Int. J. Nonlinear Anal. Appl. 16 (2025) 11, 17–30
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.35082.5237

On a class of multivalent meromorphic functions defined by combinational differential operator

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(Communicated by Mugur Alexandru Acu)

Abstract

In this paper, we define a class of meromorphically multivalent functions in $\mathbb{U}^* = \{z : z \in \mathbb{C} : 0 < |z| < 1\}$ by using a differential operator. Important properties of this class such as coefficient estimates, distortion theorem, radius of starlikeness and convexity, closure theorems, and convolution properties are obtained. We also study δ-neighboorhoods and partial sums for this class.

Keywords: meromorphic functions, multivalent functions, differential operator, coefficient estimates, convolution, neighborhoods
2020 MSC: 30C45; 30C50

1 Introduction

Let Σ_p denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \qquad (a_{p+k} \geq 0; p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $\mathbb{U}^* = \{z \ : \ z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. A function $f \in \Sigma_p$ is meromorphically starlike of order $\rho (0 \leq \rho < p)$ if

$$-\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho.$$

A function $f \in \Sigma_p$ is meromorphically convex of order $\theta (0 \leq \theta < p)$ if

$$-\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \theta.$$

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Received: August 2024 *Accepted:* October 2024

If $f \in \Sigma_p$ is given by (1.1) and $g \in \Sigma_p$ is given by

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_{p+k} z^{p+k} \qquad (b_{p+k} \geq 0; p \in \mathbb{N}). \tag{1.2}$$

then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z) \qquad (z \in \mathbb{U}^*; p \in \mathbb{N}). \tag{1.3}$$

For functions $f(z) \in \Sigma_p$, Aouf [1] defined the following differential operator:

$$\begin{aligned}
S^0_{\lambda,p} f(z) &= f(z) \\
S^1_{\lambda,p} f(z) &= (1-\lambda) f(z) + \frac{\lambda}{p} z f'(z) + \frac{2\lambda}{z^p} \\
&= \frac{1}{z^p} + \sum_{k=0}^{\infty} (\frac{p+\lambda k}{p}) a_{p+k} z^{p+k} \\
&= S_{\lambda,p} f(z) \quad (\lambda \geq 0; p \in \mathbb{N}) \\
S^2_{\lambda,p} f(z) &= S_{\lambda,p}(S^1_{\lambda,p} f(z))
\end{aligned}$$

and

$$\begin{aligned}
S^n_{\lambda,p} f(z) &= S_{\lambda,p}(S^{n-1}_{\lambda,p} f(z)) \\
&= (1-\lambda) S^{n-1}_{\lambda,p} f(z) + \frac{\lambda}{p} z (S^{n-1}_{\lambda,p} f(z))' + \frac{2\lambda}{z^p} \qquad (\lambda \geq 0; n, p \in \mathbb{N}).
\end{aligned}$$

It can be easily seen that

$$S^n_{\lambda,p} f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} (\frac{p+\lambda k}{p})^n a_{p+k} z^{p+k}, \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N}). \tag{1.4}$$

Also Orhan et al. [4] defined the differential operator $T^n_{\sigma\mu p}$ in the following way:

$$\begin{aligned}
T^0_{\sigma\mu p} f(z) &= f(z) \\
T^1_{\sigma\mu p} f(z) &= T_{\sigma\mu p} f(z) = \sigma\mu \frac{[z^{p+1} f(z)]''}{z^{p-1}} + (\sigma - \mu) \frac{[z^{p+1} f(z)]'}{z^p} + (1 - \sigma + \mu) f(z)
\end{aligned} \tag{1.5}$$

and, in general,

$$T^n_{\sigma\mu p} f(z) = T_{\sigma\mu p}(T^{n-1}_{\sigma\mu p} f(z)), \tag{1.6}$$

where $0 \leq \mu \leq \sigma$ and $n \in \mathbb{N}_0$. Recently, Sharma [6] used the generalized modified Srivastava-Gupta operators defined in [7] by using iterative combinations in ordinary and simultaneous approximation and obtained several interesting results. If the function$f(z) \in \Sigma_p$ is given by (1.1) then from (1.5) and (1.6), we obtain

$$T^n_{\sigma\mu p} f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \Psi_k(\sigma, \mu, n, p) a_{k+p} z^{k+p} \tag{1.7}$$

where

$$\Psi_k(\sigma, \mu, n, p) = [1 + (k+2p)(\sigma - \mu + (k+2p+1)\sigma\mu)]^n. \tag{1.8}$$

