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An innovative method for identifying shared fixed points of G-nonexpansive mappings on Banach spaces with a graph

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Abstract

An important goal of this study is to show that a sequence x_n made up of new iterations to fixed points of G-nonexpansive mappings on Banach space that have a graph converge weakly and strongly.

Keywords: Weak and strong convergence; G-nonexpansive mappings; Banach space.
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1 Introduction

Banach [5] is credited with proving the Banach contraction principle, a crucial basic theorem that is used to solve existence problems in a wide range of mathematical fields. The theorem is presented in Banach spaces with graphs in its most recent iteration. A generalisation of the Banach contraction principle and the notion of G-contraction were presented by Jachymski [6] in 2008 in the context of a metric space equipped with a directed graph. In 2012, Aleomraninejad et al. [1] showed how to use fixed point theory and graph theory to look at some iterative scheme results for G-contractive and G-nonexpansive mappings on graphs.

Alfuraidan and Khamsi [3] were the first to talk about the idea of G-monotone nonexpansive multivalued mappings in 2015. They are defined in a metric space with a graph. Subsequently, we established sufficient conditions for the existence of fixed points in hyperbolic metric spaces for this type of mapping. In 2015, Alfuraidan [2] came up with a new way to describe the G-contraction and said that on a Banach space with a graph, there must be fixed points of G-monotone pointwise contraction mappings.

Tiammee et al. proved in their 2015 paper [11] that the Halpern iteration process and Browder's convergence theorem for G-nonexpansive mappings in a Hilbert space with a directed graph strongly work the way they say they do. In 2016, Tripak [12] proved that the weak and strong convergence theorems for G-nonexpansive mappings of a sequence x_n made by the Ishikawa iteration are correct. These mappings were defined on a uniformly convex Banach space equipped with a directed graph and corresponded to some common fixed points. We want to show that the Ishikawa iteration can be used to find the fixed point where three G-nonexpansive mappings meet in a closed, convex subset C of a uniformly convex Banach space X. If the conditions are right, C has a directed graph. This will allow us to prove both weak and strong convergence theorems.

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2 Preliminaries

In this section, we review several common graph notation, definitions, and lemmas that are necessary for the research that will be done in this work. These include: Let (X, d) be a metric space. If $Tx = x$, a point $x \in X$ is a fixed point of a mapping T. $F(T)$, or $F(T) = x \in X : Tx = x$, denotes the set of fixed points of T. Think about a directed graph. $G = (V(G), E(G))$ is a direct graph in which all loops are included in the set of edges $E(G)$ and the graph's vertices $V(G)$. Assume that G has no parallel edges. Then, G can be thought of as a weighted graph by giving each edge the distance between its vertices.

Definition 2.1. [6] The conversion of a graph G is the graph obtained from G by reversing the direction of edges denoted by G^{-1}, and

$$E(G^{-1}) = \{(x, y) \in X \times X | (y, x) \in E(G)\}.$$

Definition 2.2. [6] Let x and y be vertices of a graph G. A path in G from x to y of length N ($N \in \mathbb{N} \cup \{0\}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N + 1$ vertices for which $x_0 = x, x_N = y$, and $(x_i, x_{i+1}) \in E(G)$, for $i = 0, 1, ..., N - 1$.

Definition 2.3. [12] A graph G is said to be connected if there is a path between any two vertices of the graph G.

Definition 2.4. [12] A directed graph $G = (V(G), E(G))$ is said to be transitive if for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, we have (x, z) are in $E(G)$.

Definition 2.5. [12] Let (X, d) be a metric space, and C be a nonempty subset of X. A mapping $T : C \to C$ is called edge-preserving if

$$(x, y) \in E(G) \to (Tx, Ty) \in E(G)$$

for all $x, y \in C$.

Definition 2.6. [12] Let C be a nonempty convex subset of a Banach space X and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$. Then a mapping $T : C \to C$ is G-nonexpansive if it satisfies the following conditions:

(i) T is edge-preserving.

(ii) $\| Tx - Ty \| \leq \| x - y \|$ whenever $(x, y) \in E(G)$, for any $x, y \in C$.

