Int. J. Nonlinear Anal. Appl. In Press, 1–7
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.31920.4735

# Uniqueness of $L$-functions with weighted sharing

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(Communicated by Mugur Alexandru Acu)

**Abstract**

$L$-functions are complex functions associated with number-theoretic objects such as number fields, elliptic curves, modular forms, and automorphic representations. The general form of an $L$-function can be represented as a Dirichlet series, an Euler product, or in terms of its analytic continuation and functional equation. One of the most famous $L$-functions is the Riemann zeta function, defined as: $\zeta(s) = 1^s + 2^{-s} + 3^{-s} + \cdots = \sum_{n=1}^{\infty} n^{-s}$, where s is a complex number. $L$- function plays a fundamental role in studying prime numbers and connects to important conjectures like the Riemann Hypothesis. In this paper, we study the uniqueness of transcendental meromorphic functions and $L$-function whose certain difference-differential polynomials share a small function and rational function with weight, where $L$-function is a function that is Dirichlet series with the Riemann zeta function as the prototype. The Selberg class $S$ of $L$-functions is the set of all Dirichlet series $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ of a complex variable $s = \sigma + it$ with $a(1) = 1$.

Keywords: Uniqueness, Meromorphic functions, Entire functions, $L$-function, Sharing values
2020 MSC: 30D35

## 1 Introduction, Definitions and Results

In this paper, by $L$-functions we always mean $L$-functions that are Dirichlet series with the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ as the prototype. The Selberg class $S$ of $L$-functions is the set of all Dirichlet series $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ of a complex variable $s = \sigma + it$ with $a(1) = 1$ ; satisfying the following axioms [13, 14]:

**(i) Ramanujan hypothesis:** $a(n) << n^{\epsilon}$ for every $\epsilon > 0$.

**(ii) Analytic continuation:** There is a nonnegative integer $m$ such that $(s-1)^m L(s)$ is an entire function of finite order.

**(iii) Functional Equation:** $L$ satisfies a functional equation of type $\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})}$, where $\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + v_j)$ with positive real number $Q, \lambda_j$, and complex numbers $v_j, \omega$ with $Re(v_j) \geq 0$ and $|\omega| = 1$.

**(iv) Euler product:** $\log L(s) = \sum_{n=1}^{\infty} (\frac{(b(n))}{n^s})$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) << n^{\theta}$ for some $\theta < \frac{1}{2}$.

The Ramanujan hypothesis implies that the Dirichlet series $L$ converges absolutely in the half-plane $Rel(s) > 1$. The degree of an $L$-function is defined as $d_L = 2\sum_{j=1}^{K} \lambda_j$ where $K$ and $\lambda_j$ are respectively the positive integers and

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*Received:* September 2023 *Accepted:* June 2024

positive real number as in axiom (iii) above. $L$-function can be analytically continued as a meromorphic function in $\mathbb{C}$.

In this paper, by meromorphic functions, we will always mean meromorphic functions in the complex plane. We shall use Nevanlinna's theory and adopt Nevanlinna's theory's standard notations to demonstrate the main findings in the current work. Nevanlinna's theory uses common notations like the characteristic function $T(r; f)$, proximity function $m(r; f)$, counting function $N(r; f)$, and reduced counting function $\overline{N}(r; f)$, which are detailed in [3, 6, 15, 16]. Any quantity satisfying $S(r; h) = o(T(r; h))$ is denoted by $S(r; h)$ for a non-constant meromorphic function h as $r \to \infty$ and $r \notin E$. If $T(r, a) = S(r, h)$, we say $a$ is a small function concerning $h$. The order of growth of $f$ is defined as $\rho(f) = \lim_{r\to\infty} \sup \dfrac{\log T(r, f)}{\log r}$. We say that $f$ is a meromorphic function of finite order if $\rho(f) < \infty$. We require the following definitions to prove our main result.

**Definition 1.1.** [5] Let $f$ be a meromorphic function defined in the complex plane. Let $n$ be a positive integer and $\alpha \in \mathbb{C}$. By $N(r, \alpha; f| \leq n)$, we denote the counting function of the $\alpha$-points of $f$ with multiplicity less than or equal to $n$ and by $\overline{N}(r, \alpha; f| \leq n)$ the reduced counting function. Also by $N(r, \alpha; f| \geq n)$, we denote the counting function of the $\alpha$-points of $f$ with multiplicity greater than or equal to $n$ and by $\overline{N}(r, \alpha; f| \geq n)$ the reduced counting function. We define $N_n(r, \alpha; f) = \overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; f| \geq 2) + ...... + \overline{N}(r, \alpha; f| \geq 2)$.

