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# Some new coincidence point results in partial $b$-metric spaces via digraphs, $\mathscr{L}$-simulation and $\theta$-functions

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**Abstract**

In the present article, we introduce the concept of $(\alpha, \theta, \xi)$-$G$-contractive mappings in partial $b$-metric spaces endowed with a digraph $G$ and discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of self mappings satisfying such contractive condition. Our main result will extend several recent results including the well-known Banach contraction theorem. Finally, we exhibit that this extension is viable which will justify the new contractive condition.

Keywords: $\mathscr{L}$-simulation function, $\theta$-function, weakly compatible maps, point of coincidence
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## 1 Introduction

The Banach contraction theorem [6] in complete metric spaces is one of the fundamental results in the field of fixed point theory. This theory has enormous applications not only in different areas of mathematics but also in economics, computer science, engineering and others. There exist a lot of generalizations of the notion of metric spaces such as $b$-metric space, introduced by Bakhtin [5], partial metric space by Matthews [24], and dislocated metric space by Hitzler et al.[19]. Combining the notions of $b$-metric and partial metric, Shukla [29] introduced another generalization, called a partial $b$-metric and established some fixed point results in partial $b$-metric spaces.

Many researchers have studied the coincidence point and common fixed point results for a pair of mappings satisfying some contractive type conditions in various spaces. In 2014, Jleli and Samet [20] presented a generalization of the Banach contraction theorem in generalized metric spaces by using $\theta$-contractions. After that, Ahmad et al.[2] extended the result of Jleli and Samet [20] to metric spaces by modifying the notion of $\theta$-contractions. In recent investigations, the study of fixed point theory combining a graph takes a vital role in many aspects. Echenique [13] studied fixed point theory by using graphs and then Espinola and Kirk [14] applied fixed point results in graph theory. In [22], Khojasteh et al. introduced the concept of $\mathscr{L}$-contractions and unified some existing metric fixed point results. Motivated by the idea given in [2, 15, 22] and some recent works on partial $b$-metric and $b$-metric spaces with a graph (see [3, 4, 7, 16, 23, 25, 26, 27, 28]), we reformulated some important coincidence point and common fixed point results in partial $b$-metric spaces endowed with a digraph by using $\mathscr{L}$-simulation and $\theta$-functions. Finally, we give some non-trivial examples to illustrate our main result.

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## 2 Some Basic Concepts

This section begins with some basic notations, definitions and necessary results that will be needed in the sequel.

**Definition 2.1.** [11] Let $X$ be a nonempty set and $b \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a $b$-metric on $X$ if the following conditions hold:

$(i)$ $d(x, y) = 0$ if and only if $x = y$;

$(ii)$ $d(x, y) = d(y, x)$ for all $x,\ y \in X$;

$(iii)$ $d(x, y) \leq b\,(d(x, z) + d(z, y))$ for all $x,\ y,\ z \in X$.

The pair $(X, d)$ is called a $b$-metric space.

It is worth noting that the class of $b$-metric spaces is effectively larger than that of the ordinary metric spaces.

**Definition 2.2.** [24] A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x,\ y,\ z \in X$:

$(p_1)$ $p(x, x) = p(y, y) = p(x, y) \Longleftrightarrow x = y$;

$(p_2)$ $p(x, x) \leq p(x, y)$;

$(p_3)$ $p(x, y) = p(y, x)$;

$(p_4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair $(X, p)$ is called a partial metric space.

It is obvious that if $p(x, y) = 0$, then from $(p_1)$ and $(p_2)$, it follows that $x = y$. However, $x = y$ does not imply $p(x, y) = 0$.

**Example 2.3.** [24] Let $X = [0, \infty)$ and let $p(x, y) = \max\{x,\ y\}$ for all $x,\ y \in X$. Then $(X, p)$ is a partial metric space but $p$ is not a metric on $X$.

**Definition 2.4.** [29] A partial $b$-metric on a nonempty set $X$ is a function $p_b : X \times X \to \mathbb{R}^+$ such that for some real number $b \geq 1$ and all $x,\ y,\ z \in X$:

$(p_{b1})$ $p_b(x, x) = p_b(y, y) = p_b(x, y) \Longleftrightarrow x = y$;

$(p_{b2})$ $p_b(x, x) \leq p_b(x, y)$;

$(p_{b3})$ $p_b(x, y) = p_b(y, x)$;

$(p_{b4})$ $p_b(x, y) \leq b[p_b(x, z) + p_b(z, y) - p_b(z, z)]$.

The pair $(X, p_b)$ is called a partial $b$-metric space.

It is clear that every partial metric space is a partial $b$-metric space with the coefficient $b = 1$ and every $b$-metric space is also a partial $b$-metric space with the same coefficient $b$. However, the reverse implications need not hold true, in general.

**Example 2.5.** [29] Let $X = \mathbb{R}^+,\ p > 1$ a constant, and $p_b : X \times X \to \mathbb{R}^+$ be defined by

$$p_b(x, y) = [\max\{x,\ y\}]^p + \mid x - y \mid^p,\ \forall x,\ y \in X.$$

Then $(X, p_b)$ is a partial $b$-metric space with coefficient $b = 2^p$, but it is neither a partial metric space nor a $b$-metric space.

**Example 2.6.** [29] Let $(X,p)$ be a partial metric space and $p_b(x,y) = (p(x,y))^p$, where $p \geq 1$ is a real number. Then $p_b$ is a partial $b$-metric with coefficient $b = 2^{p-1}$.

Let $(X,p_b)$ be a partial $b$-metric space. For each $x \in X$ and for each $\epsilon > 0$, put $B(x,\epsilon) = \{y \in X : p_b(x,y) < p_b(x,x) + \epsilon\}$. Let $\mathscr{B} = \{B(x,\epsilon) : x \in X \text{ and } \epsilon > 0\}$. Ge and Lin [17] proved that $\mathscr{B}$ is not a base for any topology on $X$. However, they proved that $\mathscr{B}$ is a subbase for some topology $\tau$ on $X$ such that $(X,\tau)$ is a $T_0$-space.

**Proposition 2.7.** [17] Let $(X,p_b)$ be a partial $b$-metric space and $(x_n)$ be a sequence in $X$. If $(x_n)$ converges to $x \in X$ with respect to $\tau$, then $\lim_{n\to\infty} p_b(x_n,x) = p_b(x,x)$.

The above proposition cannot be reversed(see [17]).

**Definition 2.8.** [29] Let $(X,p_b)$ be a partial $b$-metric space with coefficient $b \geq 1$ and let $(x_n)$ be a sequence in $X$. Then

(i) $(x_n)$ converges to a point $x \in X$ if and only if $\lim_{n\to\infty} p_b(x_n,x) = p_b(x,x)$. This will be denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(ii) $(x_n)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n,x_m)$ exists and is finite.

(iii) $(X,p_b)$ is said to be complete if every Cauchy sequence $(x_n)$ in $X$, there exists $x \in X$ such that $\lim_{n,m\to\infty} p_b(x_n,x_m) = \lim_{n\to\infty} p_b(x_n,x) = p_b(x,x)$.

**Definition 2.9.** [12] A sequence $(x_n)$ in a partial $b$-metric space $(X,p_b)$ is called 0-Cauchy if

$$\lim_{n,m\to\infty} p_b(x_n,x_m) = 0.$$

The space $(X,p_b)$ is said to be 0-complete if every 0-Cauchy sequence in $X$ converges to a point $x \in X$ such that $p_b(x,x) = 0$, i.e., $\lim_{n,m\to\infty} p_b(x_n,x_m) = \lim_{n\to\infty} p_b(x_n,x) = p_b(x,x) = 0$.

**Lemma 2.10.** [12] If $(X,p_b)$ is complete, then it is 0-complete.

The converse assertion of the above lemma may not hold, in general. The following example supports this fact.