Definition 2.7. [8] Let C be a nonempty closed convex subset of a real uniformly convex Banach space X. The mappings $T_i (i = 1, 2, 3)$ on C are said to satisfy Condition B if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that, for all $x \in C$

$$\max \{\| x - T_1 x \|, \| x - T_2 x \|, \| x - T_3 x \|\} \geq f(d(x, F))$$

where $F = F(T_1) \cap F(T_2) \cap F(T_3)$ and $F(T_i) (i = 1, 2, 3)$ are the sets of fixed points of T_i.

Definition 2.8. [12] Let C be a subset of a metric space (X, d). A mapping T is semicompact if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$.

Definition 2.9. [2] A Banach space X is said to satisfy Opal's property if the following inequality holds for any distinct elements x and y in X and for each sequence $\{x_n\}$ weakly convergent to x where $x \to \infty$ such that

$$\liminf_{n\to\infty} \| x_n - x \| < \liminf_{n\to\infty} \| x_n - y \|.$$

Definition 2.10. [10] Let X be a Banach space. A mapping T with domain D and range R in X is demiclosed at 0 if for any sequence $\{x_n\}$ in D such that $\{x_n\}$ converges weakly to $x \in D$ and $\{Tx_n\}$ converges strongly to 0 we have $Tx = 0$.

Lemma 2.11. [8] Let X be a uniformly convex Banach space, and $\{\alpha_n\}$ a sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\limsup_{n\to\infty} \| x_n \| \leqslant c$, $\limsup_{n\to\infty} \| y_n \| \leqslant c$ and $\limsup_{n\to\infty} \| \alpha x_n + (1 - \alpha_n) y_n \| = c$, for some $c \geq 0$. Then $\lim_{n\to\infty} \| x_n - y_n \| = 0$.

Lemma 2.12. [8] Let X be a Banach space, and $R > 1$ be a fixed number. Then X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that
$\| \lambda x + (1-\lambda)y \|^2 \leqslant \lambda\| x \|^2 + (1-\lambda)\| y \|^2 - \lambda(1-\lambda)g(\| x - y \|)$
for all $x, y \in B_r(0) = \{x \in X | \| x \| \leqslant R\}$ and $\lambda \in [0, 1]$.

Lemma 2.13. [8] Let X be a Banach space that satisfies Opial's property, and let $\{x_n\}$ be a sequence in X. Let $x, y \in X$ such that $\lim_{n\to\infty} \| x_n - x \|$ and $\lim_{n\to\infty} \| x_n - y \|$ exist. If $\{x_{n_j}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ converge weakly to x and y respectively, then $x = y$.

3 Main results

In this section, we use the Ishikawa iteration generated from an arbitrary x_0 for the common fixed point of three G-nonexpansive mappings in a closed convex subset C of a uniformly convex Banach space X furnished with a directed graph to show both weak and strong convergence theorems.

Consider a Banach space X that has a directed graph G such that $V(G) = C$ and $E(G)$ is convex. Let C be a nonempty closed convex subset of this space. Assume that G is a transitive graph. From C to C, the mappings $T_i (i = 1, 2, 3)$ are G-nonexpansive, and $F = F(T_1) \cap F(T_2) \cap F(T_3)$ is nonempty. Assume that the sequence $\{x_n\}$ is produced from any arbitrary $x_0 \in C$.

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T_3 x_n \\ z_n &= (1-\gamma_n)y_n + \gamma_n T_2 x_n \\ x_{n+1} &= (1-\alpha_n)z_n + \alpha_n T_1 z_n \end{aligned}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. We first begin by proposition and lemma the following useful results.

Proposition 3.1. Let $p_0 \in F$ be such that $(x_0, p_0), (y_0, p_0), (z_0, p_0), (p_0, x_0), (p_0, y_0), (p_0, z_0) \in E(G)$. Then (x_n, p_0), (y_n, p_0), $(z_n, p_0), (p_0, x_n), (p_0, y_n), (p_0, z_n), (x_n, y_n), (y_n, x_n), (z_n, y_n) \in E(G)$.

