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On strongly starlikeness of Libera operator

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Abstract

Let $\mathcal{A}$ denote the class of functions f that are analytic in the unit disk Δ with normalization $f(0) = f'(0) - 1 = 0$. In this paper conditions are determined for strongly starlikeness of the Libera operator. The results Mocanu and Nunokawa et al. are improved.

Keywords: Nunokawa's Lemma, Starlike functions, Strongly starlike functions, Libera operator
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1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and let us denote by $\mathcal{A}_n$ the class of functions $f \in \mathcal{H}$ with the normalization of the form

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ..., \quad z \in \mathbb{D},$$

with $\mathcal{A}_1 = \mathcal{A}$. Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β, $0 < \beta \leq 1$,

$$\mathcal{SS}^*(\beta) = \left\{ f \in \mathcal{A} : \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\beta\pi}{2}, \ z \in \Delta \right\},$$

which was introduced in [1, 7], and $\mathcal{SS}^*(1) \equiv \mathcal{S}^*$ is the well-known class of starlike functions in Δ. Functions in the class

$$\mathcal{R} = \{f \in \mathcal{A} : Re\{f'(z)\} > 0, \ z \in \Delta\},$$

are called functions with bounded turning. The Libera operator $L : \mathcal{A} \to \mathcal{A}$, $L[f] = F$, where

$$F(z) = \frac{2}{z} \int_0^z f(t)dt, \tag{1.1}$$

is the Libera integral operator, which has been studied by several authors on different counts. In [2] , Mocanu considered the problem of starlikeness of F and proved the following result.

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Theorem 1.1. ([2]) If $f(z)$ is analytic and $Re\{f'(z)\} > 0$ in the unit disk Δ and if the function F is given by (1.1), then $F \in \mathcal{S}^*$.

This result may be written briefly as follows.

$$L[\mathcal{R}] \subset \mathcal{S}^* = \mathcal{SS}^*(1), \tag{1.2}$$

where $L[\mathcal{R}] = \{L[f] : f \in \mathcal{R}\}$. In 1995 Mocanu [3] improved (1.2) by showing that

$$L[\mathcal{R}] \subset \mathcal{SS}^*(8/9). \tag{1.3}$$

Recently, the problem of strongly starlikeness of $L[f]$ for $f \in \mathcal{R}$ was considered also in [5]. Nunokawa et al. in [5] proved the following result which is the small improvement of the Mocanu's result (1.3).

Theorem 1.2. ([5]) If $f(z) \in \mathcal{A}$ and $Re\{f'(z)\} > 0$ in the unit disk Δ, then the function (1.1) satisfies

$$\left|\arg \frac{zF'(z)}{F(z)}\right| < \frac{\gamma\pi}{2} = 1.368..., \ z \in \Delta$$

where

$$\gamma = \frac{2}{\pi}(\frac{\pi}{2} - \log 2)(1 + \frac{\pi}{2} - \log 2) = 0.870907... \ . \tag{1.4}$$

This result may be written as

$$L[\mathcal{R}] \subset \mathcal{SS}^*(\gamma) \tag{1.5}$$

where γ is given by (1.4).

2 Main results

In this paper we go back to the problem of strongly starlikeness of Libera operator and obtain improvement of the results Mocanu and Nunokawa. In order to prove main theorem, we need the following Lemmas:

Lemma 2.1. (see [6]) If $f(z) \in \mathcal{A}$ and $|\arg\{f'(z)\}| < \dfrac{\alpha\pi}{2}$ in the unit disk Δ, then the function (1.1) satisfies

$$\left|\arg\left(\frac{F(z)}{z}\right)\right| < \frac{\alpha\pi}{2}\left(1 - \frac{2}{\pi}\log 2\right)^2, \ z \in \Delta.$$

