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Fixed point results on p-metric spaces via p-simulation functions

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Abstract

In this paper, we introduce the structure of modified b-metric spaces as a generalization of b-metric spaces. Also, we present the notions of p-contractive mappings in the modified b-metric spaces and investigate the existence of a fixed point for such mappings under various contractive conditions. We provide examples to illustrate the results presented herein.

Keywords: fixed point, complete metric space, p-metric space
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1 Introduction and preliminaries

There are a large number of generalizations of Banach contraction principle via using different forms of contractive conditions in various generalized metric spaces. Some of such generalizations are obtained via contractive conditions expressed by rational terms (see [8, 3, 1, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16]). The concept of a b-metric space, as one of the useful generalizations of standard metric spaces, was firstly used by Bakhtin [2] and Czerwik [4].

Recall (see [2, 4]) that a b-metric d on a set X is a generalization of standard metric, where the triangular inequality is replaced by

$$d(x,z) \leq b[d(x,y) + d(y,z)], x, y, z \in X,$$

for some fixed $b \geq 1$.

Remark 1.1. Let $f, g : [0,\infty) \to \mathbb{R}$ be two functions such that $f(0) = g(0)$ and $\frac{df(x)}{dx} = f'(x) \leq g'(x) = \frac{dg(x)}{dx}$. Then for $x \in [0,\infty)$ we have $f(x) \leq g(x)$.

Let Ψ denote a family of functions such that for each $\Omega \in \Psi$,
$\Omega : [0,\infty) \longrightarrow [0,\infty)$ and Ω is onto,

1. $t \leq \Omega(t)$ for all $t \in [0,\infty)$,

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2. $\frac{d\Omega}{dt} = \Omega'$ is increasing.

Lemma 1.2. Let $\Omega \in \Psi$. Then for all $x, y \in [0, \infty)$, $r \in (0, 1)$ and for every $n \in \mathbb{N}$ we have

1. $\Omega(x + y) \geq \Omega(x) + \Omega(y)$,
2. Ω is continuous and is strictly increasing,
3. $r\Omega(x) \geq \Omega(rx)$,
4. $\Omega^{-1}(x + y) \leq \Omega^{-1}(x) + \Omega^{-1}(y)$,
5. $\Omega^{-1}(rx) \geq r\Omega^{-1}(x)$,
6. $\Omega^n(x + y) \geq \Omega^n(x) + \Omega^n(y)$,
7. $\Omega^{-n}(x + y) \leq \Omega^{-n}(x) + \Omega^{-n}(y)$,
8. $|\Omega^{-1}(x) - \Omega^{-1}(y)| \leq \Omega^{-1}(|x - y|)$.

Proof .

1. If we define $g(x) = \Omega(x + b)$ and $f(x) = \Omega(x) + \Omega(b)$, then $f(0) = g(0)$ and $f'(x) = \Omega'(x) \leq \Omega'(x + b) = g'(x)$. Therefore, for all $x \in [0, \infty)$ we have $f(x) \leq g(x)$, that is, $\Omega(x + y) \geq \Omega(x) + \Omega(y)$ for all $x, y \in [0, \infty)$.
2. It is clear that Ω is continuous. Since
$$\Omega'(x) = \lim_{h\to 0} \frac{\Omega(x + h) - \Omega(x)}{h} \geq \lim_{h\to 0} \frac{\Omega(h)}{h} \geq \lim_{h\to 0} \frac{h}{h} = 1,$$
Ω is strictly increasing.
3. If we define $f(x) = \Omega(rx)$ and $g(x) = r\Omega(x)$ for all $r \in (0, 1)$, then $f(0) = g(0) = 0$ and $f'(x) = r\Omega'(rx) \leq r\Omega'(x) = g'(x)$. Therefore, for all $x \in [0, \infty)$ we have $f(x) \leq g(x)$, that is, $\Omega(rx) \leq r\Omega(x)$.
4. By (2), we know first that the inverse Ω^{-1} is strictly increasing. Hence if we replace x by $\Omega^{-1}(x)$ and y by $\Omega^{-1}(y)$ in (1), then we get
$$\Omega(\Omega^{-1}(x) + \Omega^{-1}(y)) \geq \Omega(\Omega^{-1}(x)) + \Omega(\Omega^{-1}(y)) = x + y.$$
That is, $\Omega^{-1}(x + y) \leq \Omega^{-1}(x) + \Omega^{-1}(y)$.
5. If we replace x by $\Omega^{-1}(x)$ in (3), we get $\Omega(r\Omega^{-1}(x)) \leq r\Omega(\Omega^{-1}(x))$. That is, $r\Omega^{-1}(x) \leq \Omega^{-1}(rx))$.
6. For $n = 1$ it is obvious. Suppose that (5) holds for some $n \geq 2$. Since
$$\begin{aligned} \Omega^{n+1}(x + y) &= \Omega(\Omega^n(x + y)) \\ &\geq \Omega(\Omega^n(x) + \Omega^n(y)) \\ &\geq \Omega(\Omega^n(x)) + \Omega(\Omega^n(y)) = \Omega^{n+1}(x) + \Omega^{n+1}(y), \end{aligned}$$
(5) is proved by induction.
7. Similarly, it is obtained from (4) and (5), obviously.
8. If $x > y$, then replacing x by $x - y$ in (4), we have $\Omega^{-1}(x) - \Omega^{-1}(y) \leq \Omega^{-1}(x - y)$. If $x \leq y$, then replacing y by $y - x$ in (4), we have $\Omega^{-1}(y) - \Omega^{-1}(x) \leq \Omega^{-1}(y - x)$. Hence in generally we have
$$|\Omega^{-1}(x) - \Omega^{-1}(y)| \leq \Omega^{-1}(|x - y|),$$