**Example 2.11.** The space $X = [0,\infty) \cap \mathbb{Q}$ with $p_b(x,y) = \max\{x,y\}$ is a 0-complete partial $b$-metric space with coefficient $b = 1$, but it is not complete. Moreover, the sequence $(x_n)$ with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in $(X,p_b)$, but it is not a 0-Cauchy sequence.

**Remark 2.12.** [29] In a partial $b$-metric space $(X,p_b)$, the limit of a convergent sequence need not be unique.

**Definition 2.13.** A sequence $(x_n)$ in a partial $b$-metric space $(X,p_b)$ is said to be bounded if the set $\{p_b(x_n,x_m) : n, m \in \mathbb{N}\}$ of real numbers is bounded in $\mathbb{R}$, that is, there exists $M > 0$ such that $p_b(x_n,x_m) \leq M$ for all $n, m \in \mathbb{N}$.

**Definition 2.14.** [1] Let $T$ and $S$ be self mappings of a set $X$. If $y = Tx = Sx$ for some $x$ in $X$, then $x$ is called a coincidence point of $T$ and $S$ and $y$ is called a point of coincidence of $T$ and $S$.

**Definition 2.15.** [21] The mappings $T,S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \textit{ whenever } Sx = Tx.$$

**Proposition 2.16.** [1] Let $S$ and $T$ be weakly compatible self mappings of a nonempty set $X$. If $S$ and $T$ have a unique point of coincidence $y = Sx = Tx$, then $y$ is the unique common fixed point of $S$ and $T$.

**Definition 2.17.** [10] Let $\mathscr{L}$ be the family of all mappings $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ such that

($\xi 1$) $\xi(1,1) = 1$;

($\xi 2$) $\xi(t,s) < \frac{s}{t}$ for all $t,\ s > 1$;

($\xi 3$) for any two sequences $(t_n)$, $(s_n) \subseteq (1,\infty)$ with $t_n \leq s_n$ for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1 \Rightarrow \limsup_{n\to\infty} \xi(t_n, s_n) < 1.$$

We say that $\xi \in \mathscr{L}$ is an $\mathscr{L}$-simulation function. It is to be noted that $\xi(t,t) < 1$ for all $t > 1$.

**Example 2.18.** [10] Let $\xi_b$, $\xi_w$, $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ be functions defined as follows, respectively:

*(i)* $\xi_b(t,s) = \frac{s^k}{t}$ for all $t,\ s \geq 1$, where $k \in (0,1)$.

*(ii)* $\xi_w(t,s) = \frac{s}{t\phi(s)}$ for all $t,\ s \geq 1$, where $\phi : [1,\infty) \to [1,\infty)$ is a nondecreasing and lower semicontinuous function such that $\phi^{-1}(\{1\}) = 1$.

*(iii)*

$$\xi(t,s) = \begin{cases} 1, \text{ if } (s,t) = (1,1), \\ \frac{s}{2t}, \text{ if } s < t, \\ \frac{s^\lambda}{t}, \text{ otherwise,} \end{cases}$$

for all $s,\ t \geq 1$, where $\lambda \in (0,1)$.

Then $\xi_b$, $\xi_w$, $\xi \in \mathscr{L}$.

**Definition 2.19.** [2] Let $\Theta$ be the set of all functions $\theta : (0,\infty) \to (1,\infty)$ such that

($\theta 1$) $\theta$ is nondecreasing;

($\theta 2$) for all $(t_n) \subseteq (0,\infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \Longleftrightarrow \lim_{n\to\infty} t_n = 0^+;$$

($\theta 3$) $\theta$ is continuous on $(0,\infty)$.

**Example 2.20.** [2] Let $\theta : (0,\infty) \to (1,\infty)$ be defined as $\theta(t) = e^t$ for all $t > 0$. Then $\theta \in \Theta$.

We now assign a digraph in a partial $b$-metric space as stated below.

Let $(X, p_b)$ be a partial $b$-metric space and $\Delta = \{(x,x) : x \in X\}$. Consider a digraph $G$ such that the set $V(G)$ of its vertices coincides with $X$, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We also assume that $G$ has no parallel edges. Under these assumptions, we can identify $G$ with the pair $(V(G), E(G))$. By $G^{-1}$ we denote the graph obtained from $G$ by reversing the direction of edges, i.e., $E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from $G$ by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [8, 9, 18].

**Definition 2.21.** Let $(X, p_b)$ be a partial $b$-metric space with the coefficient $b \geq 1$ and let $G = (V(G), E(G))$ be a digraph. A mapping $f : X \to X$ is called a Banach $G$-contraction or simply $G$-contraction if there exists $k \in (0, \frac{1}{b})$ such that

$$p_b(fx, fy) \leq k\, p_b(x,y)$$

for all $x,\ y \in X$ with $(x,y) \in E(G)$.

Any Banach contraction is a $G_0$-contraction, where the graph $G_0$ is defined by $E(G_0) = X \times X$. But it is valuable to note that a Banach $G$-contraction need not be a Banach contraction (see Remark 3.11).

**Remark 2.22.** If $f$ is a $G$-contraction, then $f$ is both a $G^{-1}$-contraction and a $\tilde{G}$-contraction.

## 3 Main Result

In this section we assume that $(X, p_b)$ is a partial $b$-metric space with the coefficient $b \geq 1$ and $G = (V(G), E(G))$ is a reflexive digraph which has no parallel edges.

**Definition 3.1.** Let $f,\, g : (X, p_b) \to (X, p_b)$ be mappings with the property that $p_b(fx, fy) > 0$ implies $p_b(gx, gy) > 0$. Then, the mapping $f$ is called $(\alpha, \theta, \xi)$-$G$-contractive w.r.t. the mapping $g$ if there exist three functions $\theta \in \Theta$, $\xi \in \mathscr{L}$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\xi\big(\theta(b\alpha(gx, gy)p_b(fx, fy)), \theta(p_b(gx, gy))\big) \geq 1 \tag{3.1}$$

for all $x,\, y \in X$ with $(gx, gy) \in E(\tilde{G}),\ p_b(fx, fy) > 0$ and $\alpha(gx, gy) \geq 1$.

Taking $g = I$, the identity map on $X$, the above definition gives the following definition.

**Definition 3.2.** Let $f : (X, p_b) \to (X, p_b)$ be a mapping with the property that $p_b(fx, fy) > 0$ implies $p_b(x, y) > 0$. Then, the mapping $f$ is called $(\alpha, \theta, \xi)$-$G$-contractive if there exist three functions $\theta \in \Theta$, $\xi \in \mathscr{L}$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\xi\big(\theta(b\alpha(x, y)p_b(fx, fy)), \theta(p_b(x, y))\big) \geq 1$$

for all $x,\, y \in X$ with $(x, y) \in E(\tilde{G}),\ p_b(fx, fy) > 0$ and $\alpha(x, y) \geq 1$.

Taking $G = G_0$ in Definition 3.1, we get the following.

**Definition 3.3.** Let $f,\, g : (X, p_b) \to (X, p_b)$ be mappings with the property that $p_b(fx, fy) > 0$ implies $p_b(gx, gy) > 0$. Then, the mapping $f$ is called $(\alpha, \theta, \xi)$-contractive w.r.t. the mapping $g$ if there exist three functions $\theta \in \Theta$, $\xi \in \mathscr{L}$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\xi\big(\theta(b\alpha(gx, gy)p_b(fx, fy)), \theta(p_b(gx, gy))\big) \geq 1$$

for all $x, y \in X$ with $p_b(fx, fy) > 0$ and $\alpha(gx, gy) \geq 1$.

Let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$. Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq g(X)$, there exists an element $x_1 \in X$ such that $gx_1 = fx_0$. Continuing in this way, we can construct a sequence $(gx_n)$ in $g(X)$ such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$.