Lemma 2.2. (see [4]) Let $p(z)$ be of the form

$$p(z) = 1 + \sum_{n=m\geq 1}^{\infty} a_n z^n, \quad a_m \neq 0 \ (z \in \Delta), \tag{2.1}$$

with $p(z) \neq 0$ in Δ. If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{p(z)\}| < \frac{\beta\pi}{2} \text{ in } |z| < |z_0|,$$

and

$$|\arg\{p(z_0)\}| = \frac{\beta\pi}{2},$$

for some $\beta > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{m(a^2+1)}{2a} \geq 1, \qquad \text{when} \qquad \arg\{p(z_0)\} = \frac{\beta\pi}{2},$$

and

$$k \leq -\frac{m(a^2+1)}{2a} \leq -1, \qquad \text{when} \qquad \arg\{p(z_0)\} = -\frac{\beta\pi}{2},$$

where

$$\{p(z_0)\}^{\frac{1}{\beta}} = \pm ia, \quad a > 0.$$

Now, in the following theorem, we obtain a criterion for the starlikeness of the Libera operator, which help us achieve the desired generalization.

Theorem 2.3. Let $f \in \mathcal{A}$ and suppose that for $0 < \alpha < 2$,

$$|\arg(f'(z))| < \frac{\alpha\pi}{2}. \tag{2.2}$$

If the equation, with respect to x,

$$x + \frac{2}{\pi}\tan^{-1} x = \alpha\left(1 + \left(1 - \frac{2}{\pi}\log 2\right)^2\right), \tag{2.3}$$

has a solution $\beta \in (0, 1]$, then the function given in (1.1) satisfies

$$\left|\arg\left\{\frac{zF'(z)}{F(z)}\right\}\right| < \frac{\beta\pi}{2}, \tag{2.4}$$

hence $F(z)$ is strongly starlike of order β.

Proof . Let $p(z) = \dfrac{zF'(z)}{F(z)}, \quad z \in \Delta$. If there exists a point z_0, $|z_0| < 1$, for which

$$|\arg\{p(z)\}| < \frac{\beta\pi}{2}, \quad (|z| < |z_0|)$$

and

$$|\arg\{p(z_0)\}| = \frac{\beta\pi}{2}, \quad p(z_0) = (\pm ia)^\beta, a > 0$$

then from Nunokawa's Lemma 2.2, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{a^2+1}{2a} \geq 1, \qquad \text{when} \qquad \arg\{p(z_0)\} = \frac{\beta\pi}{2},$$

and

$$k \leq -\frac{a^2+1}{2a} \leq -1, \qquad \text{when} \qquad \arg\{p(z_0)\} = -\frac{\beta\pi}{2}.$$

For the case $\arg\{p(z_0)\} = \dfrac{\beta\pi}{2}$, we have

$$\begin{aligned}
\left|\arg\left\{z_0p'(z_0) + p(z_0)^2 + p(z_0)\right\}\right| &= \left|\arg\left\{p(z_0)\left[1 + p(z_0) + \frac{z_0p'(z_0)}{p(z_0)}\right]\right\}\right| \\
&= \left|\arg\{p(z_0)\} + \arg\left\{1 + p(z_0) + \frac{z_0p'(z_0)}{p(z_0)}\right\}\right| \\
&= \left|\frac{\beta\pi}{2} + \tan^{-1}\left\{\frac{\beta k + a^\beta \sin(\beta\pi/2)}{1 + a^\beta\cos(\beta\pi/2)}\right\}\right|,
\end{aligned} \tag{2.5}$$

where $p(z_0) = (ia)^\beta$, $a > 0$ and

$$k \geq \frac{a^2+1}{2a} \geq 1.$$

Let us put

$$g(a) = \frac{\beta k + a^\beta \sin(\beta\pi/2)}{1 + a^\beta \cos(\beta\pi/2)}, \quad a > 0,$$

then it is easy to see that

$$g(a) \geq \frac{\beta + a^\beta \sin(\beta\pi/2)}{1 + a^\beta \cos(\beta\pi/2)}, \quad a > 0. \tag{2.6}$$