**Definition 1.2.** [4] Let $f$ and $g$ be two non-constant meromorphic functions share a value $\alpha$ IM. Denote by $\overline{N_*}(r, \alpha; f, g)$ the counting function of the $\alpha$-points of $f$ and $g$ with different multiplicities,where each $\alpha$-point is counted only once.

In 2010, Li [9] considered a nonconstant L-function and meromorphic function and he obtained the following result.

**Theorem 1.3.** [9].Let $a$ and $b$ be two distinct finite values and $f$ be a meromorphic function in the complex plane with finitely many poles. If $f$ and a nonconstant $L$-function $L$ share $a$ CM and $b$ IM, then $L \equiv f$.

Recently, Liu-Li-Yi [7, 8] proved the following result.

**Theorem 1.4.** [7]. Let $f$ be a nonconstant meromorphic function, let $L$ be an $L$- function, and let $n$ and $k$ be two positive integers. Suppose that $(f^n)^{(k)}$ and $(L^n)^{(k)}$ share 1 CM. If $n > 3k + 6$, then $f \equiv tL$ for a constant $t$ satisfying $t^n = 1$.

**Theorem 1.5.** [8]. Let $f$ be a nonconstant meromorphic function, let $L$ be an $L$-function, and let $n$ and $k$ be two positive integers. Suppose that $[f^n(f-1)]^{(k)}$ and $[L^n(L-1)]^{(k)}$ share 1 CM. If $n > 3k + 9$ and $k \geq 2$, then $f \equiv L$.

In 2018, W. J. Hao and J. F. Chen [2] proved the following theorem.

**Theorem 1.6.** [2] Let $f$ be a non-constant meromorphic function and $L$ be an $L$-function such that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share $(1, \infty)$, where $m, n, k$ are positive integers. If $n > m + 3k + 6$ and $k \geq 2$, then $f \equiv L$ or $f^n(f-1)^m = L^n(L-1)^m$.

**Theorem 1.7.** [2] Let $f$ be a non-constant meromorphic function and $L$ be an $L$-function such that $[f^n(f-1)^m]^{(k)}$ and $[L^n(L-1)^m]^{(k)}$ share $(1, 0)$, where $m, n, k$ are positive integers. If $n > 4m + 7k + 11$ and $k \geq 2$, then $f \equiv L$ or $f^n(f-1)^m = L^n(L-1)^m$.

We now state the following theorems, which are the main results of this paper.

**Theorem 1.8.** Let $L$ be a non-constant $L$-function and $f$ be a transcendental meromorphic function. Let $k, n, d, \mu_j(j = 1, 2, ..., d), \lambda = \sum_{j=1}^{d} \mu_j$ be positive integers such that $n > \lambda + d(2k + 4) + 4$, and $w_j \in \mathbb{C}\backslash\{0\}$, (j=1,2,...,d) be distinct constants. Also let $\rho_2(L) < 1,\ \rho_2(f) < 1$. $[L^n P(L) \prod_{j=1}^{d} L(z+w_j)^{\mu}_j]^{(k)}$ and $[f^n P(f) \prod_{j=1}^{d} f(z+w_j)^{\mu}_j]^{(k)}$ share $(\rho(z), l)$ and $f, L$ share $(\infty, 0)$, where $\rho(z)$ is a small function of $f$ and $L$. If $l = 0$ and $n > \lambda + m - 1 + (7 + 5k)(m + d + 2)$ or

$l=1$ and $n>m+\lambda+\frac{1}{2}(5k+9)(m+d+2)-1$, where $P(f)=a_0f+a_1f^{(1)}+....+a_mf^{(m)}$, then one of the following holds:

$$[L^nP(L)\prod_{j=1}^{d}L(z+w_j)_j^{\mu}]^{(k)}\equiv[f^nP(f)\prod_{j=1}^{d}f(z+w_j)_j^{\mu}]^{(k)}; \tag{1.1}$$

$$[L^nP(L)\prod_{j=1}^{d}L(z+w_j)_j^{\mu}]^{(k)}[f^nP(f)\prod_{j=1}^{d}f(z+w_j)_j^{\mu}]^{(k)}\equiv\rho^2. \tag{1.2}$$