**Definition 3.4.** Let the mappings $f,\, g : X \to X$ be such that $f(X) \subseteq g(X)$. We define $C_{fg}^{\alpha G}$ the set of all elements $x_0$ of $X$ such that for all $m,\, n = 0, 1, 2, \cdots$, $(gx_n, gx_m) \in E(\tilde{G})$ and $\alpha(gx_n, gx_m) \geq 1$, for every sequence $(gx_n)$ such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$.

Taking $g = I$, $C_{fg}^{\alpha G}$ becomes $C_f^{\alpha G}$ which is the collection of all elements $x_0$ of $X$ such that $(x_n, x_m) \in E(\tilde{G})$ and $\alpha(x_n, x_m) \geq 1$ for $m, n = 0, 1, 2, \cdots$, where $x_n = fx_{n-1},\ n = 1, 2, 3, \cdots$.

Taking $G = G_0$, $C_{fg}^{\alpha G}$ becomes $C_{fg}^{\alpha}$ which is the collection of all elements $x_0$ of $X$ such that for all $m, n = 0, 1, 2, \cdots$, $\alpha(gx_n, gx_m) \geq 1$, for every sequence $(gx_n)$ such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$.

Before presenting our main result, we state a property of the graph $G$, call it Property $(*)$.

**Property $(*)$:** If $(gx_k)$ is a sequence in $(X, p_b)$ such that $p_b(gx_k, x) \to 0$, $(gx_k, gx_{k+1}) \in E(\tilde{G})$ and $\alpha(gx_k, gx_{k+1}) \geq 1$ for all $k \geq 1$, then there exists a subsequence $(gx_{k_i})$ of $(gx_k)$ such that $(gx_{k_i}, x) \in E(\tilde{G})$ and $\alpha(gx_{k_i}, x) \geq 1$ for all $i \geq 1$.

Taking $g = I$, the above property reduces to Property $(\acute{*})$:

**Property $(\acute{*})$:** If $(x_k)$ is a sequence in a partial $b$-metric space $(X, p_b)$ such that $p_b(x_k, x) \to 0$, $(x_k, x_{k+1}) \in E(\tilde{G})$ and $\alpha(x_k, x_{k+1}) \geq 1$ for all $k \geq 1$, then there exists a subsequence $(x_{k_i})$ of $(x_k)$ such that $(x_{k_i}, x) \in E(\tilde{G})$ and $\alpha(x_{k_i}, x) \geq 1$ for all $i \geq 1$.

Taking $G = G_0$ in Property $(*)$, we get the following property:

**Property $(\dagger)$:** If $(gx_k)$ is a sequence in $(X, p_b)$ such that $p_b(gx_k, x) \to 0$ and $\alpha(gx_k, gx_{k+1}) \geq 1$ for all $k \geq 1$, then there exists a subsequence $(gx_{k_i})$ of $(gx_k)$ such that $\alpha(gx_{k_i}, x) \geq 1$ for all $i \geq 1$.

If $(X, p_b, \preceq)$ is a partially ordered partial $b$-metric space, then by taking $\alpha(x,y) = 1$ for all $x, y \in X$ and $G = G_2$, where the graph $G_2$ is defined by $E(G_2) = \{(x,y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$, the Property $(\acute{*})$ reduces to the Property (‡) which can be stated as follows:

**Property (‡):** If $(x_k)$ is a sequence in a partially ordered partial $b$-metric space $(X, p_b, \preceq)$ such that $p_b(x_k, x) \to 0$ and $x_k$, $x_{k+1}$ are comparable for all $k \geq 1$, then there exists a subsequence $(x_{k_i})$ of $(x_k)$ such that $x_{k_i}$, $x$ are comparable for all $i \geq 1$.

**Remark 3.5.** For examples of the definitions of this section, we refer to Examples 3.7, 3.9 and 3.10.

We now present our main result.

**Theorem 3.6.** Let $(X, p_b)$ be a partial $b$-metric space with the coefficient $b \geq 1$ and let $G = (V(G), E(G))$ be a digraph. Let $f, g : X \to X$ be mappings with the property that $p_b(fx, fy) > 0$ implies that $p_b(gx, gy) > 0$. Suppose that $f$ is $(\alpha, \theta, \xi)$-$G$-contractive w.r.t. the mapping $g$. Suppose also that $f(X) \subseteq g(X)$, $g(X)$ is a 0-complete subspace of $X$ and the graph $G$ has the Property $(*)$. Then $f$ and $g$ have a point of coincidence $u$(say) in $g(X)$ with $p_b(u,u) = 0$ if $C_{fg}^{\alpha G} \neq \emptyset$.
Moreover, $f$ and $g$ have a unique point of coincidence in $g(X)$ if the graph $G$ has the following property:

$(**)$ If $x$, $y$ are points of coincidence of $f$ and $g$ in $g(X)$, then $(x,y) \in E(\tilde{G})$ and $\alpha(x,y) \geq 1$.

Furthermore, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point in $g(X)$.

**Proof .** Suppose that $C_{fg}^{\alpha G} \neq \emptyset$. We choose an $x_0 \in C_{fg}^{\alpha G}$ and keep it fixed. Since $f(X) \subseteq g(X)$, there exists a sequence $(gx_n)$ in $X$ such that $gx_n = fx_{n-1}$, for $n \in \mathbb{N}$ and $(gx_n, gx_m) \in E(\tilde{G})$ and $\alpha(gx_n, gx_m) \geq 1$ for $m, n = 0, 1, 2, \cdots$.

We assume that $gx_n \neq gx_{n-1}$ for every $n \in \mathbb{N}$. In fact, if $gx_n = gx_{n-1}$ for some $n \in \mathbb{N}$ then $gx_n = fx_{n-1} = gx_{n-1}$ which implies that $gx_n$ is a point of coincidence of $f$ and $g$.

We now prove that $\lim_{n\to\infty} p_b(gx_{n-1}, gx_n) = 0$.

First we note that for all $n \in \mathbb{N}$, $(gx_{n-1}, gx_n) \in E(\tilde{G})$, $\alpha(gx_{n-1}, gx_n) \geq 1$ and $p_b(fx_{n-1}, fx_n) > 0$. Therefore, it follows from conditions (3.1) and $(\xi 2)$ that

$$\begin{aligned} 1 &\leq \xi\big(\theta(b\alpha(gx_{n-1}, gx_n)p_b(fx_{n-1}, fx_n)), \theta(p_b(gx_{n-1}, gx_n))\big) \\ &< \frac{\theta(p_b(gx_{n-1}, gx_n))}{\theta(b\alpha(gx_{n-1}, gx_n)p_b(fx_{n-1}, fx_n))}. \end{aligned}$$

This implies that

$$\theta(b\alpha(gx_{n-1}, gx_n)p_b(gx_n, gx_{n+1})) < \theta(p_b(gx_{n-1}, gx_n)).$$

Therefore, by $(\theta_1)$, it follows that, for all $n = 1, 2, \cdots$,

$$b\alpha(gx_{n-1}, gx_n)p_b(gx_n, gx_{n+1}) < p_b(gx_{n-1}, gx_n)$$

and so

$$p_b(gx_n, gx_{n+1}) \leq b\alpha(gx_{n-1}, gx_n)p_b(gx_n, gx_{n+1}) < p_b(gx_{n-1}, gx_n). \tag{3.2}$$

Hence $(p_b(gx_{n-1}, gx_n))$ is a decreasing sequence of positive real numbers, so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p_b(gx_{n-1}, gx_n) = r.$$

From condition (3.2), it follows that

$$\lim_{n\to\infty} b\alpha(gx_{n-1}, gx_n)p_b(gx_n, gx_{n+1}) = r.$$

We shall show that $r = 0$. Assume that $r > 0$. Then by using conditions $(\theta 2)$ and $(\theta 3)$, we get

$$\lim_{n\to\infty} \theta(p_b(gx_{n-1}, gx_n)) = \theta(r) > 1$$

and

$$\lim_{n\to\infty} \theta(b\alpha(gx_{n-1}, gx_n)p_b(gx_n, gx_{n+1})) = \theta(r) > 1.$$

Let $t_n = \theta(b\alpha(gx_{n-1}, gx_n)p_b(gx_n, gx_{n+1}))$ and $s_n = \theta(p_b(gx_{n-1}, gx_n))$ for all $n \in \mathbb{N}$. Then, $t_n < s_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1$. From ($\xi 3$), we obtain

$$1 \leq \limsup_{n\to\infty} \xi(t_n, s_n) < 1,$$

which is a contradiction. This implies that

$$\lim_{n\to\infty} p_b(gx_{n-1}, gx_n) = 0 \tag{3.3}$$

and so $\lim_{n\to\infty} \theta(p_b(gx_{n-1}, gx_n)) = 1$.