Proof . Let $(x_0, p_0), (y_0, p_0), (z_0, p_0) \in E(G)$. Then $(T_1 z_0, T_1 p_0) \in E(G)$ because T_1 is edge-preserving. Since $p_0 \in F$, $(T_1 z_0, p_0) \in E(G)$. By the convexity of $E(G)$ and $(T_1 z_0, p_0), (z_0, p_0) \in E(G)$ we have $(x_1, p_0) \in E(G)$. Then $(T_3 x_1, p_0) \in E(G)$, because T_3 are edge-preserving. By the convexity of $E(G)$ and $(T_3 x_1, p_0), (x_1, p_0) \in E(G)$, we have $(y_1, p_0) \in E(G)$. Then $(T_2 x_1, p_0) \in E(G)$, because T_2 are edge-preserving. Again, by the convexity of $E(G)$ and $(T_2 x_1, p_0), (y_1, p_0) \in E(G)$ we have$(z_1, p_0) \in E(G)$.

Next, we assume that $(x_k, p_0), (y_k, p_0), (z_k, p_0) \in E(G)$. Then $(T_1 z_k, p_0) \in E(G)$ since T_i's are edge-preserving. By the convexity of $E(G)$ and $(T_1 z_k, p_0), (z_k, p_0) \in E(G)$, we have $(x_{k+1}, p_0) \in E(G)$. Then $(T_3 x_{k+1}, p_0) \in E(G)$, because T_3 is edge-preserving. By the convexity of $E(G)$ and $(T_3 x_{k+1}, p_0), (x_{k+1}, p_0) \in E(G)$, we have $(y_{k+1}, p_0) \in E(G)$. Then $(T_2 x_{k+1}, p_0) \in E(G)$, because T_2 is edge-preserving. By the convexity of $E(G)$ and $(T_2 x_{k+1}, p_0), (y_{k+1}, p_0) \in E(G)$, we have $(z_{k+1}, p_0) \in E(G)$.

Hence, by induction, (x_n, p_0), (y_n, p_0), $(z_n, p_0) \in E(G)$. Using a similar argument, an assumption that (p_0, x_0), (p_0, y_0), $(p_0, z_0) \in E(G)$, we can show that (p_0, x_n), (p_0, y_n), $(p_0, z_n) \in E(G)$. Therefore, $(x_n, y_n), (y_n, x_n), (z_n, y_n) \in E(G)$ by the transitivity of G. $\square$

Lemma 3.2. Let $p_0 \in F$. Suppose that $(x_0, p_0), (y_0, p_0), (z_0, p_0), (p_0, x_0), (p_0, y_0), (p_0, z_0) \in E(G)$, for arbitrary $x_0 \in C$. Then $\lim_{n\to\infty} \| x_n - p_0 \|$ exists.

Proof . Consider

$$\begin{aligned} \| y_n - p_0 \| &= \| (1-\beta_n)x_n + \beta_n T_3 x_n - p_0 \| \\ &\leq \| (1-\beta_n)(x_n - p_0) \| + \| \beta_n (T_3 x_n - T_3 p_0) \| \\ &= \| (1-\beta_n)(x_n - p_0) \| + \beta_n \| (T_3 x_n - T_3 p_0) \| \\ &\leq \| (x_n - p_0) - \beta_n (x_n - p_0) \| + \beta_n \| (x_n - p_0) \| \\ &= (1-\beta_n) \| (x_n - p_0) \| + \beta_n \| (x_n - p_0) \| \\ &= \| x_n - p_0 \| . \end{aligned}$$

By the G-nonexpansiveness of T_i together with $\| y_n - p_0 \| \leq \| x_n - p_0 \|$ we have

$$\begin{aligned}
\| z_n - p_0 \| &= \| (1-\gamma_n)y_n + \gamma_n T_2 x_n - p_0 \| \\
&\leq \| (1-\gamma_n)(y_n - p_0) \| + \| \gamma_n (T_2 x_n - T_2 p_0) \| \\
&= \| (1-\gamma_n)(y_n - p_0) \| + \gamma_n \| (T_2 x_n - T_2 p_0) \| \\
&\leq \| (1-\gamma_n)(y_n - p_0) \| + \gamma_n \| (x_n - p_0) \| \\
&= \| (y_n - p_0) - \gamma_n (y_n - p_0) \| + \gamma_n \| (x_n - p_0) \| \\
&= (1-\gamma_n) \| (y_n - p_0) \| + \gamma_n \| (x_n - p_0) \| \\
&\leq (1-\gamma_n) \| (x_n - p_0) \| + \gamma_n \| (x_n - p_0) \| \\
&= \| x_n - p_0 \| .
\end{aligned}$$