**Theorem 1.9.** Let $L$ be a non-constant $L$-function and $f$ be a transcendental meromorphic function. Let $k,n,d,\mu_j(j=1,2...,d),\lambda=\sum_{j=1}^{d}\mu_j$ be positive integer such that $n>\lambda+d(2k+4)+4$, and $w_j\in\mathbb{C}\backslash\{0\}$, (j=1,2,...,d) be distinct constants. Also let $\rho_2(L)<1,\rho_2(f)<1$. $[L^nP(L)\prod_{j=1}^{d}L(z+w_j)_j^{\mu}]^{(k)}$ and $[f^nP(f)\prod_{j=1}^{d}f(z+w_j)_j^{\mu}]^{(k)}$ share $(R(z),l)$ and $f,L$ share $(\infty,0)$ where $R(z)$ is a rational function. If $l=0$ and $n>\lambda+m-1+(7+5k)(m+d+2)$ or $l=1$ and $n>m+\lambda+\frac{1}{2}(5k+9)(m+d+2)-1$ where $P(f)=a_0f+a_1f^{(1)}+....+a_mf^{(m)}$, then one of the following holds:

$$[L^nP(L)\prod_{j=1}^{d}L(z+w_j)_j^{\mu}]^{(k)}\equiv[f^nP(f)\prod_{j=1}^{d}f(z+w_j)_j^{\mu}]^{(k)}; \tag{1.3}$$

$$[L^nP(L)\prod_{j=1}^{d}L(z+w_j)_j^{\mu}]^{(k)}[f^nP(f)\prod_{j=1}^{d}f(z+w_j)_j^{\mu}]^{(k)}\equiv R(z)^2. \tag{1.4}$$

## 2 Lemmas

We denote H as $H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{L''}{L'}-\frac{2L'}{L-1}\right)$. For the proof of our main results, we need the following lemmas.

**Lemma 2.1.** [15]. Let $f$ be a non-constant meromorphic function and $P(f)=a_0+a_1f+....+a_nf^n$, where $a_0,a_1,...,a_n$ are complex constants and $a_n\neq0$, then $T(r,P(f))=nT(r,f)+S(r,f)$.

**Lemma 2.2.** [17]. Let $f$ be a transcendental meromorphic function of finite logarithmic order and $q,\eta$ be two non-zero complex constant. Then we have

$$\begin{aligned}T(r,f(qz+\eta)) &= T(r,f)+S_1(r,f),\\ N(r,f(qz+\eta)) &= N(r,f)+S_1(r,f),\\ N\left(r,\frac{1}{f(qz+\eta)}\right) &= N\left(r,\frac{1}{f}\right)+S_1(r,f).\end{aligned}$$

**Lemma 2.3.** [10]. Let $L$ be an $L$-function. Then $N(r,\infty;L)=S(r,L)=\bigcirc(\log r)$.

**Lemma 2.4.** [11] Let $f$ be a non-constant meromorphic function and $L$ be an $L$-function. If $f$ and $L$ share $(\infty,0)$ then $\overline{N}(r,\infty;f)=S(r,L)=\bigcirc(\log r)$.

**Lemma 2.5.** [14] Let $L$ be an $L$-function with degree $q$. Then $T(r,L)=\frac{q}{\pi}r\log r+\bigcirc(1)$.

**Lemma 2.6.** [12] Let $F$ and $G$ be two non-constant meromorphic functions sharing $(1,1)$ and $(\infty,0)$. If $H\not\equiv0$, then

$$T(r,F)\leq N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,F)+\overline{N}(r,G)+\overline{N_*}(r,F,G)+\frac{1}{2}\overline{N}(r,0;F)+S(r,F)+S(r,G).$$