Next, we shall show that $(gx_n)$ is a bounded sequence in $(X, p_b)$. We argue by contradiction. If possible, suppose that the sequence $(gx_n)$ is not bounded. Then there exists a subsequence $(gx_{n_k})$ of $(gx_n)$ such that $n_1 = 1$ and for all $k = 1, 2, \cdots$, $n_{k+1}$ is the smallest integer satisfying

$$\begin{aligned} p_b(gx_{n_{k+1}}, gx_{n_k}) &> 1 \\ \textit{and} \quad p_b(gx_l, gx_{n_k}) &\leq 1 \text{ for all } n_k \leq l \leq n_{k+1} - 1. \end{aligned}$$

We now compute that

$$\begin{aligned} 1 &< p_b(gx_{n_{k+1}}, gx_{n_k}) \\ &\leq bp_b(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + bp_b(gx_{n_{k+1}-1}, gx_{n_k}) - p_b(gx_{n_{k+1}-1}, gx_{n_{k+1}-1}) \\ &\leq bp_b(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + bp_b(gx_{n_{k+1}-1}, gx_{n_k}) \\ &\leq bp_b(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + b. \end{aligned}$$

Taking limit as $k \to \infty$ and using condition (3.3), we have

$$1 \leq \liminf_{k\to\infty} p_b(gx_{n_{k+1}}, gx_{n_k}) \leq \limsup_{k\to\infty} p_b(gx_{n_{k+1}}, gx_{n_k}) \leq b.$$

We note that $(gx_{n_{k+1}-1}, gx_{n_k-1}) \in E(\tilde{G})$, $\alpha(gx_{n_{k+1}-1}, gx_{n_k-1}) \geq 1$ and

$$p_b(gx_{n_{k+1}}, gx_{n_k}) > 1 \Rightarrow p_b(fx_{n_{k+1}-1}, fx_{n_k-1}) > 0.$$

Using conditions (3.1) and ($\xi 2$), we obtain

$$\begin{aligned} 1 &\leq \xi\big(\theta(b\alpha(gx_{n_{k+1}-1}, gx_{n_k-1})p_b(gx_{n_{k+1}}, gx_{n_k})), \theta(p_b(gx_{n_{k+1}-1}, gx_{n_k-1}))\big) \\ &< \frac{\theta(p_b(gx_{n_{k+1}-1}, gx_{n_k-1}))}{\theta(b\alpha(gx_{n_{k+1}-1}, gx_{n_k-1})p_b(gx_{n_{k+1}}, gx_{n_k}))}. \end{aligned}$$

That is,

$$\theta(b\alpha(gx_{n_{k+1}-1}, gx_{n_k-1})p_b(gx_{n_{k+1}}, gx_{n_k})) < \theta(p_b(gx_{n_{k+1}-1}, gx_{n_k-1})).$$

Therefore, by using ($\theta_1$), it follows from above that

$$\begin{aligned} b &< b\alpha(gx_{n_{k+1}-1}, gx_{n_k-1})p_b(gx_{n_{k+1}}, gx_{n_k}) \\ &< p_b(gx_{n_{k+1}-1}, gx_{n_k-1}) \\ &\leq bp_b(gx_{n_{k+1}-1}, gx_{n_k}) + bp_b(gx_{n_k}, gx_{n_k-1}) \\ &\leq b + bp_b(gx_{n_k}, gx_{n_k-1}). \end{aligned}$$

Taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} p_b(gx_{n_{k+1}-1}, gx_{n_k-1}) = b \tag{3.4}$$

and

$$\lim_{k\to\infty} \alpha(gx_{n_{k+1}-1}, gx_{n_k-1})p_b(gx_{n_{k+1}}, gx_{n_k}) = 1. \tag{3.5}$$

Let $U_k = \theta(b\alpha(gx_{n_{k+1}-1}, gx_{n_k-1})p_b(gx_{n_{k+1}}, gx_{n_k}))$ and $V_k = \theta(p_b(gx_{n_{k+1}-1}, gx_{n_k-1}))$. Then $U_k < V_k$ for all $k \in \mathbb{N}$. As $\theta$ is continuous, by using conditions (3.4) and (3.5), we have $\lim\limits_{k\to\infty} U_k = \lim\limits_{k\to\infty} V_k = \theta(b) > 1$. By using condition ($\xi3$), we have

$$1 \leq \limsup_{k\to\infty} \xi\big(U_k, V_k\big) < 1,$$

which is a contradiction. Thus $(gx_n)$ is a bounded sequence in $(X, p_b)$. Now, we shall show that $(gx_n)$ is a 0-Cauchy sequence. Let

$$R_n = \sup\{p_b(gx_i, gx_j) > 0 : i,\ j \geq n\}, n \in \mathbb{N}.$$

Since the sequence $(gx_n)$ is bounded, $R_n < +\infty$ for every $n \in \mathbb{N}$. But $(R_n)$ being a decreasing sequence of positive real numbers, there exists $R \geq 0$ such that

$$\lim_{n\to\infty} R_n = R. \tag{3.6}$$

We assume that $R > 0$. Then by the definition of $R_n$, it follows that for every natural number $k$, there exist $n_k,\ m_k \in \mathbb{N}$ such that $m_k,\ n_k \geq k$, $p_b(gx_{m_k}, gx_{n_k}) > 0$ and

$$R_k - \frac{1}{k} < p_b(gx_{m_k}, gx_{n_k}) \leq R_k.$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} p_b(gx_{m_k}, gx_{n_k}) = R > 0. \tag{3.7}$$

We note that for every $k \in \mathbb{N}$,

$$p_b(gx_{m_k}, gx_{n_k}) > 0 \Rightarrow p_b(fx_{m_k-1}, fx_{n_k-1}) > 0$$

and $\alpha(gx_{m_k-1}, gx_{n_k-1}) \geq 1,\ (gx_{m_k-1}, gx_{n_k-1}) \in E(\tilde{G})$. Using conditions (3.1) and ($\xi2$), we get

$$\begin{aligned} 1 &\leq \xi\big(\theta(b\alpha(gx_{m_k-1}, gx_{n_k-1})p_b(gx_{m_k}, gx_{n_k})), \theta(p_b(gx_{m_k-1}, gx_{n_k-1}))\big) \\ &< \frac{\theta(p_b(gx_{m_k-1}, gx_{n_k-1}))}{\theta(b\alpha(gx_{m_k-1}, gx_{n_k-1})p_b(gx_{m_k}, gx_{n_k}))}. \end{aligned}$$

That is,

$$\theta(b\alpha(gx_{m_k-1}, gx_{n_k-1})p_b(gx_{m_k}, gx_{n_k})) < \theta(p_b(gx_{m_k-1}, gx_{n_k-1})).$$

By using ($\theta_1$) and the definition of $R_n$, it follows that

$$\begin{aligned} bp_b(gx_{m_k}, gx_{n_k}) &\leq b\alpha(gx_{m_k-1}, gx_{n_k-1})p_b(gx_{m_k}, gx_{n_k}) \\ &< p_b(gx_{m_k-1}, gx_{n_k-1}) \\ &\leq R_{k-1}. \end{aligned} \tag{3.8}$$

Taking limit as $k \to \infty$ and using conditions (3.6) and (3.7), we obtain that

$$bR \leq \liminf_{k\to\infty} p_b(gx_{m_k-1}, gx_{n_k-1}) \leq \limsup_{k\to\infty} p_b(gx_{m_k-1}, gx_{n_k-1}) \leq R. \tag{3.9}$$

If $b > 1$, then it follows from condition (3.9) that $R = 0$.

