, we suppose $u \in E$

and that for some $0<\lambda<1$, $u=\lambda Tu$. Then, for $t\in\mathbb{N}_a^b$, and again by (A 1),

$$\begin{aligned} |u(t)| &= \left|\lambda\big(Tu\big)(t)\right| \\ &\le \sum_{s=a+2}^{b} |\mathcal{G}(t,s)|\,|f(s,u(s))| \\ &\le \Lambda \sum_{s=a+2}^{b} \phi(s)\psi\left(|u(s)|\right) \\ &\le \Lambda\psi\left(\|u\|\right) \sum_{s=a+2}^{b} \phi(s) \\ &\le \Lambda\Omega(b-a-1)\psi\left(\|u\|\right), \end{aligned}$$

implying that

$$\frac{\|u\|}{\Lambda\Omega(b-a-1)\psi\left(\|u\|\right)} \le 1.$$

It follows from (A 2) that $\|u\|\neq L$. If we set

$$U=\Big\{u\in E : \|u\|<L\Big\},$$

then the operator $T:\bar{U}\to E$ is completely continuous. From the choice of U, then there is no $u\in\partial U$ such that $u=\lambda Tu$, for some $0<\lambda<1$. It follows from Theorem 4.1 that T has a fixed point $u_0\in\bar{U}$, which is a desired solution of (1.1). □

Theorem 4.3. Assume that (A 1) and

(A 3) There exists $M>0$ such that

$$\frac{M}{\Theta\Omega(b-a-1)\psi\left(M\right)} > 1,$$

where

$$\Omega=\max_{t\in\mathbb{N}_{a+2}^b}\phi(t).$$

Then, the boundary value problem (1.2) has a solution defined on $\mathbb{N}_a^b$.

Proof . We first show that S maps bounded sets into bounded sets. For this purpose, for $r>0$, let

$$B_r=\{u\in F : \|u\|\le r\},$$

be a bounded subset of F. Then, by (A 1), for $t\in\mathbb{N}_a^b$ and $u\in B_r$,

$$\begin{aligned} \left|\big(Su\big)(t)\right| &\le \sum_{s=a+2}^{b} \mathcal{H}(t,s)\,|f(s,u(s))| \\ &\le \Theta \sum_{s=a+2}^{b} \phi(s)\psi\left(|u(s)|\right) \\ &\le \Theta\psi\left(\|u\|\right) \sum_{s=a+2}^{b} \phi(s) \\ &\le \Theta\Omega(b-a-1)\psi\left(r\right), \end{aligned}$$

implying that

$$\left|\big(Su\big)(t)\right| \le \Theta\Omega(b-a-1)\psi\left(r\right).$$

Thus, S maps B_r into a bounded set. Since $\mathbb{N}_a^b$ is a discrete set, it follows immediately that S maps B_r into an equicontinuous set. Therefore, by the Arzela–Ascoli theorem, S is completely continuous. Next, we suppose $u \in F$ and that for some $0 < \lambda < 1$, $u = \lambda S u$. Then, for $t \in \mathbb{N}_a^b$, and again by (A 1),

$$\begin{aligned}|u(t)| &= \left|\lambda\big(Su\big)(t)\right| \\ &\le \sum_{s=a+2}^{b} \mathcal{H}(t,s)\,|f(s,u(s))| \\ &\le \Theta \sum_{s=a+2}^{b} \phi(s)\psi\left(|u(s)|\right) \\ &\le \Theta\psi\left(\|u\|\right) \sum_{s=a+2}^{b} \phi(s) \\ &\le \Theta\Omega(b-a-1)\psi\left(\|u\|\right),\end{aligned}$$

implying that

$$\frac{\|u\|}{\Theta\Omega(b-a-1)\psi\left(\|u\|\right)} \le 1.$$

It follows from (A 2) that $\|u\| \neq M$. If we set

$$U = \Big\{u \in F : \|u\| < M\Big\},$$

then the operator $S : \bar{U} \to F$ is completely continuous. From the choice of U, then there is no $u \in \partial U$ such that $u = \lambda S u$, for some $0 < \lambda < 1$. It follows from Theorem 4.1 that S has a fixed point $u_0 \in \bar{U}$, which is a desired solution of (1.2). □

5 Examples

In this section, we provide two example to demonstrate the applicability of Theorems 4.2 and 4.3.