Making use of the differential operators $S^n_{\lambda,p} f(z)$ and $T^n_{\sigma\mu p} f(z)$ defined as in (1.4) and (1.7), respectively, we define the following differential operator for the functions $f(z) \in \Sigma_p$ as follows:

$$D^n_{\lambda,\sigma,\mu,\omega,p} f(z) = (1-\omega) S^n_{\lambda,p} f(z) + \omega T^n_{\sigma\mu p} f(z), \tag{1.9}$$

for $n \in \mathbb{N}, \lambda \geq 0, 0 \leq \mu \leq \sigma, 0 \leq \omega \leq 1$. Let $f(z)$ be given by (1.1), then using (1.4) and (1.7), the relation (1.9) can be easily written as

$$D^n_{\lambda,\sigma,\mu,\omega,p} f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} z^{p+k} \tag{1.10}$$

where

$$\Phi_k(n,\lambda,\sigma,\mu,\omega,p) = (1-\omega)(\frac{p+\lambda k}{p}) + \omega \Psi_k(\sigma,\mu,n,p) \tag{1.11}$$

and

$$\Psi_k(\sigma,\mu,n,p) = [1 + (k+2p)(\sigma - \mu + (k+2p+1)\sigma\mu)]^n, \tag{1.12}$$

for $n \in \mathbb{N}, \lambda \geq 0, 0 \leq \mu \leq \sigma, 0 \leq \omega \leq 1$. Therefore in terms of convolution (1.9) is equivalent to

$$D^n_{\lambda,\sigma,\mu,\omega,p} f(z) = (f * h)(z),$$

where

$$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \Phi_k(n,\lambda,\sigma,\mu,\omega,p) z^{p+k}.$$

Using the differential operator $D^n_{\lambda,\sigma,\mu,\omega,p} f(z)$, we define the following subclass of multivalent meromorphic functions

Definition 1.1. A function $f(z) \in \Sigma_p$ is said to be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$ if it satisfies the followig inequality:

$$\left| \frac{z^{p+2}\big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)'' + z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)' - p^2}{\gamma z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)' + \alpha(1+\gamma)p - p} \right| < \beta \tag{1.13}$$

where $0 \leq \alpha < 1, 0 < \beta \leq 1, 0 \leq \gamma \leq 1$.

Meromorphically multivalent functions have been extensively studied, for example, see Najafzadeh and Ebadian [3], Atshan and Kulkarni [2], Orhan et al. [4] and Auof [1].

2 Coefficients Estimates

Theorem 2.1. A function f defined by (1.1) is in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$ if and only if

$$\sum_{k=0}^{\infty} (p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} \leq \beta p(1-\alpha)(1+\gamma) \tag{2.1}$$

where $\Phi_k(n,\lambda,\sigma,\mu,\omega,p)$ is given by(1.11)

Proof . Assume (2.1) holds. it is enough to show that

$$M = \left| z^{p+2}\big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)'' + z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)' - p^2 \right| - \beta \left| \gamma z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)' + \alpha(1+\gamma)p - p \right| < 0.$$

For $|z| = r < 1$, by (2.1), we have

$$\begin{aligned} M &= \left| \sum_{k=0}^{\infty} (p+k)^2 \Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} z^{2p+k} \right| - \beta \left| p(1-\alpha)(1+\gamma) - \gamma \sum_{k=0}^{\infty} (p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} z^{2p+k} \right| \\ &\leq \sum_{k=0}^{\infty} (p+k)^2 \Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} r^{2p+k} - \beta p(1-\alpha)(1+\gamma) + \beta\gamma \sum_{k=0}^{\infty} (p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} r^{2p+k} \\ &< \sum_{k=0}^{\infty} (p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} - \beta p(1-\alpha)(1+\gamma) \\ &< 0. \end{aligned}$$

Hence $f \in \mathcal{H}_p(\alpha, \beta, \gamma)$. Conversely, suppose that $f(z) \in \mathcal{H}_p(\alpha, \beta, \gamma)$, then (1.13) holds true. So, we have

$$\left| \frac{z^{p+2}\big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)'' + z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)' - p^2}{\gamma z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)' + \alpha(1+\gamma)p - p} \right| = \left| \frac{\sum_{k=0}^{\infty} (p+k)^2 \Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} z^{2p+k}}{p(1-\alpha)(1+\gamma) - \gamma \sum_{k=0}^{\infty} (p+k) \Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} z^{2p+k}} \right| < \beta.$$

Since $\mathfrak{Re} z \leq |z|$, for all z, we have

$$\mathfrak{Re} \left\{ \frac{\sum_{k=0}^{\infty} (p+k)^2 \Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} z^{2p+k}}{p(1-\alpha)(1+\gamma) - \gamma \sum_{k=0}^{\infty} (p+k) \Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} z^{2p+k}} \right\} < \beta.$$

Now by letting $z \to 1^-$ through real axis, we obtain

$$\sum_{k=0}^{\infty} (p+k)[p+k+\beta\gamma] \Phi_k(n,\lambda,\sigma,\mu,\omega,p) a_{p+k} \leq \beta p(1-\alpha)(1+\gamma).$$

Hence the result follows. □

Corollary 2.2. If $f(z)$ defined by (1.1) is in the class $\mathcal{H}_p(\alpha, \beta, \gamma)$, then

$$a_{p+k} \leq \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)} z^{p+k} \tag{2.2}$$

where $\Phi_k(n,\lambda,\sigma,\mu,\omega,p)$ is given by (1.11).