**Lemma 2.7.** [12] Let $F$ and $G$ be two non-constant meromorphic functions sharing $(1,0)$ and $(\infty,0)$. If $H\not\equiv0$, then

$$\begin{aligned}T(r,F) \leq{}& N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,F)+2\overline{N}(r,G)+\overline{N_*}(r,F,G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)\\ &+S(r,F)+S(r,G);\\ T(r,G) \leq{}& N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,G)+2\overline{N}(r,F)+\overline{N_*}(r,F,G)+2\overline{N}(r,0;G)+\overline{N}(r,0;F)\\ &+S(r,F)+S(r,G).\end{aligned}$$

**Lemma 2.8.** [15] Let $F$ be a non-constant meromorphic function and $k, p$ be two positive integers. Then

$$\begin{aligned}
T(r, F^{(k)}) &\leq T(r, F) + k\overline{N}(r, F) + S(r, F);\\
N_p(r, 0; F^{(k)}) &\leq T(r, F^{(k)}) - T(r, F) + N_{p+k}(r, 0; F) + S(r, F);\\
N_p(r, 0; F^{(k)}) &\leq N_{p+k}(r, 0; F) + k\overline{N}(r, F) + S(r, F);\\
N(r, 0; F^{(k)}) &\leq N(r, 0; F) + k\overline{N}(r, F) + S(r, F).
\end{aligned}$$

## 3 Proof of the Theorem 1.8

Let $F(z) = \frac{F_1^{(k)}}{\rho(z)}$ and $L^*(z) = \frac{L_1^{(k)}}{\rho(z)}$, where $F_1 = f^n P(f) \prod_{j=1}^{d} f(z + w_j)^{\mu_j}$ and $L_1 = L^n P(L) \prod_{j=1}^{d} L(z + w_j)^{\mu_j}$, respectively. Then $F, L$ share $(1, l)$ and share $(\infty, 0)$ except for zeros and poles of $\rho(z)$.

Now

$$\begin{aligned}
(n + m + 1)T(r, f) &= T(r, f^{n+m+1})\\
&= T(r, f^n f^{m+1})\\
&= T\left(r, \frac{f^{m+1}(z)F_1(z)}{P(f)\prod_{j=1}^{d} f(z + w_j)^{\mu_j}}\right)\\
&\leq T(r, F_1(z)) + T(r, f^{m+1}) + T(r, P(f)) + T(r, \prod_{j=1}^{d} f(z + w_j)^{\mu_j})\\
&\leq T(r, F_1(z)) + (m + 1)T(r, f) + (m + 1)T(r, f) + \lambda T(r, f) + S_1(r, f)\\
&\leq T(r, F_1(z)) + (2m + \lambda + 2)T(r, f) + S_1(r, f).
\end{aligned}$$

So we get

$$(n - m - \lambda - 1)\, T(r, f) + S_1(r, f) \leq T(r, F_1(z)). \tag{3.1}$$

By Lemma 2.5, $L$ is a transcendental meromorphic function. We have from Lemma 2.8 and (3.1),

$$\begin{aligned}
N_2(r, 0; F) &\leq N_2(r, 0; F_1^{(k)}) + S(r, f)\\
&\leq T(r, F_1^{(k)}) - T(r, F_1) + N_{k+2}(r, 0, F_1) + S(r, f)\\
&\leq T\left(r, \frac{F_1^k}{\rho(z)}\right) - (n - 1 - m - \lambda)T(r, f) + N_{k+2}(r, 0, F_1) + S(r, f),\\
i.e., (n - 1 - m - \lambda)\, T(r, f) &\leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \qquad (3.2)
\end{aligned}$$

Similarly, we have

$$(n - 1 - m - \lambda)\, T(r, L) \leq T(r, L) - N_2(r, 0; L) + N_{k+2}(r, 0, L_1) + S(r, L). \tag{3.3}$$

Now we have to consider the following two cases.