Putting

$$h(x) = \frac{\beta + x \sin(\beta\pi/2)}{1 + x \cos(\beta\pi/2)}, \quad x \geq 0,$$

we have

$$h'(x) = \frac{\sin(\beta\pi/2) - \beta\cos(\beta\pi/2)}{(1 + x\cos(\beta\pi/2))^2} > 0, \quad x \geq 0,$$

because $\tan(\beta\pi/2) > \beta$. Therefore, for $x > 0$ we obtain $h(x) > h(0) = \beta$, so from (2.6) we get

$$g(a) > \beta,$$

and so

$$\tan^{-1}\left\{\frac{\beta k + a^\beta \sin(\beta\pi/2)}{1 + a^\beta \cos(\beta\pi/2)}\right\} > \tan^{-1}\beta, \quad a > 0.$$

Therefore, from (2.5), we have the following inequality

$$\begin{aligned} \left|\arg\left\{z_0 p'(z_0) + p(z_0)^2 + p(z_0)\right\}\right| &= \frac{\beta\pi}{2} + \tan^{-1}\left\{\frac{\beta k + a^\beta \sin(\beta\pi/2)}{1 + a^\beta \cos(\beta\pi/2)}\right\} \\ &> \frac{\beta\pi}{2} + \tan^{-1}\beta. \end{aligned} \tag{2.7}$$

Moreover, from Lemma 2.1, we have

$$|\arg\{P(z_0\}| = \left|\arg\left\{\frac{F(z_0)}{z_0}\right\}\right| < \frac{\alpha\pi}{2}\left(1 - \frac{2}{\pi}\log 2\right)^2, \tag{2.8}$$

where $P(z) = \dfrac{F(z)}{z}$, $z \in \Delta$. By (2.7) and (2.8), we can write

$$\begin{aligned} |\arg\{2f'(z_0)\}| &= |\arg\{2F'(z_0) + z_0 F''(z_0)\}| \\ &= \left|\arg\left\{P(z_0)\left(z_0 p'(z_0) + p(z_0)^2 + p(z_0)\right)\right\}\right| \\ &= \left|\arg\{P(z_0)\} + \arg\left\{(z_0 p'(z_0) + p(z_0)^2 + p(z_0)\right\}\right| \\ &> \frac{\beta\pi}{2} + \tan^{-1}\beta - \frac{\alpha\pi}{2}\left(1 - \frac{2}{\pi}\log 2\right)^2 = \frac{\alpha\pi}{2}, \end{aligned}$$

because β is the solution of (2.3). Therefore we have

$$\begin{aligned} |\arg\{f'(z_0)\}| &= |\arg\{2f'(z_0)\}| \\ &> \frac{\alpha\pi}{2}. \end{aligned} \tag{2.9}$$

This contradicts the hypothesis and for the case $\arg\{p(z_0)\} = -\dfrac{\beta\pi}{2}$, applying the same method as the above, we also have (2.9). This is also a contradiction and it completes the proof. □

If we put $\alpha = 1$ in Theorem 2.3 we obtain the following result.

Corollary 2.4. If $f \in \mathcal{A}$ and $Re\{f'(z)\} > 0$ in the unit disk Δ, then the function (1.1) satisfies

$$\left|\arg \frac{zF'(z)}{F(z)}\right| < \frac{\beta\pi}{2}, \ z \in \Delta,$$

where $\beta = 0.860004....$

Corollary 2.4 may be written shortly as follows:

$$L[\mathcal{R}] \subset \mathcal{SS}^*(\beta) \tag{2.10}$$

where $\beta = 0.860004... < \gamma = 0.870907...$ which shows that the result (2.10) improves the result (1.5) obtained by Nunokawa et al.

In the Corollary below there are examples of the choice α and β which satisfies Theorem 2.3.

Corollary 2.5. **(i)** If $f \in \mathcal{A}$ and

$$|\arg(f'(z))| < \frac{\alpha\pi}{2},$$

with $\alpha \approx 0.6059$, then $f \in \mathcal{SS}^*(1/2)$.

(ii) If $f \in \mathcal{A}$ and

$$|\arg(f'(z))| < \frac{\alpha\pi}{2},$$

with $\alpha \approx 0.7933$, then $f \in \mathcal{SS}^*(2/3)$.

(iii) If $f \in \mathcal{A}$ and

$$|\arg(f'(z))| < \frac{\alpha\pi}{2},$$

with $\alpha \approx 1.1432$, then $f \in \mathcal{SS}^*(1) = \mathcal{S}^*$.

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