By the G-nonexpansiveness of T_i together with $\| z_n - p_0 \| \leq \| x_n - p_0 \|$ we have

$$\begin{aligned}
\| x_{n+1} - p_0 \| &= \| (1-\alpha_n)z_n + \alpha_n T_1 z_n - p_0 \| \\
&\leq \| (1-\alpha_n)(z_n - p_0) \| + \| \alpha_n (T_1 z_n - T_1 p_0) \| \\
&= \| (1-\alpha_n)(z_n - p_0) \| + \alpha_n \| (T_1 z_n - T_1 p_0) \| \\
&\leq \| (1-\alpha_n)(z_n - p_0) \| + \alpha_n \| (z_n - p_0) \| \\
&= \| (z_n - p_0) - \alpha_n (z_n - p_0) \| + \alpha_n \| (z_n - p_0) \| \\
&= (1-\alpha_n) \| (z_n - p_0) \| + \alpha_n \| (z_n - p_0) \| \\
&= \| z_n - p_0 \| \\
&\leq \| x_n - p_0 \| .
\end{aligned}$$

We have $\| x_{n+1} - p_0 \| \leq \| x_n - p_0 \|$. Then $\{\| x_n - p_0 \|\}$ is decreasing. Thus $\lim_{n\to\infty} \| x_n - p_0 \|$ exists. In particular, the sequence $\{x_n\}$ is bounded. □

Lemma 3.3. Let X be uniformly convex Banach space, and let C be a closed convex subset of X ,$\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, and $(x_0, p_0), (y_0, p_0), (z_0, p_0), (p_0, x_0), (p_0, y_0), (p_0, z_0) \in E(G)$, for arbitrary $p_0 \in F$ and $x_0 \in C$. Then $\lim_{n\to\infty} \| x_n - T_1 x_n \| = 0$, $\lim_{n\to\infty} \| x_n - T_2 x_n \| = 0$, and $\lim_{n\to\infty} \| x_n - T_3 x_n \| = 0$.

Proof . Let $\lim_{n\to\infty} \| x_n - p_0 \| = k$. If $k = 0$, then by the G-nonexpansiveness of T_i we have

$$\begin{aligned}
\| x_n - T_i x_n \| &\leq \| x_n - p_0 \| + \| p_0 - T_i x_n \| \\
&\leq \| x_n - p_0 \| + \| p_0 - x_n \| .
\end{aligned}$$

If $k > 0$, by the G-nonexpansiveness of T_i and lemma 2.11, we have

$$\begin{aligned}
\| x_{n+1} - p_0 \|^2 &= \| (1-\alpha_n)z_n + \alpha_n T_1 z_n - p_0 \|^2 \\
&= \| (1-\alpha_n)z_n + \alpha_n p_0 - \alpha_n p_0 + \alpha_n T_1 z_n - p_0 \|^2 \\
&= \| (1-\alpha_n)z_n - (1-\alpha_n)p_0 - \alpha_n p_0 + \alpha_n T_1 z_n \|^2 \\
&= \| (1-\alpha_n)(z_n - p_0) + \alpha_n (T_1 z_n - p_0) \|^2 \\
&\leq \alpha_n \| T_1 z_n - p_0 \|^2 + (1-\alpha_n) \| z_n - p_0 \|^2 - \alpha_n(1-\alpha_n) g(\| (T_1 z_n - p_0) - (z_n - p_0) \|) \\
&= \alpha_n \| T_1 z_n - p_0 \|^2 + (1-\alpha_n) \| z_n - p_0 \|^2 - \alpha_n(1-\alpha_n) g(\| T_1 z_n - z_n \|) \\
&= \alpha_n \| T_1 z_n - T_1 p_0 \|^2 + (1-\alpha_n) \| z_n - p_0 \|^2 - \alpha_n(1-\alpha_n) g(\| T_1 z_n - z_n \|) \\
&\leq \alpha_n \| z_n - p_0 \|^2 + (1-\alpha_n) \| z_n - p_0 \|^2 - \alpha_n(1-\alpha_n) g(\| T_1 z_n - z_n \|) \\
&= \alpha_n \| z_n - p_0 \|^2 + \| z_n - p_0 \|^2 - \alpha_n \| z_n - p_0 \|^2 - \alpha_n(1-\alpha_n) g(\| T_1 z_n - z_n \|) \\
&= \| z_n - p_0 \|^2 - \alpha_n(1-\alpha_n) g(\| T_1 z_n - z_n \|) \\
&\leq \| z_n - p_0 \|^2 - \delta^2 g(\| T_1 z_n - z_n \|) \\
&\leq \| x_n - p_0 \|^2 - \delta^2 g(\| T_1 z_n - z_n \|).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \delta^2 g(\| T_1 z_n - z_n \|) \le \lim_{n\to\infty} \| x_n - p_0 \|^2 - \lim_{n\to\infty} \| x_{n+1} - p_0 \|^2 = 0.$$