If $b = 1$, then $\lim\limits_{k\to\infty} p_b(gx_{m_k-1}, gx_{n_k-1}) = R > 0$. Also, condition (3.8) ensures that

$$\lim_{k\to\infty} \alpha(gx_{m_k-1}, gx_{n_k-1})p_b(gx_{m_k}, gx_{n_k}) = R.$$

Let $\mathcal{U}_k = \theta(\alpha(gx_{m_k-1}, gx_{n_k-1})p_b(gx_{m_k}, gx_{n_k}))$ and $\mathcal{V}_k = \theta(p_b(gx_{m_k-1}, gx_{n_k-1}))$. Then $\mathcal{U}_k < \mathcal{V}_k$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} \mathcal{U}_k = \lim_{k\to\infty} \mathcal{V}_k = \theta(R) > 1$. It now follows from condition $(\xi_3)$ that

$$1 \leq \limsup_{k\to\infty} \xi\big(\mathcal{U}_k, \mathcal{V}_k\big) < 1,$$

which is a contradiction and so we have $R = 0$. Hence, we deduce that

$$\lim_{n,m\to\infty} p_b(gx_n, gx_m) = 0.$$

Therefore, $(gx_n)$ is a 0-Cauchy sequence in $g(X)$. As $g(X)$ is 0-complete, there exists an $u = g\nu \in g(X)$ for some $\nu \in X$ such that $gx_n \to u$ and $p_b(u, u) = 0$. Therefore,

$$\lim_{n,m\to\infty} p_b(gx_n, gx_m) = \lim_{n\to\infty} p_b(gx_n, u) = p_b(u, u) = 0.$$

By Property $(*)$ there exists a subsequence $(gx_{n_i})$ of $(gx_n)$ such that $(gx_{n_i}, g\nu) \in E(\tilde{G})$ and $\alpha(gx_{n_i}, g\nu) \geq 1$, for all $i \geq 1$. Next, we shall show that $f$ and $g$ have a point of coincidence in $g(X)$. Let us consider the collection $P = \{p_b(fx_{n_i}, f\nu) > 0 : i \in \mathbb{N}\}$. For $p_b(fx_{n_i}, f\nu) \in P$, we obtain from condition (3.1) that

$$\begin{aligned} 1 &\leq \xi\big(\theta(b\alpha(gx_{n_i}, g\nu)p_b(fx_{n_i}, f\nu)), \theta(p_b(gx_{n_i}, g\nu))\big) \\ &< \frac{\theta(p_b(gx_{n_i}, g\nu))}{\theta(b\alpha(gx_{n_i}, g\nu)p_b(fx_{n_i}, f\nu))}. \end{aligned}$$

That is,

$$\theta(b\alpha(gx_{n_i}, g\nu)p_b(fx_{n_i}, f\nu)) < \theta(p_b(gx_{n_i}, g\nu)).$$

Therefore, by using $(\theta_1)$, it follows that

$$bp_b(fx_{n_i}, f\nu) \leq b\alpha(gx_{n_i}, g\nu)p_b(fx_{n_i}, f\nu) < p_b(gx_{n_i}, g\nu).$$

If $p_b(fx_{n_i}, f\nu) \notin P$, then

$$0 = p_b(fx_{n_i}, f\nu) \leq p_b(gx_{n_i}, g\nu).$$

Thus,

$$bp_b(fx_{n_i}, f\nu) \leq p_b(gx_{n_i}, g\nu) \text{ for all } i \in \mathbb{N}.$$

Now,

$$\begin{aligned} 0 &\leq p_b(f\nu, g\nu) \\ &\leq bp_b(f\nu, fx_{n_i}) + bp_b(fx_{n_i}, g\nu) - p_b(fx_{n_i}, fx_{n_i}) \\ &\leq p_b(gx_{n_i}, g\nu) + bp_b(fx_{n_i}, g\nu) \\ &= p_b(gx_{n_i}, g\nu) + bp_b(gx_{n_i+1}, g\nu). \end{aligned}$$

Taking limit as $i \to \infty$, we obtain that

$$p_b(f\nu, g\nu) = 0.$$

Hence, we get $f\nu = g\nu = u$. Therefore, $u$ is a point of coincidence of $f$ and $g$.

For uniqueness, we assume that there exists $u^* \in X$ such that $fx = gx = u^*$ for some $x \in X$ with $p_b(u^*, u^*) = 0$ and $u \neq u^*$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$ and $\alpha(u, u^*) \geq 1$. Then,

$$\begin{aligned} 1 &\leq \xi\big(\theta(b\alpha(g\nu, gx)p_b(f\nu, fx)), \theta(p_b(g\nu, gx))\big) \\ &= \xi\big(\theta(b\alpha(u, u^*)p_b(u, u^*)), \theta(p_b(u, u^*))\big) \\ &< \frac{\theta(p_b(u, u^*))}{\theta(b\alpha(u, u^*)p_b(u, u^*))}. \end{aligned}$$

That is,

$$\theta(b\alpha(u,u^*)p_b(u,u^*)) < \theta(p_b(u,u^*)).$$

This gives that $bp_b(u,u^*) \leq b\alpha(u,u^*)p_b(u,u^*) < p_b(u,u^*)$, a contradiction. Hence, $u = u^*$. Therefore, $f$ and $g$ have a unique point of coincidence in $g(X)$.

If $f$ and $g$ are weakly compatible, then by Proposition 2.16, $f$ and $g$ have a unique common fixed point in $g(X)$. □

We give some examples to illustrate our main result.

**Example 3.7.** Let $X = [0,\infty)$ and define $p_b : X \times X \to \mathbb{R}^+$ by $p_b(x,y) = [\max\{x, y\}]^2 + \mid x - y \mid^2$ for all $x, y \in X$. Then $(X, p_b)$ is a 0-complete partial $b$-metric space with the coefficient $b = 4$. Let $G$ be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{n}) : n = 1, 2, 3, \cdots\}$. Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{5}, \text{ if } x \neq \frac{2}{5}, \\ 1, \text{ if } x = \frac{2}{5} \end{cases}$$

and $gx = 5x$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$ and $p_b(fx, fy) > 0 \Rightarrow p_b(gx, gy) > 0$. Let $\alpha : X \times X \to [0,\infty)$ be defined by

$$\alpha(x,y) = \begin{cases} 1, \text{ if } x,\, y \in [0,1], \\ \frac{1}{2}, \text{ otherwise.} \end{cases}$$

We note that there exists $x_0(= 0) \in X$ such that $x_0 \in C_{fg}^{\alpha G}$. Let $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ be defined by $\xi(t,s) = \frac{s^{\frac{1}{2}}}{t}$, for all $t,\, s \geq 1$ and $\theta : (0,\infty) \to (1,\infty)$ be defined by $\theta(t) = e^t$ for all $t > 0$.