3 Distortion Theorem

Theorem 3.1. If $f(z)$ defined by (1.1) is in the class $\mathcal{H}_p(\alpha, \beta, \gamma)$, then for $0 < |z| = r < 1$

$$\frac{1}{r^p} - \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^p \tag{3.1}$$

and

$$\frac{p}{r^{p+1}} - \frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^{p-1} \leq |f'(z)| \leq \frac{p}{r^{p+1}} + \frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^{p-1} \tag{3.2}$$

where

$$\Phi_0(n,\lambda,\sigma,\mu,\omega,p) = \left[\Big(1-\omega\Big) + \omega\Big[1 + 2p(\sigma - \mu + (2p+1)\sigma\mu\Big]^n \right]. \tag{3.3}$$

The bounds are attained for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} z^p. \tag{3.4}$$

Proof . In view of Theorem 2.1 we have

$$p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)\sum_{k=0}^{\infty}a_{p+k} \le \sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k} \le \beta p(1-\alpha)(1+\gamma)$$

which is equivalent to

$$\sum_{k=0}^{\infty}a_{p+k} \le \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}. \tag{3.5}$$

Thus for $0<|z|=r<1$, we get

$$\begin{aligned}|f(z)| &\le \frac{1}{r^p}+\sum_{k=0}^{\infty}a_{p+k}r^{p+k}\\ &\le \frac{1}{r^p}+r^p\sum_{k=0}^{\infty}a_{p+k}\\ &\le \frac{1}{r^p}+\frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}r^p\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}|f(z)| &\ge \frac{1}{r^p}-\sum_{k=0}^{\infty}a_{p+k}r^{p+k}\\ &\ge \frac{1}{r^p}-r^p\sum_{k=0}^{\infty}a_{p+k}\\ &\ge \frac{1}{r^p}-\frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}r^p\end{aligned} \tag{3.7}$$

which,together,yield (3.1).Furthermore, it follows from Theorem 2.1 that

$$\sum_{k=0}^{\infty}(p+k)a_{p+k} \le \frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}. \tag{3.8}$$

Hence

$$\begin{aligned}|f'(z)| &\le \frac{p}{r^{p+1}}+\sum_{k=0}^{\infty}(p+k)a_{p+k}r^{p+k-1}\\ &\le \frac{p}{r^{p+1}}+r^{p-1}\sum_{k=0}^{\infty}(p+k)a_{p+k}\\ &\le \frac{p}{r^{p+1}}+\frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}r^{p-1}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}|f'(z)| &\ge \frac{p}{r^{p+1}}-\sum_{k=0}^{\infty}(p+k)a_{p+k}r^{p+k-1}\\ &\ge \frac{p}{r^{p+1}}-r^{p-1}\sum_{k=0}^{\infty}(p+k)a_{p+k}\\ &\ge \frac{p}{r^{p+1}}-\frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}r^{p-1},\end{aligned} \tag{3.10}$$

which together yield (3.2). It can be easily seen that the function $f(z)$ defined by (3.4) is extremal for Theorem 3.1. □

4 Radius of Starlikeness and Convexity

Theorem 4.1. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$ then $f(z)$ is meromorphically p-valent starlike of order $\rho(0\le\rho<p)$ in $0<|z|<r(n,\lambda,\sigma,\mu,\omega,p,\rho,\alpha,\beta,\gamma)$, where

$$r(n,\lambda,\sigma,\mu,\omega,p,\rho,\alpha,\beta,\gamma)=\inf_k\left[\frac{(p-\rho)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)(3p+k-\rho)}\right]^{\dfrac{1}{2p+k}}, \tag{4.1}$$

$k\ge 0, p\in\mathbb{N}, n\in\mathbb{N}_0$, the result is sharp.

Proof . It is sufficient to show that

$$\left|\frac{(zf'(z))+pf(z)}{f(z)}\right|\le p-\rho,$$

for $0<|z|<r(n,\lambda,\sigma,\mu,\omega,p,\rho,\alpha,\beta,\gamma)$. Note that

$$\begin{aligned}\left|\frac{zf'(z)+pf(z)}{f(z)}\right| &= \left|\frac{\sum_{k=0}^{\infty}(2p+k)a_{p+k}z^{p+k}}{z^{-p}+\sum_{k=0}^{\infty}a_{p+k}z^{p+k}}\right| \\ &\le \frac{\sum_{k=0}^{\infty}(2p+k)a_{p+k}r^{2p+k}}{1-\sum_{k=0}^{\infty}a_{p+k}r^{2p+k}}.\end{aligned}$$

Thus, $\left|\dfrac{zf'(z)+pf(z)}{f(z)}\right|\le p-\rho$, if

$$\sum_{k=0}^{\infty}\frac{(3p+k-\rho)}{(p-\rho)}a_{p+k}r^{2p+k}\le 1. \tag{4.2}$$