Hence $\lim_{n\to\infty} g(\| T_1 z_n - z_n \|) = 0$. Since g is strictly increasing and continuous at 0,

$$\lim_{n\to\infty} \| T_1 z_n - z_n \| = 0. \tag{3.1}$$

By the G-nonexpansiveness of T_i and lemma 2.11, we have

$$\begin{aligned}
\| x_{n+1} - p_0 \| &= \| (1-\alpha_n) z_n + \alpha_n T_1 z_n - p_0 \| \\
&= \| (1-\alpha_n) z_n + \alpha_n p_0 - \alpha_n p_0 + \alpha_n T_1 z_n - p_0 \| \\
&= \| (1-\alpha_n) z_n - (1-\alpha_n) p_0 - \alpha_n p_0 + \alpha_n T_1 z_n \| \\
&= \| (1-\alpha_n)(z_n - p_0) + \alpha_n(-p_0 + T_1 z_n) \| \\
&= \| (1-\alpha_n)(z_n - p_0) + \alpha_n(T_1 z_n - p_0) \| \\
&= \| (1-\alpha_n)(z_n - p_0) + \alpha_n(T_1 z_n - T_1 p_0) \| \\
&\le \| (1-\alpha_n)(z_n - p_0) \| + \| \alpha_n(T_1 z_n - T_1 p_0) \| \\
&= \| (1-\alpha_n)(z_n - p_0) \| + \alpha_n \| (T_1 z_n - T_1 p_0) \| \\
&\le \| (1-\alpha_n)(z_n - p_0) \| + \alpha_n \| (z_n - p_0) \| \\
&= (1-\alpha_n) \| (z_n - p_0) \| + \alpha_n \| (z_n - p_0) \| \\
&= \| z_n - p_0 \| .
\end{aligned}$$

Thus $k = \liminf_{n\to\infty} \| x_{n+1} - p_0 \| \le \liminf_{n\to\infty} \| z_n - p_0 \|$. Since $\| z_n - p_0 \| \le \| x_n - p_0 \|$, $\limsup_{n\to\infty} \| z_n - p_0 \| \le \limsup_{n\to\infty} \| x_n - p_0 \| = k$. Consider

$$\begin{aligned}
k &= \lim_{n\to\infty} \| z_n - p_0 \| \\
&= \lim_{n\to\infty} \| (1-\gamma_n) y_n + \gamma_n T_2 x_n - p_0 \| \\
&= \lim_{n\to\infty} \| (1-\gamma_n)(y_n - p_0) + \gamma_n(T_2 x_n - p_0) \| .
\end{aligned}$$

Since $\limsup_{n\to\infty} \| T_2 x_n - p_0 \| = \limsup_{n\to\infty} \| T_2 x_n - T_2 p_0 \| \le \limsup_{n\to\infty} \| x_n - p_0 \| = k$, $\limsup_{n\to\infty} \| y_n - p_0 \| \le \limsup_{n\to\infty} \| x_n - p_0 \| = k$ and by lemma 2.11, we have

$$\lim_{n\to\infty} \| T_2 x_n - y_n \| = 0. \tag{3.2}$$

By the G-nonexpansiveness of T_i and lemma 2.11, we have

$$\begin{aligned}
\| z_n - p_0 \| &= \| (1-\gamma_n) y_n + \gamma_n T_2 x_n - p_0 \| \\
&= \| y_n - \gamma_n y_n + \gamma_n T_2 x_n - p_0 \| \\
&= \| y_n - p_0 + \gamma_n(T_2 x_n - y_n) \| \\
&\le \| y_n - p_0 \| + \| \gamma_n(T_2 x_n - y_n) \| \\
&= \| y_n - p_0 \| + \gamma_n \| T_2 x_n - y_n \| \\
&\le \| y_n - p_0 \| + \| T_2 x_n - y_n \| .
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} \| z_n - p_0 \| &\le \lim_{n\to\infty} (\| y_n - p_0 \| + \| T_2 x_n - y_n \|) \\
&= \lim_{n\to\infty} \| y_n - p_0 \| + \lim_{n\to\infty} \| T_2 x_n - y_n \| \\
&= \lim_{n\to\infty} \| y_n - p_0 \| .
\end{aligned}$$