as desired. □

Remark 1.3. For every $\Omega \in \Psi$ and for all $t \in [0, \infty)$ we have $\Omega^{-1}(t) \leq t \leq \Omega(t)$ and $\Omega^{-1}(0) = 0 = \Omega(0)$.

For example, if $\Omega : [0, \infty) \longrightarrow [0, \infty)$ defined by $\Omega(t) = e^t - 1$, $\Omega(t) = te^t$ or $\Omega(t) = t^2 + 2t$ for all $t \in [0, \infty)$, then it is easy to see that $\Omega \in \Psi$.

Demmaa *et al.* [5] gave the definition of b-simulation function in the setting of b-metric space.

Definition 1.4. Let (X, d, b) be a $b-$metric space. A b-simulation function is a function $\xi : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:
(i) $\xi(t, s) < s - t$, for all $t, s > 0$,
(ii) if t_n, s_n are sequences in $(0, \infty)$ such that
$$0 < \lim_{n\longrightarrow\infty} t_n \leq \liminf_{n\longrightarrow\infty} s_n \leq \limsup_{n\longrightarrow\infty} s_n \leq b \lim_{n\longrightarrow\infty} t_n < \infty,$$
then
$$\limsup_{n\longrightarrow\infty} \xi(bt_n, s_n) < 0.$$

The following are some examples of b-simulation functions [5].

Example 1.5. [5] Let $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by
(i) $\xi(t,s) = \lambda s - t$ for all $t,s \in [0,\infty)$, where $\lambda \in [0,1)$,
(ii) $\xi(t,s) = \psi(s) - \varphi(t)$ for all $t,s \in [0,\infty)$, where $\varphi,\psi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \varphi(t)$ for all $t > 0$,
(iii) $\xi(t,s) = s\frac{f(t,s)}{g(t,s)}t$ for all $t,s \in [0,\infty)$, where $f,g : [0,\infty) \times [0,\infty) \to (0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t,s > 0$,
(iv) $\xi(t,s) = s - \varphi(s) - t$ for all $t,s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$,
(v) $\xi(t,s) = s\varphi(s) - t$ for all $t,s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is such that $\lim\limits_{t\to r^+} \varphi(t) \ < 1$ for all $r > 0$.

Each of the functions considered in (i)-(v) is a b-simulation function.

Demma et al. [5] gave the following theorem.

Theorem 1.6. [5] Let (X,d,b) be a complete b-metric space and let $T : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that

$$\xi(bd(Tx,Ty), d(x,y)) \geq 0 \quad for \quad all \quad x,y \in X.$$

Then T has a unique fixed point.

Now, we introduce the concept of extended b-metric spaces as follows.

Definition 1.7. Let X be a (nonempty) set. A function $\widetilde{d} : X \times X \to \mathbb{R}^+$ is a p-metric if there exists $\Omega \in \Psi$ such that for all $x,y,z \in X$, the following conditions hold:

($\widetilde{d}_1$) $\widetilde{d}(x,y) = 0$ iff $x = y$,

($\widetilde{d}_2$) $\widetilde{d}(x,y) = \widetilde{d}(y,x)$,

($\widetilde{d}_3$) $\widetilde{d}(x,z) \leq \Omega(\widetilde{d}(x,y)) + \Omega(\widetilde{d}(y,z))$.

In this case, the triple $(X,\widetilde{d},\Omega)$ is called a p-metric space, or an extended b-metric space.

A b-metric [4] is a p-metric with $\Omega(t) = bt$ for some fixed $b \geq 1$. Also every metric is a p-metric for every $\Omega \in \Psi$.