Theorem 2.1 ensures that

$$\sum_{k=0}^{\infty}\frac{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{\beta p(1-\alpha)(1+\gamma)}a_{p+k}\le 1, \tag{4.3}$$

in view of (4.3), it follows that (4.2) will be true if

$$\frac{(3p+k-\rho)}{(p-\rho)}r^{2p+k}\le\frac{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{\beta p(1-\alpha)(1+\gamma)} \tag{4.4}$$

or if

$$r\le\left[\frac{(p-\rho)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)(3p+k-\rho)}\right]^{\dfrac{1}{2p+k}}. \tag{4.5}$$

Setting $r(n,\lambda,\sigma,\mu,\omega,p,\rho,\alpha,\beta,\gamma)$ in (4.5), the result follows. The result is sharp with the extremal function $f(z)$ given by (2.2). ☐

Theorem 4.2. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$. Then $f(z)$ is meromorphically p-valent convex of order $\theta(0\le\theta<p)$ in $0<|z|<r(n,\lambda,\sigma,\mu,\omega,p,\theta,\alpha,\beta,\gamma)$, where

$$r(n,\lambda,\sigma,\mu,\omega,p,\theta,\alpha,\beta,\gamma)=\inf_k\left[\frac{p(p-\theta)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)(3p+k-\theta)}\right]^{\dfrac{1}{2p+k}}, \tag{4.6}$$

$k\ge 0, p\in\mathbb{N}, n\in\mathbb{N}_0$, the result is sharp.

Proof . It suffices to show that

$$\left|\frac{(zf'(z))' + pf'(z)}{f'(z)}\right| \le p - \theta,$$

for $0 < |z| < r(n, \lambda, \sigma, \mu, \omega, p, \theta, \alpha, \beta, \gamma)$. Note that

$$\left|\frac{(zf'(z))' + pf'(z)}{f'(z)}\right| = \left|\frac{\sum_{k=0}^{\infty}(p+k)(2p+k)a_{p+k}z^{p+k-1}}{-pz^{-(p+1)} + \sum_{k=0}^{\infty}(p+k)a_{p+k}z^{p+k-1}}\right|$$
$$\le \frac{\sum_{k=0}^{\infty}(p+k)(2p+k)a_{p+k}r^{2p+k}}{p - \sum_{k=0}^{\infty}(p+k)a_{p+k}r^{2p+k}}.$$

Thus, $\left|\frac{(zf'(z))' + pf'(z)}{f'(z)}\right| \le p - \theta$, if

$$\sum_{k=0}^{\infty} \frac{(p+k)(3p+k-\theta)}{p(p-\theta)} a_{p+k} r^{2p+k} \le 1. \tag{4.7}$$

Theorem 2.1 ensures that

$$\sum_{k=0}^{\infty} \frac{(p+k)[p+k+\beta\gamma]\Phi_k(n, \lambda, \sigma, \mu, \omega, p)}{\beta p(1-\alpha)(1+\gamma)} a_{p+k} \le 1, \tag{4.8}$$

so in view of (4.8) it follows that (4.7) will be true if

$$\frac{(p+k)(3p+k-\theta)}{p(p-\theta)} r^{2p+k} \le \frac{(p+k)[p+k+\beta\gamma]\Phi_k(n, \lambda, \sigma, \mu, \omega, p)}{\beta p(1-\alpha)(1+\gamma)}, \tag{4.9}$$

or, if

$$r \le \left[\frac{p(p-\theta)\Phi_k(n, \lambda, \sigma, \mu, \omega, p)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)(3p+k-\theta)}\right]^{\dfrac{1}{2p+k}}. \tag{4.10}$$

Therefore, we obtain the $r(n, \lambda, \sigma, \mu, \omega, p, \theta, \alpha, \beta, \gamma)$ in (4.6). The result is sharp for the extremal function $f(z)$ given by (2.2). $\square$

5 Closure Theorems

Theorem 5.1. Let

$$f_{p-1} = \frac{1}{z^p} \tag{5.1}$$

and

$$f_{p+k} = \frac{1}{z^p} + \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)\Phi_k(n, \lambda, \sigma, \mu, \omega, p)(p+k+\beta\gamma)} z^{p+k}, \tag{5.2}$$

where, $k \ge 0, p \in \mathbb{N}, n \in \mathbb{N}_0$, and $z \in U^*$. Then $f(z)$ is in the class $\mathcal{H}_p(\alpha, \beta, \gamma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=-1}^{\infty} c_{p+k} f_{p+k}(z) \tag{5.3}$$

where $c_{p+k} \ge 0$ and $\sum_{k=-1}^{\infty} c_{p+k} = 1$.