For $x = 0$, $y = \frac{1}{5n}$, $n \in \mathbb{N}$, we have $gx = 0$, $gy = \frac{1}{n}$, $fx = 0$, $fy = \frac{1}{25n}$ and so $(gx, gy) \in E(\tilde{G})$, $\alpha(gx, gy) = 1$. We now compute that

$$p_b(fx, fy) = p_b(0, \frac{1}{25n}) = \frac{2}{625n^2} > 0, \quad p_b(gx, gy) = p_b(0, \frac{1}{n}) = \frac{2}{n^2}.$$

So $(\theta(p_b(gx, gy)))^{\frac{1}{2}} = e^{\frac{1}{n^2}}$, $\theta(b\alpha(gx, gy)p_b(fx, fy)) = e^{\frac{8}{625n^2}}$. Since $\frac{1}{n^2} > \frac{8}{625n^2} \Rightarrow e^{\frac{1}{n^2}} > e^{\frac{8}{625n^2}}$, we have

$$\begin{aligned} \xi(\theta(b\alpha(gx,gy)p_b(fx,fy)), \theta(p_b(gx,gy))) &= \frac{(\theta(p_b(gx,gy)))^{\frac{1}{2}}}{\theta(b\alpha(gx,gy)p_b(fx,fy))} \\ &= \frac{e^{\frac{1}{n^2}}}{e^{\frac{8}{625n^2}}} \\ &> 1. \end{aligned}$$

Moreover, for $0 < x = y \leq \frac{1}{5}$, we have $(gx, gy) \in E(\tilde{G})$, $\alpha(gx, gy) = 1$ and $p_b(fx, fy) = p_b(\frac{x}{5}, \frac{x}{5}) = \frac{x^2}{25} > 0$, $p_b(gx, gy) = p_b(5x, 5x) = 25x^2$. So $(\theta(p_b(gx, gy)))^{\frac{1}{2}} = e^{\frac{25x^2}{2}}$, $\theta(b\alpha(gx, gy)p_b(fx, fy)) = e^{\frac{4x^2}{25}}$. From $\frac{25x^2}{2} > \frac{4x^2}{25}$, we have $e^{\frac{25x^2}{2}} > e^{\frac{4x^2}{25}}$, and so,

$$\begin{aligned} \xi(\theta(b\alpha(gx,gy)p_b(fx,fy)), \theta(p_b(gx,gy))) &= \frac{(\theta(p_b(gx,gy)))^{\frac{1}{2}}}{\theta(b\alpha(gx,gy)p_b(fx,fy))} \\ &= \frac{e^{\frac{25x^2}{2}}}{e^{\frac{4x^2}{25}}} \\ &> 1. \end{aligned}$$

Therefore,

$$\xi(\theta(b\alpha(gx,gy)p_b(fx,fy)), \theta(p_b(gx,gy))) > 1$$

for all $x,\, y \in X$ with $(gx, gy) \in E(\tilde{G})$, $\alpha(gx, gy) \geq 1$ and $p_b(fx, fy) > 0$.

Any sequence $(gx_n)$ with the property $p_b(gx_n, x) \to 0$, $(gx_n, gx_{n+1}) \in E(\tilde{G})$ and $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \geq 1$ must be either the zero sequence or a sequence of the following form

$$gx_n = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \frac{1}{n}, & \text{if } n \text{ is even} \end{cases}$$

where the words 'odd' and 'even' are interchangeable. Also, $p_b(gx_n, x) \to 0$ ensures that $x = 0$ and consequently, it follows that Property $(*)$ holds true. Moreover, $f$ and $g$ are weakly compatible. Thus, we have all the conditions of Theorem 3.6 and 0 is the unique common fixed point of $f$ and $g$ in $g(X)$ with $p_b(0, 0) = 0$.

**Remark 3.8.** It is worth mentioning that in the example given above, $f$ is not a Banach $G$-contraction. If we take $x = y = \frac{2}{5}$, then

$$p_b(fx, fy) = p_b(1, 1) = 1 = \frac{25}{4}.\frac{4}{25} > k\, p_b(x, y)$$

for any $k \in (0, \frac{1}{b})$.

To examine the necessity of the weak compatibility condition in Theorem 3.6, let us consider the following example.

**Example 3.9.** Let $X = \mathbb{R}$ and define $p_b : X \times X \to \mathbb{R}^+$ by $p_b(x, y) =\mid x - y \mid^3$ for all $x,\, y \in X$. Then $(X, p_b)$ is a 0-complete partial $b$-metric space with the coefficient $b = 4$. Let $G$ be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x, y) : (x, y) \in [0, 1] \times [0, 1]\}$. Let $f,\, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{3}, \text{ if } x \neq \frac{2}{3}, \\ 1, \text{ if } x = \frac{2}{3} \end{cases}$$

and $gx = 5x - 14$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$ and $p_b(fx, fy) > 0$, these imply that $p_b(gx, gy) > 0$. Let $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, \text{ if } x,\, y \in [0, 1], \\ \frac{1}{2}, \text{ otherwise.} \end{cases}$$

Then it is easy to verify that there exists $x_0 (= 3) \in C^{\alpha G}_{fg}$. Let $\xi : [1, \infty) \times [1, \infty) \to \mathbb{R}$ be defined by $\xi(t, s) = \frac{s^{\frac{1}{3}}}{t}$ for all $t,\, s \geq 1$ and $\theta : (0, \infty) \to (1, \infty)$ be defined by $\theta(t) = e^t$, for all $t > 0$.

If $\frac{14}{5} \leq x,\, y \leq 3,\, x \neq y$, then $(gx, gy) \in E(\tilde{G})$, $\alpha(gx, gy) = 1$ and $p_b(fx, fy) > 0$. We now compute that $bp_b(fx, fy) = 4p_b(\frac{x}{3}, \frac{y}{3}) = \frac{4}{27} \mid x - y \mid^3 = \frac{4}{3375} p_b(gx, gy)$. From $\frac{1}{3} p_b(gx, gy) > \frac{4}{3375} p_b(gx, gy)$, we have $e^{\frac{1}{3} p_b(gx,gy)} > e^{\frac{4}{3375} p_b(gx,gy)}$, and then

$$\begin{aligned} \xi(\theta(b\alpha(gx, gy)p_b(fx, fy)), \theta(p_b(gx, gy))) &= \frac{(\theta(p_b(gx, gy)))^{\frac{1}{3}}}{\theta(b\alpha(gx, gy)p_b(fx, fy))} \\ &= \frac{e^{\frac{1}{3} p_b(gx,gy)}}{e^{\frac{4}{3375} p_b(gx,gy)}} \\ &> 1. \end{aligned}$$

Therefore,

$$\xi(\theta(b\alpha(gx, gy)p_b(fx, fy)), \theta(p_b(gx, gy))) > 1$$

for all $x,\, y \in X$ with $(gx, gy) \in E(\tilde{G})$, $\alpha(gx, gy) \geq 1$ and $p_b(fx, fy) > 0$. Moreover, any sequence $(gx_n)$ with the property that $(gx_n, gx_{n+1}) \in E(\tilde{G})$ and $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \geq 1$ must be a sequence in $[0, 1]$. Also, $p_b(gx_n, x) \to 0 \Rightarrow \mid gx_n - x \mid \to 0 \Rightarrow x \in [0, 1]$ which ensures that Property $(*)$ holds true. Furthermore, $f(3) = g(3) = 1$ but $g(f(3)) \neq f(g(3))$, i.e., $f$ and $g$ are not weakly compatible. However, all other conditions of Theorem 3.6 are fulfilled. We observe that 1 is the unique point of coincidence of $f$ and $g$ without being any common fixed point.

The following example shows that Theorem 3.6 remains invalid without the Property $(*)$.