Example 1.8. Let (X,d) be a b-metric space with coefficient $b \geq 1$ and let $\widetilde{d}(x,y) = \sinh(d(x,y))$. We show that $\widetilde{d}$ is a p-metric with $\Omega(t) = \sinh(2bt)$ for all $t \geq 0$ (and $\Omega^{-1}(u) = \frac{1}{2b}\sinh^{-1}(2bu)$ for $u \geq 0$). Obviously, the conditions ($\widetilde{d}_1$) and ($\widetilde{d}_2$) of Definition 1.7 are satisfied. Since $\sinh(x)$ is an increasing function, for all $x,y \geq 0$, we have

$$\sinh(x+y) \leq \sinh(2\max\{x,y\}) \leq \sinh(2x) + \sinh(2y).$$

Therefore, for all $x,y,z \in X$, we have

$$\begin{aligned}
\widetilde{d}(x,z) &= \sinh(d(x,z)) \\
&\leq \sinh(bd(x,y) + bd(y,z)) \leq \sinh(b\sinh(d(x,y)) + b\sinh(d(y,z))) \\
&= \sinh(b\widetilde{d}(x,y) + b\widetilde{d}(y,z)) \\
&\leq \sinh(2b\widetilde{d}(x,y)) + \sinh(2b\widetilde{d}(y,z)) \\
&= \Omega(\widetilde{d}(x,y)) + \Omega(\widetilde{d}(y,z)).
\end{aligned}$$

So the condition ($\widetilde{p}_3$) of Definition 1.7 is also satisfied and $\widetilde{d}$ is a p-metric. Note that $\sinh|x-y|$ is not a metric on $\mathbb{R}$, e.g.,

$$\sinh 5 \approx 74.203 \not\leq 3.627 + 10.0179 \approx \sinh 2 + \sinh 3.$$

Similarly, although $d(x,y) = (x-y)^2$ is a b-metric on $\mathbb{R}$ with $b = 2$, there is no $b \neq 1$ such that $\widehat{d}(x,y) = \sinh(x-y)^2$ is a b-metric with parameter b. Indeed, putting $z = 0$ and $y = 1$ we have $\sinh x^2 \leq b(\sinh(x-1)^2 + \sinh 1)$ which does not hold for any fixed b and x sufficiently large.

Definition 1.9. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space. A sequence $\{x_n\}$ in X is said to be

(1) p-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $\widetilde{d}(x_n, x_m) < \varepsilon$;

(2) p-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$, $\widetilde{d}(x, x_n) < \varepsilon$.

(3) A p-metric space X is called complete if every p-Cauchy sequence is p-convergent in X.

Lemma 1.10. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space. If a sequence $\{x_n\}$ in X p-converges to x, then x is unique.

Proof . Let $\{x_n\}$ p-converge to x and y. Then using the rectangle inequality in the p-metric space it is easy to see that

$$\widetilde{d}(x, y) \leq \Omega(\widetilde{d}(x, x_n)) + \Omega(\widetilde{d}(y, x_n)).$$

Taking the limit as $n \to \infty$ in the above inequality we obtain $\widetilde{d}(x, y) = 0$ and so $x = y$. □

Lemma 1.11. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space. If a sequence $\{x_n\}$ in X is p-convergent to x, then $\{x_n\}$ is a p-Cauchy sequence.

Proof . Since $\lim_{n\to\infty} x_n = x$, using the rectangle inequality in the p-metric space, it is easy to see that

$$\widetilde{d}(x_n, x_m) \leq \Omega(\widetilde{d}(x_n, x)) + \Omega(\widetilde{d}(x, x_m)).$$

Taking the limit as $n, m \to \infty$ in the above inequality we obtain

$$\lim_{n,m\to\infty} \widetilde{d}(x_n, x_m) = 0.$$

Hence $\{x_n\}$ is a p-Cauchy sequence. □

We will need the following simple lemma about the p-convergent sequences.

Lemma 1.12. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space with function Ω.

(1) Suppose that $\{x_n\}$ and $\{y_n\}$ are p-convergent to x and y, respectively. Then we have

$$\Omega^{-2}(\widetilde{d}(x, y)) \leq \liminf_{n\longrightarrow\infty} \widetilde{d}(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} \widetilde{d}(x_n, y_n) \leq \Omega^2(\widetilde{d}(x, y)).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} \widetilde{d}(x_n, y_n) = 0$.

(2) Suppose that $\{x_n\}$ is p-convergent to x and $z \in X$ is arbitrary. Then we have

$$\Omega^{-1}(\widetilde{d}(x, z)) \leq \liminf_{n\longrightarrow\infty} \widetilde{d}(x_n, z) \leq \limsup_{n\longrightarrow\infty} \widetilde{d}(x_n, z) \leq \Omega(\widetilde{d}(x, z)).$$

Proof . (1) Using the rectangle inequality in the p-metric space it is easy to see that

$$\begin{aligned}\widetilde{d}(x, y) &\leq \Omega(\widetilde{d}(x, x_n)) + \Omega(\widetilde{d}(y, x_n)) \\ &\leq \Omega(\widetilde{d}(x, x_n)) + \Omega\big[\Omega(\widetilde{d}(y, y_n)) + \Omega(\widetilde{d}(x_n, y_n))\big]\end{aligned}$$

and

$$\begin{aligned}\widetilde{d}(x_n, y_n) &\leq \Omega(\widetilde{d}(x_n, x)) + \Omega(\widetilde{d}(y_n, x)) \\ &\leq \Omega(\widetilde{d}(x_n, x)) + \Omega\big[\Omega(\widetilde{d}(y_n, y)) + \Omega(\widetilde{d}(x, y))\big].\end{aligned}$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the desired result.