Proof . Let $f(z)=\sum_{k=-1}^{\infty} c_{p+k} f_{p+k}(z)$, where $c_{p+k}\geq 0$ and $\sum_{k=-1}^{\infty} c_{p+k}=1$. Then

$$\begin{aligned} f(z) &= \sum_{k=-1}^{\infty} c_{p+k} f_{p+k}(z) \\ &= \frac{1}{z^p}+\sum_{k=0}^{\infty} c_{p+k} \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)} z^{p+k}. \end{aligned} \tag{5.4}$$

We have

$$\begin{aligned} \sum_{k=0}^{\infty} c_{p+k} \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)} \times \frac{(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)} &= \sum_{k=0}^{\infty} c_{p+k}=1-c_{p-1} \\ &\leq 1. \end{aligned} \tag{5.5}$$

Then Theorem 2.1 shows $f(z)\in\mathcal{H}_p(\alpha,\beta,\gamma)$. Conversely, suppose $f(z)\in\mathcal{H}_p(\alpha,\beta,\gamma)$.Then by Corollary 2.2, we have

$$a_{p+k}\leq \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}.$$

Set

$$c_{p+k}=\frac{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{\beta p(1-\alpha)(1+\gamma)} a_{p+k} \quad \text{and} \quad c_{p-1}=1-\sum_{k=0}^{\infty} c_{p+k}.$$

Then

$$\begin{aligned} f(z) &= \frac{1}{z^p}+\sum_{k=0}^{\infty} a_{p+k} z^{p+k} \\ &= \frac{1}{z^p}+\sum_{k=0}^{\infty} \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)} c_{p+k} z^{p+k} \\ &= \frac{1}{z^p}+\sum_{k=0}^{\infty}(f_{p+k}(z)-\frac{1}{z^p})c_{p+k} \\ &= \frac{1}{z^p}\left(1-\sum_{k=0}^{\infty} c_{p+k}\right)+\sum_{k=0}^{\infty} c_{p+k} f_{p+k}(z) \\ &= \frac{1}{z^p} c_{p-1}+\sum_{k=0}^{\infty} c_{p+k} f_{p+k}(z) \\ &= \sum_{k=-1}^{\infty} c_{p+k} f_{p+k}(z). \end{aligned}$$

This completes the proof of Theorem 5.1. □

Theorem 5.2. The class $\mathcal{H}_p(\alpha,\beta,\gamma)$ is closed under convex linear combinations.

Proof . Let each of the functions

$$f_j(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty} a_{p+k,j} z^{p+k} \qquad (a_{p+k,j}\geq 0; j=1,2), \tag{5.6}$$

be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$.It is sufficient to show that the function $h(z)$ defined by

$$h(z)=(1-t)f_1(z)+tf_2(z) \ (0\leq t\leq 1), \tag{5.7}$$

is also in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$. Since

$$h(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}[(1-t)(a_{p+k,1}+ta_{p+k,2}]z^{p+k}\ (0\leq t\leq 1), \tag{5.8}$$

by Theorem 2.1, we have

$$\begin{aligned}&\sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)[(1-t)a_{p+k,1}+ta_{p+k,2}]\\&=(1-t)\sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k,1}+t\sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k,2}\\&\leq(1-t)\beta p(1-\alpha)(1+\gamma)+t\beta p(1-\alpha)(1+\gamma)=\beta p(1-\alpha)(1+\gamma),\end{aligned}$$

which shows that $h(z)\in\mathcal{H}_p(\alpha,\beta,\gamma)$,hence the result follows. □

6 Convolution Properties

Theorem 6.1. Let the functions $f_j(z)$, for $j=1,2$, defined by (5.6) be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$. Then $(f_1*f_2)(z)\in\mathcal{H}_p(\phi,\beta,\gamma)$, where

$$\phi=1-\frac{p\beta(1+\gamma)(1-\alpha)^2}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}, \tag{6.1}$$

such that $\Phi_0(n,\lambda,\sigma,\mu,\omega,p)$ is given by (3.3). The result is sharp for the functions $f_j(z)$, for $j=1,2$, given by

$$f_j(z)=\frac{1}{z^p}+\frac{p\beta(1+\gamma)(1-\alpha)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}z^p. \tag{6.2}$$

Proof . Employing the technique used earlier by Schlid and Silverman[5], we will find the largest ϕ such that

$$\sum_{k=0}^{\infty}\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\phi)}a_{p+k,1}a_{p+k,2}\leq 1, \tag{6.3}$$

for $f_j(z)\in\mathcal{H}_p(\alpha,\beta,\gamma)$, for $j=1,2$. Since $f_j(z)\in\mathcal{H}_p(\alpha,\beta,\gamma)$, for $j=1,2$, we readily see that

$$\sum_{k=0}^{\infty}\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}a_{p+k,j}\leq 1, \tag{6.4}$$

by applying the Cauchy-Schwarz inequality,we obtain

$$\sum_{k=0}^{\infty}\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\sqrt{a_{p+k,1}a_{p+k,2}}\leq 1. \tag{6.5}$$