Since $\| x_{n+1} - p_0 \| \leq \| z_n - p_0 \|$, implies that

$$k = \lim_{n\to\infty} \| x_{n+1} - p_0 \| \leq \lim_{n\to\infty} \| z_n - p_0 \| \leq \lim_{n\to\infty} \| y_n - p_0 \| .$$

Since $\| y_n - p_0 \| \leq \| x_n - p_0 \|$, we have

$$\lim_{n\to\infty} \| y_n - p_0 \| \leq \lim_{n\to\infty} \| x_n - p_0 \| = k.$$

Consider

$$\begin{aligned} k &= \lim_{n\to\infty} \| y_n - p_0 \| \\ &= \lim_{n\to\infty} \| (1-\beta_n)x_n + \beta_n T_3 x_n - p_0 \| \\ &= \lim_{n\to\infty} \| (1-\beta_n)(x_n - p_0) + \beta_n (T_3 x_n - p_0) \| . \end{aligned}$$

Since $\limsup_{n\to\infty} \| T_3 x_n - p_0 \| = \limsup_{n\to\infty} \| T_3 x_n - T_3 p_0 \| \leq \limsup_{n\to\infty} \| x_n - p_0 \| = k$,

$$\limsup_{n\to\infty} \| x_n - p_0 \| \leq k$$

and by Lemma 2.11, we have

$$\lim_{n\to\infty} \| T_3 x_n - x_n \| = 0. \tag{3.3}$$

By the G-nonexpansiveness of T_i and equation 3.3 we have

$$\begin{aligned} \| y_n - x_n \| &= \| (1-\beta_n)x_n + \beta_n T_3 x_n - x_n \| \\ &= \| \beta_n (T_3 x_n - x_n) \| \\ &\leq \| T_3 x_n - x_n \| \\ &= 0. \end{aligned}$$

Thus

$$\| y_n - x_n \| = 0. \tag{3.4}$$

By the G-nonexpansiveness of T_i and equation 3.2 and 3.4, we have

$$\begin{aligned} \| y_n - T_2 y_n \| &= \| y_n - T_2 x_n + T_2 x_n - T_2 y_n \| \\ &\leq \| y_n - T_2 x_n \| + \| T_2 x_n - T_2 y_n \| \\ &\leq \| y_n - T_2 x_n \| + \| x_n - y_n \| \\ &= 0. \end{aligned}$$

Thus

$$\| y_n - T_2 y_n \| = 0. \tag{3.5}$$

By the G-nonexpansiveness of T_i, equations 3.4 and 3.5, we have

$$\begin{aligned} \| x_n - T_2 x_n \| &= \| x_n - y_n + y_n - T_2 y_n + T_2 y_n - T_2 x_n \| \\ &\leq \| x_n - y_n \| + \| y_n - T_2 y_n \| + \| T_2 y_n - T_2 x_n \| \\ &\leq \| x_n - y_n \| + \| y_n - T_2 y_n \| + \| y_n - x_n \| \\ &= 0. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \| x_n - T_2 x_n \| = 0. \tag{3.6}$$

By the G-nonexpansiveness of T_i and equation 3.2, we have

$$\begin{aligned}\| x_n - z_n \| &=\| x_n - ((1-\gamma_n)y_n + \gamma_n T_2 x_n) \| \\ &\leq\| x_n - y_n \| + \| \gamma_n y_n - \gamma_n T_2 x_n) \| \\ &= \gamma_n \| y_n - T_2 x_n \| \\ &\leq\| y_n - T_2 x_n \| \\ &= 0.\end{aligned}$$

Thus

$$\| x_n - z_n \|= 0. \tag{3.7}$$

By the G-nonexpansiveness of T_i and equation 3.2, we have

$$\begin{aligned}\| z_n - y_n \| &=\| (1-\gamma_n)y_n + \gamma_n T_2 x_n - y_n \| \\ &= \gamma_n \| T_2 x_n - y_n \| \\ &\leq\| T_2 x_n - y_n \| \\ &= 0.\end{aligned}$$