(2) Using the rectangle inequality we see that

$$\widetilde{d}(x,z) \leq \Omega(\widetilde{d}(x,x_n)) + \Omega(\widetilde{d}(x_n,z)).$$

Taking the lower limit as $n \to \infty$ in the above inequality we have

$$\widetilde{d}(x,z) \leq \Omega(\liminf_{n\to\infty} \widetilde{d}(x_n,z)),$$

and hence

$$\Omega^{-1}(\widetilde{d}(x,z)) \leq \liminf_{n\to\infty} \widetilde{d}(x_n,z).$$

Also

$$\widetilde{d}(x_n,z) \leq \Omega(\widetilde{d}(x_n,x)) + \Omega(\widetilde{d}(z,x)).$$

Taking the upper limit as $n \to \infty$ in the above inequality we obtain the desired result. □

2 Fixed point results on p-metric spaces via p-simulation functions

We start with the following lemma.

Lemma 2.1. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} \widetilde{d}(x_n, y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$, then $\lim_{n\to\infty} y_n = t$.

Proof . By the triangle inequality in the p-metric space, we have

$$\widetilde{d}(y_n,t) \leq \Omega(\widetilde{d}(y_n,x_n)) + \Omega(\widetilde{d}(x_n,t)).$$

Now, by taking the upper limit when $n \to \infty$ in the above inequality we get

$$\limsup_{n\longrightarrow\infty} \widetilde{d}(y_n,t) \leq \Omega(\limsup_{n\longrightarrow\infty} \widetilde{d}(x_n,y_n)) + \Omega(\limsup_{n\longrightarrow\infty} \widetilde{d}(x_n,t)) = 0.$$

Hence $\lim_{n\to\infty} y_n = t$.□

Now, we give the definition of p-simulation function in the setting of p-metric space.

Definition 2.2. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space. A p-simulation function is a function $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(i) $\xi(t,s) < s - t$, for all $t, s > 0$,

(ii) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \lim_{n\to\infty} t_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq \Omega(\lim_{n\to\infty} t_n) < \infty,$$

then

$$\limsup_{n\to\infty} \xi(\Omega(t_n), s_n) < 0.$$

If we define $\Omega(t) = bt$ for every $b \geq 1$, then it is easy to see that every b-simulation function is a p-simulation function.

Let $\mathcal{Z}$ be the family of all p-simulation functions $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$. Firstly, we present the following definition which will be used in our main results.

Definition 2.3. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space and $\xi \in \mathcal{Z}$. We say that $T : X \longrightarrow X$ is an almost $\mathcal{Z}$-contraction if there is a constant $\theta \geq 0$ such that

$$\xi(\Omega(\widetilde{d}(Tx, Ty)), M(x, y) + \theta N(x, y)) \geq 0 \tag{2.1}$$

for all $x, y \in X$, where

$$N(x, y) = \min\{\widetilde{d}(x, Tx), \widetilde{d}(y, Ty), \widetilde{d}(x, Ty), \widetilde{d}(y, Tx)\}$$

and

$$M(x, y) = \max\left\{\widetilde{d}(x, y), q\widetilde{d}(x, Tx), q\widetilde{d}(y, Ty), \Omega^{-1}\left(\frac{\widetilde{d}(x, Ty) + \widetilde{d}(y, Tx)}{2}\right)\right\},$$

where $0 < q < 1$.

Remark 2.4. If T is an almost $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}$, then

$$\Omega(\widetilde{d}(Tx, Ty)) < M(x, y) + \theta N(x, y) \tag{2.2}$$

for all $x, y \in X$.

The following are our main results.

Lemma 2.5. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space and let $T : X \longrightarrow X$ be a mapping. Suppose that there exists a p-simulation function ξ such that

$$\xi(\Omega(\widetilde{d}(Tx, Ty)), M(x, y) + \theta N(x, y)) \geq 0, \quad \text{for all } x, y \in X.$$

Let $\{x_n\}$ be a sequence of Picard of initial point at $x_0 \in X$. Suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then

$$\lim_{n \longrightarrow \infty} d(x_{n-1}, x_n) \ = 0.$$

Proof . We prove that

$$\lim_{n \to \infty} \widetilde{d}(x_n, x_{n+1}) = 0. \tag{2.3}$$

Since

$$\begin{aligned} N(x_{n-1}, x_n) &= \min\{\widetilde{d}(x_{n-1}, Tx_{n-1}), \widetilde{d}(x_n, Tx_n), \widetilde{d}(x_{n-1}, Tx_n), \widetilde{d}(x_n, Tx_{n-1})\} \\ &= \min\{\widetilde{d}(x_{n-1}, x_n), \widetilde{d}(x_n, x_{n+1}), \widetilde{d}(x_{n-1}, x_{n+1}), \widetilde{d}(x_n, x_n)\} \\ &= 0, \end{aligned}$$