This implies that we need only to show that

$$\frac{a_{p+k,1}a_{p+k,2}}{(1-\phi)}\leq\frac{\sqrt{a_{p+k,1}a_{p+k,2}}}{(1-\alpha)}, \tag{6.6}$$

or equivalently that

$$\sqrt{a_{p+k,1}a_{p+k,2}}\leq\frac{1-\phi}{(1-\alpha)}. \tag{6.7}$$

Hence by the inequality (6.5), it is sufficient to prove that

$$\frac{p\beta(1+\gamma)(1-\alpha)}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}\leq\frac{1-\phi}{1-\alpha}, \tag{6.8}$$

which implies that

$$\phi \leq 1 - \frac{p\beta(1+\gamma)(1-\alpha)^2}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}. \tag{6.9}$$

Now, define the function $\Lambda(k)$ by

$$\Lambda(k) = 1 - \frac{p\beta(1+\gamma)(1-\alpha)^2}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}. \tag{6.10}$$

We note that $\Lambda(k)$ is an increasing function of k, therefore we conclude that

$$\phi \leq \Lambda(0) = 1 - \frac{p\beta(1+\gamma)(1-\alpha)^2}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}, \tag{6.11}$$

therefore the proof is complete. □

Theorem 6.2. Let the functions $f_1(z), f_2(z)$ defined by (5.6) be in the classes $\mathcal{H}_p(\alpha_1,\beta,\gamma), \mathcal{H}_p(\alpha_2,\beta,\gamma)$, respectively. Then $(f_1 * f_2)(z) \in \mathcal{H}_p(\eta,\beta,\gamma)$, where

$$\eta = 1 - \frac{p\beta(1+\gamma)(1-\alpha_1)(1-\alpha_2)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}, \tag{6.12}$$

such that $\Phi_0(n,\lambda,\sigma,\mu,\omega,p)$ is given by (3.3), and the result is sharp for the functions $f_j(z)(j=1,2)$ given by

$$f_1(z) = \frac{1}{z^p} + \frac{p\beta(1+\gamma)(1-\alpha_1)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} z^p, \tag{6.13}$$

and

$$f_2(z) = \frac{1}{z^p} + \frac{p\beta(1+\gamma)(1-\alpha_2)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} z^p. \tag{6.14}$$

Proof . Use the arguments similar to those in the proof of Theorem 6.1. □

Theorem 6.3. If $f_1(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,1} z^{p+k}$ and $f_2(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,2} z^{p+k}$ be in $\mathcal{H}_p(\alpha,\beta,\gamma)$, where $0 \leq a_{p+k,2} \leq 1$, $k = 0,1,2,...$, and $p \in \mathbb{N}$), then $(f_1 * f_2)(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$.

Proof . Since

$$\sum_{k=0}^{\infty} \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} a_{p+k,1}a_{p+k,2} \leq \sum_{k=0}^{\infty} \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} a_{p+k,1} \leq 1,$$

then by Theorem 2.1, $(f_1 * f_2)(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$. □

Corollary 6.4. If $f(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$, then the integral operator

$$\mathcal{F}_{c,p}(z) = \frac{c}{z^{p+c}} \int_0^z t^{c+p-1} f(t)dt, \quad c > 0, \tag{6.15}$$

is also in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$.

Proof . It is easy to check that

$$\mathcal{F}_{c,p}(z) = f(z) * \Big(\frac{1}{z^p} + \sum_{k=0}^{\infty} \frac{c}{c+2p+k} z^{p+k}\Big). \tag{6.16}$$

Since $0 < \frac{c}{c+2p+k} \leq 1$, by Theorem 6.3, the proof is trivial. □

Theorem 6.5. Let the functions $f_j(z)(j=1,2)$ defined by (5.6) be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$, and

$$p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)-2p\beta(1+\gamma)(1-\alpha)\geq 0, \tag{6.17}$$

then the function $h(z)$ defined by

$$h(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}(a_{p+k,1}^2+a_{p+k,2}^2)z^{p+k}, \tag{6.18}$$

belongs to the class $\mathcal{H}_p(\alpha,\beta,\gamma)$, where $\Phi_0(n,\lambda,\sigma,\mu,\omega,p)$ is given by (3.3).

Proof . Since $f_1(z)\in\mathcal{H}_p(\alpha,\beta,\gamma)$, we have

$$\sum_{k=0}^{\infty}\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}a_{p+k,1}\leq 1, \tag{6.19}$$

and so

$$\sum_{k=0}^{\infty}\left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right]^2a_{p+k,1}^2\leq 1. \tag{6.20}$$

Similarly, since $f_2(z)\in\mathcal{H}_p(\alpha,\beta,\gamma)$, we have

$$\sum_{k=0}^{\infty}\left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right]^2a_{p+k,2}^2\leq 1. \tag{6.21}$$

Hence

$$\sum_{k=0}^{\infty}\frac{1}{2}\left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right]^2(a_{p+k,1}^2+a_{p+k,2}^2)\leq 1. \tag{6.22}$$