**Example 3.10.** Let $X = [0, \infty)$ and define $p_b : X \times X \to \mathbb{R}^+$ by $p_b(x, y) =\mid x - y \mid^2$ for all $x, y \in X$. Then $(X, p_b)$ is a 0-complete partial $b$-metric space with the coefficient $b = 2$. Let $G$ be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x, y) : (x, y) \in (0, 1] \times (0, 1]\}$. Let $f,\, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{8}, \text{ if } x \neq 0, \\ 1, \text{ if } x = 0 \end{cases}$$

and $gx = \frac{x}{2}$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$ and $p_b(fx, fy) > 0 \Rightarrow p_b(gx, gy) > 0$. Let $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x,\, y \in [0, 1], \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Then there exists $x_0 (= \frac{1}{2}) \in X$ such that $x_0 \in C^{\alpha G}_{fg}$. Let $\xi : [1, \infty) \times [1, \infty) \to \mathbb{R}$ be defined by $\xi(t, s) = \frac{s^{\frac{3}{4}}}{t}$ for all $t, s \geq 1$ and $\theta : (0, \infty) \to (1, \infty)$ be defined by $\theta(t) = e^t$ for all $t > 0$. For $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, $\alpha(gx, gy) \geq 1$ and $p_b(fx, fy) > 0$, we have $x \neq y$, $x, y \in (0, 2]$ and $bp_b(fx, fy) = \frac{1}{8} p_b(gx, gy)$. Then $\frac{3}{4} p_b(gx, gy) > \frac{1}{8} p_b(gx, gy)$ implies $e^{\frac{3}{4} p_b(gx,gy)} > e^{\frac{1}{8} p_b(gx,gy)}$, and we obtain

$$\begin{aligned} \xi(\theta(b\alpha(gx, gy)p_b(fx, fy)), \theta(p_b(gx, gy))) &= \frac{(\theta(p_b(gx, gy)))^{\frac{3}{4}}}{\theta(b\alpha(gx, gy)p_b(fx, fy))} \\ &= \frac{e^{\frac{3}{4} p_b(gx,gy)}}{e^{\frac{1}{8} p_b(gx,gy)}} \\ &> 1. \end{aligned}$$

We now show that the Property $(*)$ does not hold true. For $x_n = \frac{2}{n}$, $gx_n = \frac{1}{n}$ and hence $p_b(gx_n, 0) \to 0$. Also, $(gx_n, gx_{n+1}) \in E(\tilde{G})$ and $\alpha(gx_n, gx_{n+1}) = 1$ for all $n \in \mathbb{N}$. But there exists no subsequence $(gx_{n_i})$ of $(gx_n)$ such that $(gx_{n_i}, 0) \in E(\tilde{G})$. We observe that $f$ and $g$ have no point of coincidence in $g(X)$. This proves that Theorem 3.6 remains invalid without the Property $(*)$.

**Remark 3.11.** In Example 3.10, $f$ is not a Banach contraction. In fact, for $x = 0$, $y = 1$, we have

$$p_b(fx, fy) = p_b\left(1, \frac{1}{8}\right) = \left(\frac{7}{8}\right)^2 = \left(\frac{7}{8}\right)^2 p_b(x, y) > k\, p_b(x, y)$$

for any $k \in (0, \frac{1}{b})$.

But $f$ is a Banach $G$-contraction since for all $x, y \in X$ with $(x, y) \in E(G)$, we have

$$p_b(fx, fy) = \frac{1}{64}\, p_b(x, y),$$

where $\frac{1}{64} \in (0, \frac{1}{b})$.

## 4 An Application

In this section we apply Theorem 3.6 to study the existence and uniqueness of solution of an integral equation. The main aim of this section is to present an existence and uniqueness theorem for unique solution of the following integral equation

$$x(t) = \int_0^{\xi} K(t, r, x(r))dr, \tag{4.1}$$

where $\xi > 0$ and $K : [0, \xi] \times [0, \xi] \times \mathbb{R} \to \mathbb{R}$, $x : [0, \xi] \to \mathbb{R}$ are functions.

Let $X = C[0, \xi]$ be the set of all real valued continuous functions defined on $[0, \xi]$. We define $p_b : X \times X \to \mathbb{R}^+$ by

$$p_b(x, y) = \sup_{0 \leq t \leq \xi} \mid x(t) - y(t) \mid^p \text{ for all } x, y \in X,$$

where $p > 1$. Then it is easy to verify that $(X, p_b)$ is a 0-complete partial $b$-metric space with the coefficient $b = 2^{p-1}$. In the following theorem $X$ represents this partial $b$-metric space.

**Theorem 4.1.** Suppose that $X = C[0, \xi]$ and the following hypotheses hold:

(*i*) $K : [0, \xi] \times [0, \xi] \times \mathbb{R} \to \mathbb{R}$ is continuous;

$(ii)$ for all $t,\, r \in [0,\xi]$, there exists a continuous function $\eta : [0,\xi]\times[0,\xi] \to \mathbb{R}$ such that

$$| K(t,r,x(r)) - K(t,r,y(r)) | \le \beta^{\frac{1}{p}}\eta(t,r) \mid x(r)-y(r) \mid \text{ for all } x,\, y \in X \tag{4.2}$$

and

$$\sup_{0\le t\le \xi}\int_0^{\xi}\eta(t,r)dr \le 1, \tag{4.3}$$

where $0<\beta<\frac{1}{b}$.

Then the integral equation (4.1) has a unique solution in $X$.

**Proof .** Let $f : X \to X$ be defined by $(fx)(t) = \int_0^{\xi} K(t,r,x(r))dr$ for all $x \in X$ and for all $t \in [0,\xi]$. Then the existence of a solution to the integral equation (4.1) is equivalent to the existence of a fixed point of $f$.

Utilizing conditions (4.2) and (4.3), for all $x,\, y \in X$ and $t \in [0,\xi]$, we compute that

$$\begin{aligned}
| (fx)(t) - (fy)(t) |^p &= \left|\int_0^{\xi}(K(t,r,x(r)) - K(t,r,y(r)))dr\right|^p \\
&\le \left(\int_0^{\xi} | K(t,r,x(r)) - K(t,r,y(r)) | \, dr\right)^p \\
&\le \left(\int_0^{\xi} \beta^{\frac{1}{p}}\eta(t,r) \mid x(r)-y(r)\mid dr\right)^p \\
&= \beta\left(\int_0^{\xi}\eta(t,r)(| x(r)-y(r) |^p)^{\frac{1}{p}}dr\right)^p \\
&\le \beta\left(\int_0^{\xi}\eta(t,r)(p_b(x,y))^{\frac{1}{p}}dr\right)^p \\
&= \beta p_b(x,y)\left(\int_0^{\xi}\eta(t,r)dr\right)^p \\
&\le \beta p_b(x,y).
\end{aligned}$$

Therefore,

$$p_b(fx,fy) = \sup_{0\le t\le\xi} | (fx)(t) - (fy)(t) |^p \le \beta p_b(x,y), \text{ for all } x,\, y \in X, \tag{4.4}$$

where $0<\beta<\frac{1}{b}$. We note that $p_b(fx,fy)>0$ implies that $p_b(x,y)>0$. Let $\xi : [1,\infty)\times[1,\infty) \to \mathbb{R}$ be defined by $\xi(t,s) = \frac{s^k}{t}$ for all $t,\, s \ge 1$, where $k = b\beta \in (0,1)$ and $\theta : (0,\infty) \to (1,\infty)$ be defined by $\theta(t) = e^t$, for all $t>0$. Let us take $g = I$, the identity map on $X$, $G = G_0$, where $G_0$ is the complete graph $(X, X\times X)$ and $\alpha(x,y) = 1$ for all $x,\, y \in X$. It now follows from condition (4.4) that for $p_b(fx,fy)>0$, we have

$$e^{b\alpha(gx,gy)p_b(fx,fy)} \le e^{b\beta p_b(gx,gy)} = \left(e^{p_b(gx,gy)}\right)^{b\beta} = \left(e^{p_b(gx,gy)}\right)^{k},$$

where $k = b\beta \in (0,1)$. This implies that

$$\theta(b\alpha(gx,gy)p_b(fx,fy)) \le (\theta(p_b(gx,gy)))^k .$$

This proves that

$$\xi(\theta(b\alpha(gx,gy)p_b(fx,fy)), \theta(p_b(gx,gy))) \ge 1$$

for all $x,\, y \in X$ with $p_b(fx,fy)>0$. Thus all the hypotheses of Theorem 3.6 holds good and hence $f$ has a unique fixed point $x$ (say) in $X$. This means that $x$ is the unique solution for the integral equation (4.1). □

## 5 Some Consequences of the Main Result

In this section we exhibit some important fixed point results which will justify the extension of our main result.