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{\widetilde{d}(x_{n-1}, x_n), q\widetilde{d}(x_{n-1}, Tx_{n-1}), q\widetilde{d}(x_n, Tx_n), \Omega^{-1}\left(\frac{\widetilde{d}(x_{n-1}, Tx_n) + \widetilde{d}(x_n, Tx_{n-1})}{2}\right)\right\} \\ &= \max\left\{\widetilde{d}(x_{n-1}, x_n), q\widetilde{d}(x_{n-1}, x_n), q\widetilde{d}(x_n, x_{n+1}), \Omega^{-1}\left(\frac{\widetilde{d}(x_{n-1}, x_{n+1}) + \widetilde{d}(x_n, x_n)}{2}\right)\right\} \\ &= \max\left\{\widetilde{d}(x_{n-1}, x_n), q\widetilde{d}(x_n, x_{n+1}), \Omega^{-1}\left(\frac{\widetilde{d}(x_{n-1}, x_{n+1})}{2}\right)\right\}. \end{aligned}$$

So

$$\widetilde{d}(x_{n-1}, x_{n+1}) \quad \leq \quad \Omega(\widetilde{d}(x_{n-1}, x_n)) + \Omega(\widetilde{d}(x_n, x_{n+1})).$$

That is,

$$\Omega^{-1}\left(\frac{\widetilde{d}(x_{n-1}, x_{n+1})}{2}\right) \quad \leq \quad \Omega^{-1}\left[\frac{\Omega(\widetilde{d}(x_{n-1}, x_n)) + \Omega(\widetilde{d}(x_n, x_{n+1}))}{2}\right].$$

We prove that $\widetilde{d}(x_n, x_{n+1}) \leq \widetilde{d}(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$. If $\widetilde{d}(x_n, x_{n+1}) > \widetilde{d}(x_{n-1}, x_n)$, for some $n \in \mathbb{N}$, then we get

$$\Omega^{-1}\left(\frac{\widetilde{d}(x_{n-1}, x_{n+1})}{2}\right) \leq \widetilde{d}(x_n, x_{n+1}).$$

Hence

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{\widetilde{d}(x_{n-1}, x_n), q\widetilde{d}(x_n, x_{n+1}), \Omega^{-1}\left(\frac{\widetilde{d}(x_{n-1}, x_{n+1})}{2}\right)\right\} \\ &= l\widetilde{d}(x_n, x_{n+1}) \end{aligned}$$

for all $q \leq l \leq 1$. Using (2.1), for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} 0 &\leq \xi(\Omega(\widetilde{d}(x_n, x_{n+1})), l\widetilde{d}(x_n, x_{n+1})) \\ &< l\widetilde{d}(x_n, x_{n+1}) - \Omega(\widetilde{d}(x_n, x_{n+1})), \end{aligned}$$

which is a contradiction. Therefore,

$$M(x_{n-1}, x_n) = \max\left\{\widetilde{d}(x_{n-1}, x_n), q\widetilde{d}(x_n, x_{n+1}), \Omega^{-1}\left(\frac{\widetilde{d}(x_{n-1}, x_{n+1})}{2}\right)\right\} = \widetilde{d}(x_{n-1}, x_n).$$

Using (2.1), for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} 0 &\leq \xi(\Omega(\widetilde{d}(x_n, x_{n+1})), \widetilde{d}(x_{n-1}, x_n)) \\ &< \widetilde{d}(x_{n-1}, x_n) - \Omega(\widetilde{d}(x_n, x_{n+1})). \end{aligned}$$

It follows from the above inequality that

$$0 < \widetilde{d}(x_n, x_{n+1}) \leq \Omega(\widetilde{d}(x_n, x_{n+1})) < \widetilde{d}(x_{n-1}, x_n) \tag{2.4}$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\{\widetilde{d}(x_n, x_{n+1})\}$ is decreasing and so $r \geq 0$ such that $\lim_{n\to\infty} \widetilde{d}(x_n, x_{n+1}) = r$.

Assume that $r > 0$. Take the sequences $\{t_n\}$ and $\{s_n\}$ as $t_n = \widetilde{d}(x_n, x_{n+1})$ and

$$s_n = M(x_{n-1}, x_n) + \theta N(x_{n-1}, x_n) = \widetilde{d}(x_{n-1}, x_n).$$

Since $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = r$, by the axiom (ξ_2), we deduce

$$0 \leq \limsup_{n\to\infty} \; \xi(\Omega(\widetilde{d}(x_n, x_{n+1})), \widetilde{d}(x_{n-1}, x_n)) < 0,$$

which is a contradiction. Thus $r = 0$, that is, (2.3) holds. □

Lemma 2.6. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space and let $T : X \longrightarrow X$ be a mapping. Suppose that there exists a p-simulation function ξ such that (2.1) holds. Let $\{x_n\}$ be a sequence of Picard of initial point at $x_0 \in X$. Suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a bounded sequence.