In view of Theorem 2.1, it is sufficient to show that

$$\sum_{k=0}^{\infty}\left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right](a_{p+k,1}^2+a_{p+k,2}^2)\leq 1. \tag{6.23}$$

Thus the inequality (6.23) holds if, for $k=0,1,2,...$

$$\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\leq\frac{1}{2}\left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right]^2, \tag{6.24}$$

or equivalently if

$$(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)-2p\beta(1+\gamma)(1-\alpha)\geq 0, \tag{6.25}$$

for $k=0,1,2,\ldots$. The left hand side of (6.25) is an increasing function of k, hence it holds for all k if

$$p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)-2p\beta(1+\gamma)(1-\alpha)\geq 0, \tag{6.26}$$

which is true by our assumption. Hence the proof is complete. □

Theorem 6.6. Let the functions $f_j(z)(j=1,2)$ defined by (5.6), be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$. Then the function $h(z)$ defined by (6.18) belongs in the class $\mathcal{H}_p(\tau,\beta,\gamma)$,where

$$\tau=1-\frac{2p\beta(1+\gamma)(1-\alpha)^2}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}, \tag{6.27}$$

such that $\Phi_0(n,\lambda,\sigma,\mu,p)$ is given by (3.3). The result is sharp for the functions$f_j(z)(j=1,2)$ defined by (6.2),

Proof . Since

$$\sum_{k=0}^{\infty}\frac{\Big[(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)\Big]^2}{\Big[p\beta(1+\gamma)(1-\alpha)\Big]^2}a_{p+k,j}^2 \leq \left[\sum_{k=0}^{\infty}\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}a_{p+k,j}\right]^2 \leq 1, \tag{6.28}$$

for $f_j(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$, where $j=1,2$, so we get

$$\sum_{k=0}^{\infty}\frac{\Big[(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)\Big]^2}{2\Big[p\beta(1+\gamma)(1-\alpha)\Big]^2}(a_{p+k,1}^2+a_{p+k,2}^2) \leq 1. \tag{6.29}$$

We have to find the largest τ such that

$$\frac{1}{1-\tau} \leq \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{2p\beta(1+\gamma)(1-\alpha)^2}, \tag{6.30}$$

that is

$$\tau \leq 1-\frac{2p\beta(1+\gamma)(1-\alpha)^2}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}. \tag{6.31}$$

If we define $L(k)$ by

$$L(k) = 1-\frac{2p\beta(1+\gamma)(1-\alpha)^2}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}, \tag{6.32}$$

that it is easy to see that $L(k)$ is an increasing function of k, thus we conclude that

$$\tau \leq L(0) = 1-\frac{2p\beta(1+\gamma)(1-\alpha)^2}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}, \tag{6.33}$$

which completes the proof. □

7 Neighborhoods and Partial Sums

Definition 7.1. For every $\delta > 0$ and a non-negative sequence $\mathcal{S} = \{s_k\}_{k=0}^{\infty}$, where

$$s_k = \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} \tag{7.1}$$
$$(k \geq 0, p \in \mathbb{N}, 0 \leq \alpha < 1, 0 < \beta \leq 1, 0 \leq \gamma \leq 1, \lambda \geq 0, 0 \leq \mu \leq \sigma),$$

the δ-neighborhood of a function $f \in \Sigma_p$ is defined by

$$\mathcal{N}_\delta(f) = \Big\{g \in \Sigma_p \ : \ g(z) = \frac{1}{z^p}+\sum_{k=0}^{\infty}b_{p+k}z^{p+k} \text{ and } \sum_{k=0}^{\infty}s_k|b_{p+k}-a_{p+k}| \leq \delta\Big\}. \tag{7.2}$$

Theorem 7.2. Let $f \in \mathcal{H}_p(\alpha,\beta,\gamma)$ be given by (1.1). If f satisfies

$$\frac{f(z)+\epsilon z^{-p}}{1+\epsilon} \in \mathcal{H}_p(\alpha,\beta,\gamma) \ (\epsilon \in \mathbb{C}, |\epsilon| < \delta, \delta > 0), \tag{7.3}$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{H}_p(\alpha,\beta,\gamma). \tag{7.4}$$

Proof . It is not difficult to see that a function f belongs to $\mathcal{H}_p(\alpha,\beta,\gamma)$ if and only if

$$\frac{z^{p+2}\big(D^n_{\lambda,\sigma,\mu,\delta,p}f(z)\big)''+z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\delta,p}f(z)\big)'-p^2}{\beta\gamma z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\delta,p}f(z)\big)'+\beta\alpha(1+\gamma)p-\beta p}\neq\nu\quad(\nu\in\mathbb{C},|\nu|=1), \tag{7.5}$$

which is equivalent to

$$\frac{(f*h)(z)}{z^{-p}}\neq 0, \tag{7.6}$$

where $h(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}c_{p+k}z^{p+k}$, such that

$$c_{p+k}=\frac{(p+k)(p+k-\nu\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\nu\beta(1+\gamma)(1-\alpha)},\quad\text{and}\quad |c_{p+k}|\leq\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}.$$