**Case 1.** Let $H \not\equiv 0$. In this case we have to consider the following two subcases.

**Subcase 1.1.** Let $l = 0$, hence by Lemmas 2.3, 2.4, and 2.7 and the inequality (3.2), we have

$$\begin{aligned}
(n - 1 - m - \lambda)\, T(r, f) &\leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f)\\
&\leq N_2(r, 0; F) + N_2(r, 0; L) + 3\overline{N}(r, \infty, F) + 2\overline{N}(r, \infty; L) + \overline{N}(r, 0, L)\\
&\quad - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, L)\\
&\leq N_2(r, 0; L_1^{(k)}) + 2\overline{N}(r, 0; F_1^{(k)}) + \overline{N}(r, 0; L_1^{(k)}) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, L)\\
&\leq N_{k+2}(r, 0; L_1) + 2N_{k+1}(r, 0; F_1) + N_{k+1}(r, 0; L_1) + N_{k+2}(r, 0; F_1)\\
&\quad + S(r, f) + S(r, L). \qquad (3.4)
\end{aligned}$$

Now

$$\begin{aligned} N_{k+2}(r,0;L_1) &\leq (k+2)\overline{N}(r,0;L_1)\\ &\leq (k+2)\overline{N}(r,0;L^nP(L)\prod_{j=1}^{d}L(z+w_j)^{\mu_j})\\ &\leq (k+2)\overline{N}(r,0;L^n)+(k+2)\overline{N}(r,0;\prod_{j=1}^{d}L(z+w_j)^{\mu_j})+(k+2)\overline{N}(r,0;P(L))\\ &\leq (k+2)T(r,L)+(k+2)dT(r,L)+(k+2)(m+1)T(r,L). \end{aligned} \tag{3.5}$$

Using (3.5) in (3.4), we get

$$\begin{aligned} (n-1-m-\lambda)\ T(r,f) \leq&(k+2)(m+d+2)T(r,L)+(k+1)(m+d+2)T(r,L)\\ &+(3k+4)(m+d+2)T(r,f)+S(r,f)+S(r,L)\\ \leq&(2k+3)(m+d+2)T(r,L)+(3k+4)(m+d+2)T(r,f)+S(r,f)+S(r,L). \end{aligned} \tag{3.6}$$

Similarly, we have

$$(n-1-m-\lambda)\ T(r,L)\leq (2k+3)(m+d+2)T(r,f)+(3k+4)(m+d+2)T(r,L)+S(r,f)+S(r,L). \tag{3.7}$$

From (3.6) and (3.7), we get

$$(n-1-m-\lambda)\ \{T(r,f)+T(r,L)\} \leq (5k+7)(m+d+2)T(r,f)+T(r,L)+S(r,f)+S(r,L),$$

which gives a contradiction as $n>\lambda+m+1+(5k+7)(m+d+2)$.

**Subcase 1.2** Let $l=1$. By Lemmas 2.3, 2.4 and 2.6 and the inequality (3.2), we have

$$\begin{aligned} (n-1-m-\lambda)\ T(r,f) \leq&T(r,F)-N_2(r,0;F)+N_{k+2}(r,0;F_1)+S(r,f)\\ \leq&N_2(r,0;L)+\frac{3}{2}\overline{N}(r,F)+\overline{N}(r,L)+\overline{N_*}(r,\infty;F,L)\\ &+\frac{1}{2}\overline{N}(r,0;F)+N_{k+2}(r,0;F_1)+S(r,f)+S(r,L)\\ \leq&N_2(r,0;L_1^{(k)})+\frac{1}{2}N_{k+1}(r,0;F_1)+N_{k+2}(r,0;F_1)+S(r,f)+S(r,L)\\ \leq&N_{k+2}(r,0;L_1)+\frac{1}{2}N_{k+1}(r,0;F_1)+N_{k+2}(r,0;F_1)+S(r,f)+S(r,L)\\ \leq&(k+2)(m+d+2)T(r,L)+[\frac{1}{2}(k+1)+(k+2)](m+d+2)T(r,f)+S(r,f)+S(r,L)\\ \leq&(k+2)(m+d+2)T(r,L)+\frac{1}{2}(3k+5)(m+d+2)T(r,f)+S(r,f)+S(r,L). \end{aligned} \tag{3.8}$$

Similarly, we have

$$(n-1-m-\lambda)\ T(r,L)\leq (k+2)(m+d+2)T(r,f)+\frac{1}{2}(3k+5)(m+d+2)T(r,L)+S(r,f)+S(r,L). \tag{3.9}$$

From (3.8) and (3.9), we arrive at a contradiction as $n>m+\lambda+1+\frac{1}{2}(5k+9)(m+d+2)$.