Proof . On the contrary, assume that $\{x_n\}$ is not bounded. Then there is a subsequence $\{x_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer greater than n_k such that

$$\widetilde{d}(x_{n_{k+1}}, x_{n_k}) > 1$$

and

$$\widetilde{d}(x_m, x_{n_k}) \leq 1$$

for $n_k \leq m \leq n_{k+1}-1$. By the triangular inequality, we have

$$\begin{aligned}\Omega(1) &< \Omega(\widetilde{d}(x_{n_{k+1}}, x_{n_k})) < \widetilde{d}(x_{n_{k+1}-1}, x_{n_k-1})\\ &\leq \Omega(\widetilde{d}(x_{n_{k+1}-1}, x_{n_k})) + \Omega(\widetilde{d}(x_{n_k}, x_{n_k-1}))\\ &\leq \Omega(1) + \Omega(\widetilde{d}(x_{n_k}, x_{n_k-1})).\end{aligned}$$

Letting $k \to \infty$ in the last inequality and using (2.3), we obtain

$$\lim_{k\to\infty} \Omega(\widetilde{d}(x_{n_k}, x_{n_{k+1}}) = \Omega(1). \tag{2.5}$$

That is, $\lim_{k\to\infty} \widetilde{d}(x_{n_k}, x_{n_{k+1}}) = 1$ and

$$\lim_{k\to\infty} \widetilde{d}(x_{n_{k+1}-1}, x_{n_k-1}) = \Omega(1). \tag{2.6}$$

So

$$N(x_{n_{k+1}-1}, x_{n_k-1}) \quad = \quad \min\{\widetilde{d}(x_{n_{k+1}-1}, x_{n_k+1}), \widetilde{d}(x_{n_k-1}, x_{n_k}), \widetilde{d}(x_{n_{k+1}-1}, x_{n_k}), \widetilde{d}(x_{n_k-1}, x_{n_k+1})\}.$$

Letting $k \to \infty$ and using (2.3), we obtain

$$\lim_{k\to\infty} N(x_{n_{k+1}-1}, x_{n_k-1}) = 0. \tag{2.7}$$

Also,

$$M(x_{n_{k+1}-1}, x_{n_k-1}) \qquad = \max\left\{\begin{array}{l}\widetilde{d}(x_{n_{k+1}-1}, x_{n_k-1}), q\widetilde{d}(x_{n_{k+1}-1}, x_{n_{k+1}}), q\widetilde{d}(x_{n_k-1}, x_{n_k}),\\ \Omega^{-1}\left(\frac{\widetilde{d}(x_{n_{k+1}-1}, x_{n_k}) + \widetilde{d}(x_{n_k-1}, x_{n_k+1})}{2}\right)\end{array}\right\}.$$

So

$$\widetilde{d}(x_{n_{k+1}-1}, x_{n_k}) \leq \Omega(\widetilde{d}(x_{n_{k+1}-1}, x_{n_{k+1}})) + \Omega(\widetilde{d}(x_{n_{k+1}}, x_{n_k})).$$

Letting $k \to \infty$ and using (2.3), we obtain

$$\lim_{k\to\infty} \widetilde{d}(x_{n_{k+1}-1}, x_{n_k}) \leq \Omega(1).$$

Similarly, we can prove that

$$\lim_{k\to\infty} \widetilde{d}(x_{n_k-1}, x_{n_k+1}) \leq \Omega(1).$$

Therefore, using Equation 2.6, we obtain

$$\lim_{k\to\infty} M(x_{n_{k+1}-1}, x_{n_k-1}) = \Omega(1).$$

By (2.1), we have

$$\begin{aligned}0 &\leq \xi(\Omega(\widetilde{d}(Tx_{n_{k+1}-1}), Tx_{n_k-1})), M(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k})\\ &< M(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k-1}) - \widetilde{d}(x_{n_{k+1}}, x_{n_k}),\end{aligned}$$

which implies that

$$\Omega(\widetilde{d}(x_{n_{k+1}}, x_{n_k})) < M(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k-1}). \tag{2.8}$$

Choose the sequences $\{t_k\}$ and $\{s_k\}$ as $t_k = \widetilde{d}(x_{n_{k+1}}, x_{n_k})$ and

$$s_k = M(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k-1}).$$

By (2.5) and (2.6), $\lim_{k\to\infty} t_k = 1$ and $\lim_{k\to\infty} = s_k = \Omega(1)$. Thus we can apply the axiom (ξ_2) to these sequences, that is,

$$\limsup_{k\longrightarrow\infty} \xi(\Omega(\widetilde{d}(x_{n_{k+1}}, x_{n_k})), M(x_{n_{k+1}-1}, x_{n_k-1}) + \theta N(x_{n_{k+1}-1}, x_{n_k-1}))) < 0,$$

which contradicts to (2.8). This proves that $\{x_n\}$ is a bounded sequence. □

Lemma 2.7. Let $(X, \widetilde{d}, \Omega)$ be a p-metric space and let $T : X \longrightarrow X$ be a mapping. Suppose that there exists a p-simulation function ξ such that (2.1) holds. Let $\{x_n\}$ be a sequence of Picard of initial point at $x_0 \in X$. Suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a p-Cauchy sequence.