Thus

$$\| z_n - y_n \|= 0. \tag{3.8}$$

By the G-nonexpansiveness of T_i, equations 3.1, 3.7 and 3.8, we have

$$\begin{aligned}\| x_n - T_1 y_n \| &=\| x_n - z_n + z_n - T_1 z_n + T_1 z_n - T_1 y_n \| \\ &\leq\| x_n - z_n \| + \| z_n - T_1 z_n \| + \| T_1 z_n - T_1 y_n \| \\ &\leq\| x_n - z_n \| + \| z_n - T_1 z_n \| + \| z_n - y_n \| \\ &= 0.\end{aligned}$$

Thus

$$\| x_n - T_1 y_n \|= 0. \tag{3.9}$$

By the G-nonexpansiveness of T_i and equations 3.9, 3.4, we have

$$\begin{aligned}\| x_n - T_1 x_n \| &=\| x_n - T_1 y_n + T_1 y_n - T_1 x_n \| \\ &\leq\| x_n - T_1 y_n \| + \| T_1 y_n - T_1 x_n \| \\ &\leq\| x_n - T_1 y_n \| + \| y_n - x_n \| \\ &= 0.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \| x_n - T_1 x_n \|= 0. \tag{3.10}$$

By equation 3.3, 3.6 and 3.10, we have $\lim_{n\to\infty} \| x_n - T_1 x_n \|= 0$, $\lim_{n\to\infty} \| x_n - T_2 x_n \|= 0$, $\lim_{n\to\infty} \| x_n - T_3 x_n \|= 0$. □

Lemma 3.4. Suppose that X satisfies the Opial's property and let $(x_0, p_0), (y_0, p_0), (z_0, p_0), (p_0, x_0), (p_0, y_0), (p_0, z_0) \in E(G)$, for $p_0 \in F$ and arbitrary $x_0 \in C$. Then $I - T_i$'s, for $i = 1, 2, 3$ are demiclosed.

Proof . Suppose that $\{x_n\}$ is a sequence in C that converges weakly to v. From Lemma 3.3 we get $\lim_{n\to\infty} \| x_n - T_i x_n \|= 0$. Suppose for contradiction that $v \neq T_i v$. By Opial's property, we have

$$\begin{aligned}\limsup_{n\to\infty} \| x_n - v \| &< \limsup_{n\to\infty} \| x_n - T_i v \| \\ &\leq \limsup_{n\to\infty}(\| x_n - T_i x_n \| + \| T_i x_n - T_i v \|) \\ &\leq \limsup_{n\to\infty} \| x_n - v \|,\end{aligned}$$

a contradiction. Hence, $v = T_i v$. This implies that $I - T_i$ is demiclosed. □

Theorem 3.5. Suppose that X is uniformly convex, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [\delta, 1-\delta]$, for some $\delta \in (0, \frac{1}{2}), T_i$ $(i = 1, 2, 3)$ satisfies Condition B and $(x_0, p), (y_0, p), (z_0, p), (p, x_0), (p, y_0), (p, z_0) \in E(G)$, for each $p \in F$ and arbitrary $x_0 \in C$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i.

Proof . Let $p \in F$. From Lemma 3.2 we have
1) $\{x_n\}$ is bounded,
2) $\lim_{n\to\infty} \| x_n - p \|$ exists ,
3) $\| x_{n+1} - p \| \leq \| x_n - p \|$ for all $n \geq 1$.

We imply that $d(x_{n+1}, F) \leq d(x_n, F)$. Since T_i satisfies Condition B and $\lim_{n\to\infty} \| x_n - T_i x_n \| = 0$, we get $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Thus $\lim_{n\to\infty} d(x_n, F) = 0$ exists. Hence, there are a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset F$ such that

$$\| x_{n_k} - p_k \| \leq \frac{1}{2^k}.$$

Put $n_{k+1} = n_k + d$ for some $k \geq 1$. Then

$$\| x_{n_{k+1}} - p_k \| \leq \| x_{n_k+d-1} - p_k \| \leq \| x_{n_k} - p_k \| \leq \frac{1}{2^k}.$$