Example 5.1. Consider the boundary value problem

$$\begin{cases}\big(\nabla^{1.5}_{\rho(0)}u\big)(t) + tu^2(t) = 0, & t \in \mathbb{N}_2^5,\\ \big(\nabla u\big)(1) = 0, & \big(\nabla u\big)(5) = 0.\end{cases} \tag{5.1}$$

Here $a = 0$, $b = 5$, $\nu = 1.5$ and $f(t,r) = tr^2$. Clearly,

$$|f(t,r)| \le \phi(t)\psi\left(|r|\right), \quad (t,r) \in \mathbb{N}_2^5 \times \mathbb{R},$$

where

$$\phi(t) = t, \quad t \in \mathbb{N}_2^5,$$

and

$$\psi\left(|r|\right) = |r|^2 = r^2, \quad r \in \mathbb{R}.$$

Also, $\phi : \mathbb{N}_2^5 \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing function. Thus, the assumption (A 1) of Theorem 4.2 holds. Further, we have

$$\Omega = \max_{t \in \mathbb{N}_2^5} \phi(t) = 5.$$

Now, we calculate Λ. The Green's function associated with the boundary value problem (5.1) is given by

$$\mathcal{G}(t,s) = \begin{cases}\mathcal{G}_1(t,s), & t \in \mathbb{N}_0^{\rho(s)},\\ \mathcal{G}_2(t,s), & t \in \mathbb{N}_s^5,\end{cases} \tag{5.2}$$

where

$$\mathcal{G}_1(t,s) = \frac{G_{0.5}(t)}{\big(\nabla G_{\nu-1}\big)(5)} H_{-0.5}(5,\rho(s)), \tag{5.3}$$

and

$$\mathcal{G}_2(t,s) = \frac{G_{0.5}(t)}{\big(\nabla G_{0.5}\big)(5)} H_{-0.5}(5,\rho(s)) - H_{0.5}(t,\rho(s)). \tag{5.4}$$

It follows from Lemma 3.5 that

$$\max_{s\in\mathbb{N}_2^5} |\mathcal{G}(t,s)| = \tilde{G}(t), \quad t\in\mathbb{N}_0^5,$$

where

$$\tilde{G}(t) = \begin{cases} \mathcal{G}_1(0,5), & t=0, \\ \mathcal{G}_1(1,5), & t=1, \\ \max\Big\{\mathcal{G}_1(t,5), |\mathcal{G}_2(t,2)|, |\mathcal{G}_2(t,5)|\Big\}, & t\in\mathbb{N}_2^4, \\ \max\Big\{|\mathcal{G}_2(5,2)|, |\mathcal{G}_2(5,5)|\Big\}, & t=5. \end{cases} \tag{5.5}$$

Then,

$$\Lambda = \max_{t\in\mathbb{N}_0^5} \tilde{G}(t) = \max_{(t,s)\in\mathbb{N}_0^5\times\mathbb{N}_2^5} |\mathcal{G}(t,s)| = 12.5.$$

There exists $0 < L < \frac{1}{250}$ such that

$$\frac{L}{(12.5)(5)(4)L^2} > 1,$$

implying that the assumption (A 2) of Theorem 4.2 holds. Therefore, by Theorem 4.2, the boundary value problem (1.1) has a solution defined on $\mathbb{N}_0^5$.

Example 5.2. Consider the boundary value problem

$$\begin{cases} \big(\nabla^{1.5}_{\rho(0)}u\big)(t) = tu^2(t), & t\in\mathbb{N}_2^5, \\ u(0)=0, \quad \big(\nabla u\big)(1) = \big(\nabla u\big)(5), \end{cases} \tag{5.6}$$