**Theorem 5.1.** Let $(X, p_b)$ be a 0-complete partial $b$-metric space with the coefficient $b \geq 1$ and let $G = (V(G), E(G))$ be a digraph. Let $f : X \to X$ be a mapping with the property that $p_b(fx, fy) > 0$ implies that $p_b(x, y) > 0$. Suppose that $f$ is $(\alpha, \theta, \xi)$-$G$-contractive and the graph $G$ has the Property $(\acute{*})$. Then $f$ has a fixed point $u$(say) in $X$ with $p_b(u, u) = 0$ if $C_f^{\alpha G} \neq \emptyset$.

Moreover, $f$ has a unique fixed point in $X$ if the graph $G$ has the following property:

$(\acute{**})$ If $x$, $y$ are fixed points of $f$ in $X$, then $(x, y) \in E(\tilde{G})$ and $\alpha(x, y) \geq 1$.

**Proof .** The proof follows from Theorem 3.6 by taking $g = I$, the identity map on $X$. □

**Theorem 5.2.** Let $(X, p_b)$ be a partial $b$-metric space with the coefficient $b \geq 1$ and let $f$, $g : X \to X$ be mappings with the property that $p_b(fx, fy) > 0$ implies that $p_b(gx, gy) > 0$. Suppose that $f$ is $(\alpha, \theta, \xi)$-contractive w.r.t. the mapping $g$. Suppose also that $f(X) \subseteq g(X)$, $g(X)$ is a 0-complete subspace of $X$ and $\alpha$ has the Property (†). Then $f$ and $g$ have a point of coincidence $u$(say) in $g(X)$ with $p_b(u, u) = 0$ if $C_{fg}^{\alpha} \neq \emptyset$. Moreover, $f$ and $g$ have a unique point of coincidence in $g(X)$ if $\alpha$ has the following property:

If $x$, $y$ are points of coincidence of $f$ and $g$ in $g(X)$, then $\alpha(x, y) \geq 1$.

Furthermore, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point in $g(X)$.

**Proof .** The proof can be obtained from Theorem 3.6 by considering $G = G_0$, where $G_0$ is the complete graph $(X, X \times X)$. □

**Theorem 5.3.** Let $(X, p_b)$ be a 0-complete partial $b$-metric space and let $f : X \to X$ be a mapping with the property that $p_b(fx, fy) > 0$ implies that $p_b(x, y) > 0$. Suppose that there exist $\xi \in \mathscr{L}$ and $\theta \in \Theta$ such that

$$\xi(\theta(bp_b(fx, fy)), \theta(p_b(x, y))) \geq 1,$$

for all $x$, $y \in X$ and $p_b(fx, fy) > 0$. Then $f$ has a unique fixed point $u$(say) in $X$ with $p_b(u, u) = 0$.

**Proof .** The proof follows from Theorem 3.6 by taking $g = I,\ G = G_0$ and $\alpha(x, y) = 1$ for all $x$, $y \in X$. □

**Theorem 5.4.** Let $(X, p_b)$ be a partial $b$-metric space and let $f$, $g : X \to X$ be mappings with the property that $p_b(fx, fy) > 0$ implies that $p_b(gx, gy) > 0$. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that

$$\theta(bp_b(fx, fy)) \leq (\theta(p_b(gx, gy)))^k$$

for all $x$, $y \in X$ and $p_b(fx, fy) > 0$. If $f(X) \subseteq g(X)$ and $g(X)$ is a 0-complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence $u$(say) in $g(X)$ with $p_b(u, u) = 0$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point in $g(X)$.

**Proof .** The proof can be obtained from Theorem 3.6 by considering $G = G_0, \xi = \xi_b$ and $\alpha(x, y) = 1$, for all $x, y \in X$. □

**Theorem 5.5.** Let $(X, p_b)$ be a 0-complete partial $b$-metric space and let $f : X \to X$ be a mapping with the property that $p_b(fx, fy) > 0$ implies that $p_b(x, y) > 0$. Suppose that there exists $\theta \in \Theta$ such that

$$\theta(bp_b(fx, fy)) \leq \frac{\theta(p_b(x, y))}{\phi(\theta(p_b(x, y)))},$$

for all $x$, $y \in X$ and $p_b(fx, fy) > 0$, where $\phi : [1, \infty) \to [1, \infty)$ is a nondecreasing and lower semicontinuous function such that $\phi^{-1}(\{1\}) = 1$. Then $f$ has a unique fixed point in $X$ with $p_b(u, u) = 0$.

**Proof .** The proof follows from Theorem 3.6 by taking $G = G_0, g = I, \xi = \xi_w$ and $\alpha(x, y) = 1$ for all $x, y \in X$. □

The following is the $b$-metric version of Banach contraction theorem.

**Theorem 5.6.** Let $(X, d)$ be a complete $b$-metric space and let $f : X \to X$ be a mapping satisfying

$$d(fx, fy) \leq \beta d(x, y),$$

for all $x$, $y \in X$, where $\beta \in (0, \frac{1}{b})$ is a constant. Then $f$ has a unique fixed point in $X$.

**Proof .** The conclusion of the theorem follows from Theorem 3.6 by taking $G = G_0$, $g = I$, $\xi = \xi_b, \theta(t) = e^t$, for all $t > 0$ and $\alpha(x, y) = 1$, for all $x$, $y \in X$. □

**Remark 5.7.** Theorem 5.6 shows that our main result is a generalization of the well known Banach contraction theorem.

**Theorem 5.8.** Let $(X, d)$ be a complete $b$-metric space endowed with a partial ordering $\preceq$ and let $f : X \to X$ be a mapping. Suppose that there exist $\xi \in \mathscr{L}$ and $\theta \in \Theta$ such that

$$\xi(\theta(bd(fx, fy)), \theta(d(x, y))) \geq 1$$

for all comparable elements $x$, $y \in X$ and $d(fx, fy) > 0$. Suppose also that the triple $(X, d, \preceq)$ has the Property (‡). If there exists $x_0 \in X$ such that $x_n, x_m$ are comparable for all $n$, $m = 0, 1, 2, \cdots$, where $x_n = fx_{n-1}$, for all $n \in \mathbb{N}$, then $f$ has a fixed point in $X$. Moreover, $f$ has a unique fixed point in $X$ if the following property holds:

If $x$, $y$ are fixed points of $f$ in $X$, then $x$ and $y$ are comparable.

**Proof .** The proof can be obtained from Theorem 3.6 by taking $g = I$, $\alpha(x, y) = 1$ for all $x$, $y \in X$ and $G = G_2$, where the graph $G_2$ is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. □

**Remark 5.9.** It is worth noting that we can obtain several important fixed point results in metric, partial metric and $b$-metric spaces by suitable choices of $\xi$, $\theta$ and $\alpha$.

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