**Case 2.** Let $H\equiv 0$. Then, we have

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{L''}{L'}-\frac{2L'}{L-1}\right)\equiv 0.$$

Integrating both sides we get

$$\log F' - \log(F-1)^2 - \log L' + \log(L-1)^2 = \log b$$

$$i.e., \quad \frac{F'(L-1)^2}{(F-1)^2 L'} = b$$

$$i.e., \quad \frac{F'}{(F-1)^2} - \frac{bL'}{(L-1)^2} = 0$$

$$i.e., \quad (F-1) = \frac{(L-1)}{b - c(L-1)},$$

where $b \neq 0$ and $c$ are constants. Now we have to consider the following subcases.

**Subcase 2.1** Let $c = 0$. Then from (3.9), we have

$$F - 1 = \frac{L-1}{b}. \tag{3.10}$$

If $b \neq 1$, then from (3.10), we get

$$\overline{N}(r, 0; F) = \overline{N}(r, 1-b; L). \tag{3.11}$$

By lemma 2.3 and 2.8, using second fundamental theorem of Nevanlinna and from inequality (3.3), we have

$$\begin{aligned}(n-1-m-\lambda)\; T(r, L) &\leq T(r, L) - N_2(r, 0; L) + N_{k+2}(r, 0; L_1) + S(r, L) \\ &\leq \overline{N}(r, 0; L) + \overline{N}(r, 1-b; L) + \overline{N}(r, L) - N_2(r, 0; L) + N_{k+2}(r, 0; L_1) + S(r, L) \\ &\leq \overline{N}(r, 0; L) + \overline{N}(r, 0; F) - N_2(r, 0; L) + N_{k+2}(r, 0; L_1) + S(r, L) \\ &\leq \overline{N}(r, 0; L_1^{(k)}) + \overline{N}(r, 0; F_1^{(k)}) - N_2(r, 0; L_1^{(k)}) + N_{k+2}(r, 0; L_1) + S(r, L) \\ &\leq (k+1)(m+d+2)T(r, L) + (k+1)(m+d+2)T(r, f) + S(r, f) + S(r, L). \quad (3.12)\end{aligned}$$

Similarly, we have

$$(n-1-m-\lambda)\; T(r, f) \leq (k+1)(m+d+2)T(r, f) + (k+1)(m+d+2)T(r, L) + S(r, f) + S(r, L). \tag{3.13}$$

From the inequalities (3.12) and (3.13), we arrive at a contradiction as $n > m + \lambda + 1 + 2(k+1)(m+\lambda+2)$. Hence $b = 1$ and therefore we get from (3.10),

$$[L^n P(L) \prod_{j=1}^{d} L(z+w_j)_j^{\mu}]^{(k)} = [f^n P(f) \prod_{j=1}^{d} f(z+w_j)_j^{\mu}]^{(k)}.$$

**Subcase 2.2** $c \neq 0$ and $b = c$ . If $c = -1$, then from (3.11) we have $FL \equiv 1$. Hence

$$[L^n P(L) \prod_{j=1}^{d} L(z+w_j)_j^{\mu}]^{(k)} [f^n P(f) \prod_{j=1}^{d} f(z+w_j)_j^{\mu}]^{(k)} \equiv \rho^2.$$

If $c \neq -1$, then from (3.10), we have $\frac{1}{F} = \frac{cL}{(1+c)L-1}$. Hence $\overline{N}(r, 0; F) = N(r, \frac{1}{1+c}; L)$. Now proceeding as in Subcase 2.1, we arrive at a contradiction.

**Subcase 2.3.** $c \neq 0$ and $b \neq c$. Then from (3.11), we have

$$L + \frac{b-c}{c} = \frac{-b}{c^2} \frac{1}{F - (1 + \frac{1}{c})}. \tag{3.14}$$

As $L$ has at most one pole $z = 1$ by (3.15), we have $F - (1 + \frac{1}{c})$ has at most one zero. By using the same method as in Subcase 2.1, we arrive at a contradiction. This completes the proof of the Theorem 1.8.

## 4 Proof of the Theorem 1.9

Since $f$ and $L$ are transcendental entire function and $R(z)$ is a rational function, $R(z)$ is a small function of $f$ and $L$. Hence by Theorem 1.1, we get the required result.


## Acknowledgment

This study was funded by UGC India (MANF-2018-19-WES-97001). The authors are thankful to the anonymous reviewers for their valuable suggestions.


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