Proof . Consider the sequence $\{C_n\} \subset [0, \infty)$ given by

$$C_n = \sup\{\widetilde{d}(x_i, x_j) : i, j \geq n\}.$$

It is clear that $\{C_n\}$ is a positive decreasing sequence and hence there is some $C \geq 0$ such that $\lim_{n\to\infty} C_n = C$. If $C > 0$, then, by the definition of C_n, for every $k \in \mathbb{N}$, there exist n_k and m_k such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < \widetilde{d}(x_{m_k}, x_{n_k}) \leq C_k.$$

Thus

$$\lim_{k\to\infty} \widetilde{d}(x_{m_k}, x_{n_k}) = C. \tag{2.9}$$

Using (2.1) and the triangular inequality, we have

$$\widetilde{d}(x_{m_k}, x_{n_k}) \leq \Omega(\widetilde{d}(x_{m_k}, x_{n_k})) \leq \widetilde{d}(x_{m_k-1}, x_{n_k-1}) \leq C_{k-1}.$$

Taking $k \to \infty$ and using (2.3) and (2.9), we obtain

$$C < \lim_{k\to\infty} \widetilde{d}(x_{m_k-1}, x_{n_k-1}) \leq C.$$

That is,

$$\lim_{k\to\infty} \widetilde{d}(x_{m_k-1}, x_{n_k-1}) = C. \tag{2.10}$$

Additionally, with the aid of (2.3), we have

$$\lim_{k\to\infty} N(x_{m_k-1}, x_{n_k-1}) = 0. \tag{2.11}$$

Taking the sequences $\{t_k = \widetilde{d}(x_{m_k}, x_{n_k})\}$ and $\{s_k = M(x_{m_k-1}, x_{n_k-1}) + \theta N(x_{m_k-1}, x_{n_k-1})\}$, and considering (2.9)–(2.11), we get $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = C$. Then, by (2.1) and (ξ_2), we obtain

$$0 \leq \limsup_{k\longrightarrow\infty} \xi(\Omega(\widetilde{d}(x_{m_k}, x_{n_k})), M(x_{m_k-1}, x_{n_k-1}) + \theta N(x_{m_k-1}, x_{n_k-1})) < 0,$$

which is a contradiction and so $C = 0$. That is, $\{x_n\}$ is a p-Cauchy sequence. □

Theorem 2.8. Let $(X, \widetilde{d}, \Omega)$ be a complete p-metric space and $T : X \to X$ be an almost $\mathcal{Z}$-contraction with respect to a function $\xi \in \mathcal{Z}$. Then T has a unique fixed point, and for every initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

Proof . Take $x_0 \in X$ and consider the Picard sequence $\{x_n = T^n x_0 = Tx_{n-1}\}_{n\geq 0}$. If $x_{n_0} = x_{n_0+1}$, for some n_0, then x_{n_0} is a fixed point of T. Hence, for the rest of the proof, we assume that $\widetilde{d}(x_n, x_{n+1}) > 0$, for all $n \geq 0$. Now, by Lemma 2.7, the sequence $\{x_n\}$ is Cauchy and since $(X, \widetilde{d}, \Omega)$ is a complete p-metric space, there exists some $u \in X$

such that $\lim_{n\to\infty} x_n = u$. We shall show that the point u is a fixed point of T. Suppose that $Tu \neq u$. Then $\widetilde{d}(u, Tu) > 0$. By (2.1), we obtain

$$\begin{aligned} 0 &\leq \xi(\Omega(\widetilde{d}(Tx_n, Tu)), M(x_n, u) + \theta N(x_n, u)) \\ &\leq M(x_n, u) + \theta N(x_n, u) - \Omega(\widetilde{d}(Tx_n, Tu)). \end{aligned}$$

That is,

$$\Omega(\widetilde{d}(Tx_n, Tu)) < M(x_n, u) + \theta N(x_n, u).$$

It is easy to see that $\lim_{n\to\infty} N(x_n, u) = 0$. Also, since

$$M(x_n, u) = \max\left\{\widetilde{d}(x_n, u), q\widetilde{d}(x_n, Tx_n), q\widetilde{d}(u, Tu), \Omega^{-1}\left(\frac{\widetilde{d}(Tx_n, Tu) + \widetilde{d}(u, Tx_n)}{2}\right)\right\},$$

by using Lemma 2.6, we obtain

$$\begin{aligned} \lim_{n\longrightarrow\infty} M(x_n, u) &= \max\left\{0, 0, q\widetilde{d}(u, Tu), \Omega^{-1}\left(\frac{\lim_{n\longrightarrow\infty}\widetilde{d}(Tx_n, Tu) + \lim_{n\longrightarrow\infty}\widetilde{d}(u, Tx_n)}{2}\right)\right\} \\ &\leq \max\left\{q\widetilde{d}(u, Tu), \Omega^{-1}\left(\Omega\left(\frac{\widetilde{d}(u, Tu)}{2}\right)\right)\right\} \\ &\leq \max\{q\widetilde{d}(u, Tu), \widetilde{d}(u, Tu)\} = l\widetilde{d}(u, Tu), \end{aligned}$$

where $q \leq l \leq 1$. Thus,

$$\begin{aligned} \widetilde{d}(u, Tu) &\leq \Omega(\lim_{n\longrightarrow\infty}\widetilde{d}(Tx_n, Tu)) \\ &\leq \limsup_{n\longrightarrow\infty}\,[M(x_n, u) + \theta N(x_n, u) - \Omega(\widetilde{d}(x_{n+1}, Tu)] = l\widetilde{d}(u, Tu) \\ &< \widetilde{d}(u, Tu), \end{aligned}$$

which implies that $\widetilde{d}(u, Tu) = 0$, that is, u is a fixed point of T.