Thus

$$\begin{aligned} \| p_{k+1} - p_k \| &= \| p_{k+1} - x_{n_k} + x_{n_k} - p_k \| \\ &\leq \| p_{k+1} - x_{n_k} \| + \| x_{n_k} - p_k \| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &= \frac{3}{2^{k+1}}. \end{aligned}$$

So that $\{p_k\}$ is a Cauchy sequence. We assume that $p_k \to v \in C$ as $n \to \infty$. Since F is closed, $v \in F$. Hence $x_{n_k} \to v$ as $k \to \infty$. Since $\lim_{n\to\infty} \| x_n - v \|$ exists, the conclusion follows. □

Theorem 3.6. Suppose that X is uniformly convex, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [\delta, 1-\delta]$, for some $\delta \in (0, \frac{1}{2})$, T_i, for $i = 1, 2, 3$ is semicompact, $\{x_n\}$ dominates C and $(x_0, p_0), (y_0, p_0), (z_0, p_0), (p_0, x_0), (p_0, y_0), (p_0, z_0) \in E(G)$, for $p_0 \in F$ and arbitrary $x_0 \in C$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i.

Proof . Suppose that T_2 is semicompact, by Lemma 3.2 and Lemma 3.3 we have a sequence $\{x_n\}$ is bounded for all $n \geq 1$ and $\lim_{n\to\infty} \| x_n - T_i x_n \| = 0$, by T_2 is semicompact there exist $v \in C$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to v$ as $k \to \infty$ and $\lim_{n\to\infty} \| x_{n_k} - T_i x_{n_k} \| = 0$. Consider

$$\begin{aligned} \| v - T_i v \| &= \| v - x_{n_k} + x_{n_k} - T_i x_{n_k} + T_i x_{n_k} - T_i v \| \\ &\leq \| v - x_{n_k} \| + \| x_{n_k} - T_i x_{n_k} \| + \| T_i x_{n_k} - T_i v \| \\ &\leq \| v - x_{n_k} \| + \| x_{n_k} - T_i x_{n_k} \| + \| x_{n_k} - v \| . \end{aligned}$$

Since $x_{n_k} \to v$ and $\lim_{n\to\infty} \| x_{n_k} - T_i x_{n_k} \| = 0$, we have $\| v - T_i v \| = 0$, that is $v = T_i v$. Thus, $v \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, it follows by repeating the same argument as in the proof of Theorem 3.5 that $\{x_n\}$ converges strongly to a common fixed point of T_i. □

Theorem 3.7. Suppose that X is uniformly convex. Then, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [\delta, 1-\delta]$, for some $\delta \in (0, \frac{1}{2})$. If X satisfies Opial's property, $I - T_i$ is demiclosed at zero for each i, F is dominated by x_0, and $(x_0, p_0), (y_0, p_0), (z_0, p_0), (p_0, x_0), (p_0, y_0), (p_0, z_0) \in E(G)$, for $p_0 \in F$ and arbitrary $x_0 \in C$. Then $\{x_n\}$ converges weakly to a common fixed point of T_i.

Proof . By Lemma 3.2 for each $v \in F$, $\lim_{n\to\infty} \| x_n - v \|$ exists. Let $\{x_{n_k}\}$ and $\{x_{n_j}\}$ be subsequences of the sequence $\{x_n\}$ with $\{x_{n_k}\}$ converges weakly to v_1 and $\{x_{n_j}\}$ converges weakly to v_2. Notice that, by Lemma 3.3, $\lim_{n\to\infty} \| x_n - T_i x_n \| = 0$. Thus

$$\lim_{k\to\infty} \| x_{n_k} - T_i x_{n_k} \| = 0.$$

This imply that $\| x_{n_k} - T_i x_{n_k} \| \to 0$ as $n \to \infty$ and $\lim_{j\to\infty} \| x_{n_j} - T_i x_{n_j} \| = 0$. Then $\| x_{n_j} - T_i x_{n_j} \| \to 0$ as $n \to \infty$. Since $I - T_i$ is demiclosed at zero and X satisfies Opial's property, $T_i v_1 = v_1$ and $T_i v_2 = v_2$. By Lemma 3.4, we have $v_1, v_2 \in F$. In particular, $v_1 = v_2$ by Lemma 2.12. Thus, $\{x_n\}$ converges weakly to a common fixed point of T_i. □

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