Here $a=0$, $b=5$, $\nu=1.5$ and $f(t,r) = -tr^2$. Clearly,

$$|f(t,r)| \le \phi(t)\psi\left(|r|\right), \quad (t,r)\in\mathbb{N}_2^5\times\mathbb{R},$$

where

$$\phi(t) = t, \quad t\in\mathbb{N}_2^5,$$

and

$$\psi\left(|r|\right) = |r|^2 = r^2, \quad r\in\mathbb{R}.$$

Also, $\phi : \mathbb{N}_2^5 \to [0,\infty)$ and $\psi : [0,\infty)\to[0,\infty)$ is a nondecreasing function. Thus, the assumption (A 1) of Theorem 4.3 holds. Further, we have

$$\Omega = \max_{t\in\mathbb{N}_2^5} \phi(t) = 5.$$

Now, we calculate Θ. The Green's function associated with the boundary value problem (5.1) is given by

$$\mathcal{H}(t,s) = \begin{cases} \mathcal{H}_1(t,s), & t\in\mathbb{N}_0^{\rho(s)}, \\ \mathcal{H}_2(t,s), & t\in\mathbb{N}_s^5, \end{cases} \tag{5.7}$$

where

$$\mathcal{H}_1(t,s) = \frac{H_{0.5}(t,0)H_{-0.5}(5,\rho(s))}{1-H_{-0.5}(5,0)}, \tag{5.8}$$

and

$$\mathcal{H}_2(t,s) = \frac{H_{0.5}(t,0)H_{-0.5}(5,\rho(s))}{1-H_{-0.5}(5,0)} - H_{0.5}(t,\rho(s)). \tag{5.9}$$

It follows from Lemma 3.7 that

$$\max_{t\in\mathbb{N}_0^5}\mathcal{H}(t,s)=\mathcal{H}(5,s),\quad s\in\mathbb{N}_0^5,$$

where

$$\mathcal{H}(5,s)=\frac{H_{0.5}(5,0)H_{-0.5}(5,\rho(s))}{1-H_{-0.5}(5,0)}+H_{0.5}(5,\rho(s)),\quad s\in\mathbb{N}_0^5. \tag{5.10}$$

Then,

$$\Theta=\max_{s\in\mathbb{N}_0^5}\mathcal{H}(5,s)=\max_{(t,s)\in\mathbb{N}_0^5\times\mathbb{N}_2^5}\mathcal{H}(t,s)=4.3870.$$

There exists $0<M<\frac{1}{88}$ such that

$$\frac{M}{(4.3870)(5)(4)M^2}>1,$$

implying that the assumption (A 3) of Theorem 4.3 holds. Therefore, by Theorem 4.3, the boundary value problem (1.2) has a solution defined on $\mathbb{N}_0^5$.

References

[1] R.P. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[2] H.M. Ahmed and M. Alessandra Ragusa, *Nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential*, Bull. Malays. Math. Sci. Soc. **45** (2022), no. 6, 3239–3253.

[3] K. Ahrendt and C. Kissler, *Green's function for higher-order boundary value problems involving a nabla Caputo fractional operator*, J. Difference Equ. Appl. **25** (2019), no. 6, 788–800.

[4] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser Boston Inc., Boston, MA, 2001.

[5] C. Chen, M. Bohner, and B. Jia, *Existence and uniqueness of solutions for nonlinear Caputo fractional difference equations*, Turk. J. Math. **44** (2020), no. 3, 857–869.

[6] Y. Gholami and K. Ghanbari, *Coupled systems of fractional ∇-difference boundary value problems*, Differ. Equ. Appl. **8** (2016), no. 4, 459–470.

[7] C. Goodrich and A.C. Peterson, *Discrete Fractional Calculus*, Springer, Cham, 2015.

[8] L. Gu, Q. Zhong, and Z. Shao, *On multiple positive solutions for singular fractional boundary value problems with Riemann–Stieltjes integrals*, J. Funct. Spaces **2023** (2023), 1–7.

[9] A. Ikram, *Lyapunov inequalities for nabla Caputo boundary value problems*, J. Difference Equ. Appl. **25** (2019), no. 6, 757–775.

[10] J.M. Jonnalagadda, *Solutions of fractional nabla difference equations - existence and uniqueness*, Opuscula Math. **36** (2016), no. 2, 215–238.

[11] J.M. Jonnalagadda, *On two-point Riemann–Liouville type nabla fractional boundary value problems*, Adv. Dyn. Syst. Appl. **13** (2018), no. 2, 141–166.

[12] J.M. Jonnalagadda and N.S. Gopal, *Green's function for a discrete fractional boundary value problem*, Differ. Equ. Appl. **14** (2022), no. 2, 163–178.

[13] Z. Laadjal, T. Abdeljawad, and F. Jarad, *Some results for two classes of two-point local fractional proportional boundary value problems*, Filomat **37** (2023), no. 21, 7199–7216.