Suppose that there are two distinct fixed points $u, v \in X$ of the mapping T. Then $\widetilde{d}(u, v) > 0$. Also,

$$N(u, v) = \min\{\widetilde{d}(u, Tu), \widetilde{d}(v, Tv), \widetilde{d}(u, Tv), \widetilde{d}(v, Tu)\} = 0$$

and

$$M(u, v) = \max\left\{\widetilde{d}(u, v), q\widetilde{d}(u, Tu), q\widetilde{d}(v, Tv), \Omega^{-1}\left(\frac{\widetilde{d}(u, Tv) + \widetilde{d}(v, Tu)}{2}\right)\right\} = \widetilde{d}(u, v).$$

Therefore, it follows from (2.1) and (ξ_2) that

$$\begin{aligned} 0 &\leq \xi(\Omega(\widetilde{d}(Tu, Tv)), M(u, v) + \theta N(u, v)) \\ &= \xi(\Omega(\widetilde{d}(u, v)), \widetilde{d}(u, v)) \\ &< \widetilde{d}(u, v) - \Omega(\widetilde{d}(u, v)), \end{aligned}$$

which is a contradiction. Thus the fixed point of T in X is unique. □

By a similar argument to Theorem 2.8, we can prove the following theorem.

Theorem 2.9. Let $(X, \widetilde{d}, \Omega)$ be a complete p-metric space and let $T : X \longrightarrow X$ be a mapping. Suppose that there exists a p-simulation function ξ such that

$$\xi(\Omega(\widetilde{d}(Tx, Ty)), \widetilde{d}(x, y)) \geq 0$$

for all $, y \in X$. Then T has a unique fixed point.

Corollary 2.10. Let $(X, \widetilde{d}, \Omega)$ be a complete p-metric space and let $T : X \to X$ be a mapping. Suppose that there exists $\lambda \in (0, 1)$ such that

$$\Omega(\widetilde{d}(Tx, Ty)) \leq \lambda M(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof . The result follows from Theorem 2.8, by taking the p-simulation function

$$\xi(t, s) = \lambda s - t$$

for all $t, s \geq 0$.□

Corollary 2.11. Let $(X, \widetilde{d}, \Omega)$ be a complete p-metric space and let $T : X \to X$ be a mapping. Suppose that there exists $\lambda \in]0, 1[$ such that

$$\Omega(\widetilde{d}(Tx, Ty)) \leq \lambda \widetilde{d}(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof . The result follows from Theorem 2.9, by taking the p-simulation function

$$\xi(t, s) = \lambda s - t$$

for all $t, s \geq 0$.□

The following corollary gives the result of Demmaa et al. [5].

Corollary 2.12. Let (X, d, b) be a complete b-metric space and let $T : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that

$$\xi(bd(Tx, Ty), d(x, y)) \geq 0$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof . The result follows from Theorem 2.9, by taking $\Omega(t) = bt$. □

Example 2.13. Let $X = [0, \infty)$ and $\widetilde{d} : X \times X \longrightarrow \mathbb{R}$ be defined by $\widetilde{d}(x, y) = sinh|x-y|$. If we define $\Omega(t) = sinh(2t)$, then $(X, \widetilde{d}, \Omega)$ is a complete p-metric space. Define a mapping $T : X \longrightarrow X$ by $Tx = \frac{1}{4}sinh^{-1}(x)$. By Lemma 1.2, for all $x, y \in X$ with $\frac{1}{2} \leq q < 1$, we have

$$\begin{aligned}
\Omega(\widetilde{d}(Tx, Ty)) &= \sinh(2\widetilde{d}(Tx, Ty)) \\
&= \sinh(2(\sinh|\frac{1}{4}(\sinh^{-1}(x) - \sinh^{-1}(y))|)) \\
&\leq \sinh(2(\frac{1}{4}\sinh(|\sinh^{-1}(x) - \sinh^{-1}(y)|))) \\
&\leq \sinh(\frac{1}{2}(\sinh(\sinh^{-1}(|x - y|)))) \\
&\leq \frac{1}{2}\sinh(|x - y|) \\
&\leq q\widetilde{d}(x, y).
\end{aligned}$$

Hence all the conditions of Corollary 2.11 are satisfied. So T has a unique fixed point $x = 0$.

3 Conclusions

We have introduced the structure of modified b-metric spaces as a generalization of b-metric spaces. Also, we have presented the notions of p-contractive mappings in the modified b-metric spaces and have investigated the existence of fixed point for such mappings under various contractive conditions. Moreover, we have provided an examples to illustrate the results presented herein.

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