Furthermore,under the hypotheses (7.3),using (7.6) we obtain

$$\frac{1}{z^{-p}}\Big(\frac{f(z)+\epsilon z^{-p}}{1+\epsilon}*h(z)\Big)\neq 0. \tag{7.7}$$

Now assume that $\left|\frac{(f*h)(z)}{z^{-p}}\right|<\delta$.Then by (7.7) we have

$$\left|\frac{1}{1+\epsilon}\frac{(f*h)(z)}{z^{-p}}+\frac{\epsilon}{1+\epsilon}\right|\geq\frac{|\epsilon|}{|1+\epsilon|}-\frac{1}{|1+\epsilon|}\left|\frac{(f*h)(z)}{z^{-p}}\right|>\frac{|\epsilon|-\delta}{|1+\epsilon|}>0.$$

This is a contradiction as $|\epsilon|<\delta$. Therefore $\left|\frac{(f*h)(z)}{z^{-p}}\right|\geq\delta$. If we now let

$$g(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}b_{p+k}z^{p+k}\in\mathcal{N}_\delta(f),$$

then we have

$$\begin{aligned}\delta-\left|\frac{(g*h)(z)}{z^{-p}}\right|\leq\left|\frac{(f-g)*h(z)}{z^{-p}}\right|&=\left|\sum_{k=0}^{\infty}(a_{p+k}-b_{p+k})c_{p+k}z^{2p+k}\right|\\&\leq\sum_{k=0}^{\infty}|a_{p+k}-b_{p+k}||c_{p+k}||z|^{2p+k}\\&<\sum_{k=0}^{\infty}\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}|a_{p+k}-b_{p+k}|\\&\leq\delta.\end{aligned}$$

Thus, $\frac{(g*h)(z)}{z^{-p}}\neq 0$ which implies that $g(z)\in\mathcal{H}_p(\alpha,\beta,\gamma)$, and the proof is complete. □

Theorem 7.3. Let $f\in\Sigma_p$ be given by (1.1) and the partial sums $k_0(z)$ and $k_q(z)$ be defined by $k_0(z)=\frac{1}{z^p}$ and $k_q(z)=\frac{1}{z^p}+\sum_{k=0}^{q-1}a_{p+k}z^{p+k}\quad(q>0)$, also suppose that

$$\sum_{k=0}^{\infty}\theta_{p+k}a_{p+k}\leq 1 \tag{7.8}$$

where $\theta_{p+k}=\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}$. Then, for $q>0$, we have

$$\mathfrak{Re}\Big\{\frac{f(z)}{k_q(z)}\Big\}>1-\frac{1}{\theta_q} \tag{7.9}$$

and

$$\mathfrak{Re}\left\{\frac{k_q(z)}{f(z)}\right\} > \frac{\theta_q}{1+\theta_q}. \tag{7.10}$$

Proof . Under the hypotheses of the theorem we can see from (7.8) that

$$\theta_{p+k+1} > \theta_{p+k} > 1 \quad (k = 0, 1, 2, ...).$$

Therefore making use of (7.8) again we have

$$\sum_{k=0}^{q-1} a_{p+k} + \theta_q \sum_{k=q}^{\infty} a_{p+k} \le \sum_{k=0}^{\infty} \theta_{p+k} a_{p+k} \le 1. \tag{7.11}$$

Let

$$w(z) = \theta_q \left[\frac{f(z)}{k_q(z)} - \left(1 - \frac{1}{\theta_q}\right)\right] = 1 + \frac{\theta_q \sum\limits_{k=q}^{\infty} a_{p+k} z^{2p+k}}{1 + \sum\limits_{k=0}^{q-1} a_{p+k} z^{2p+k}}. \tag{7.12}$$

Applying (7.11) and (7.12) we find

$$\left|\frac{w(z)-1}{w(z)+1}\right| = \left|\frac{\theta_q \sum\limits_{k=q}^{\infty} a_{p+k} z^{2p+k}}{2 + 2\sum\limits_{k=0}^{q-1} a_{p+k} z^{2p+k} + \theta_q \sum\limits_{k=q}^{\infty} a_{p+k} z^{2p+k}}\right| \le \frac{\theta_q \sum\limits_{k=q}^{\infty} a_{p+k}}{2 - 2\sum\limits_{k=0}^{q-1} a_{p+k} - \theta_q \sum\limits_{k=q}^{\infty} a_{p+k}} \le 1, \tag{7.13}$$

which shows that $\mathfrak{Re} w(z) > 0$. From (7.12) we immediately obtain (7.9). Similarly by letting

$$\varphi(z) = (1+\theta_q)\left[\frac{k_q(z)}{f(z)} - \frac{\theta_q}{1+\theta_q}\right],$$

we can prove (7.10), therefore the proof is complete. □

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