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Hyperoperations on metric spaces: A new frontier

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Abstract

This paper introduces novel hyperoperations on metric spaces. While generally non-commutative, these hyperoperations satisfy the weak commutative property. We further establish necessary and sufficient conditions for associativity. Finally, we obtain a class of Γ-H_v-groups.

Keywords: Metric space, H_v-group, Hypergroup, Semihypergroup
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1 Introduction

The early 20th century witnessed a profound marriage of algebraic and geometric concepts, particularly in the realm of metric spaces. This fruitful union can be traced back to the pioneering work of Hausdorff, who in 1914 introduced the notion of a topological space [8]. This abstraction liberated mathematicians from the constraints of specific geometric configurations, allowing for the study of spaces based solely on their topological properties. Building on this foundation, the 1930s saw significant contributions from John von Neumann and Gelfand with the development of Banach spaces – complete normed vector spaces [6, 11]. This theory elegantly combined algebraic structures like vector spaces with the concept of metric distance, providing a powerful framework for analyzing linear operators in infinite-dimensional settings. Furthermore, the emergence of functional analysis in 1930, spearheaded by Stefan Banach [1], marked another milestone in this confluence. This field leverages algebraic techniques to investigate the properties of functions defined on metric spaces, with applications extending to various branches of mathematics and physics.

The latter half of the 20th century witnessed continued exploration of this rich interplay. Theories like C*-algebras and von Neumann algebras emerged, offering a robust framework for studying quantum mechanics and other areas of mathematical physics. In conclusion, the connection between algebra and metric spaces has flourished as a vibrant research area for over a century, impacting numerous areas of mathematics and beyond.

The field of algebraic hyperstructures, born from the seminal work of Marty in 1934 with the introduction of hypergroups, has witnessed a burgeoning literature. A testament to this growth is the abundance of research articles and dedicated monographs, including the notable works by Corsini and Leoreanu [4], Corsini [2], and Davvaz [5]. Notably, these hyperstructures offer a fertile ground for generalizing classical algebraic structures.

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Connections between hypergraphs and hyperstructures have been explored by several researchers, particularly in the context of defining hyperoperations based on hypergraph properties [3]. Nikkhah, Davvaz and Mirvakili [9] extended this connection by introducing a new type of hyperoperation on hypergraphs. Additionally, Hamidizadeh, Mirvakili and Manaviyat [7], defined a hyperoperation by using open and closed balls in metric spaces.

This paper builds on this research by introducing novel hyperoperations on metric spaces. We investigate the necessary and sufficient conditions for the resulting hypergroupoids to be hypergroups. Our work contributes to the growing body of knowledge on the relationship between hyperstructures and metric spaces.

Algebraic hyperstructures represent a generalization of classical algebraic structures. In contrast to classical structures, where the composition of two elements is an element, hyperstructures define the composition of two elements as a set. Specifically, let $P^*(X)$ denote the set of all non-empty subsets of a given set X. A hypergroupoid is a pair $(X, *)$, where X is a non-empty set and $*$ is a hyperoperation defined as $* : X \times X \longrightarrow P^*(X)$, such that $(x, y) \mapsto x * y$. If $A, B \in P^*(X)$, then $A * B = \cup\{a * b | a \in A, b \in B\}$, $x * B = \{x\} * B$, and $A * y = A * \{y\}$. In cases where $A = \emptyset$ or $B = \emptyset$, we define $A * B = \emptyset$.

A hypergroupoid $(X, *)$ is classified as a semihypergroup (H_v-semigroup) if it satisfies the (weak) associative axiom, i.e.,

$$x * (y * z) = (x * y) * z \quad (x * (y * z) \cap (x * y) * z \neq \emptyset),$$

for all $x, y, z \in X$. It is called reproductive if $x * X = X * x = X$, for all $x \in X$. A hypergroup (H_v-group) is a reproductive semihypergroup (H_v-semigroup). If $(X, *)$ is commutative (Weak commutative), i.e., $x * y = y * x$ ($x * y \cap y * x \neq \emptyset$), for all $x, y \in X$, it is referred to as a join space. In such cases, the following implication holds for all elements a, b, c, d of X:

$$a/b \cap c/d \neq \emptyset \Rightarrow a * d \cap b * c \neq \emptyset.$$

Let $(H, *)$ be a non-commutative hypergroupoid. Set $a/b = \{x \mid a \in x * b\}$ and $b \setminus a = \{x \mid a \in b * x\}$. We say that it is a transposition hypergroupoid if for all elements a, b, c, d of X:

$$a/b \cap d \setminus c \neq \emptyset \Rightarrow a * d \cap b * c \neq \emptyset.$$

Numerous authors have studied the relationship between hypergraphs and hyperstructures, as presented in [3, 9]. Let $G = (V, E)$ be an undirected connected graph. Corsini and Leoreanu-Fotea [3] defined hyperoperation $\circ$ on V such that for every $x, y \in V$, $x \circ y = N(x) \cup N(y)$, where $N(x)$ is the neighborhood of vertex x in graph G. Hamidizadeh, Mirvakili and Manaviyat [7], defined a hyperoperation by using open and closed balls in metric spaces instead $N(x)$ in an undirected graph (Notice that every connected undirected graph is a metric space).

We define a metric space (X, d) as follows: A function $d : X \times X \to R$ is defined as a metric on X if it satisfies the following properties for all $x, y, z \in X$:

(M1) $d(x, y) \geq 0$,

(M2) $d(x, y) = 0$ if and only if $x = y$,

(M3) $d(x, y) = d(y, x)$, and

(M4) $d(x, y) + d(y, z) \geq d(x, z)$.

The real number $d(x, y)$ represents the distance between x and y, and the set X together with the metric d is referred to as a metric space (X, d) [10]. Let $\mathcal{X}_* = (X, *)$ and $\mathcal{X}_\circ = (X, \circ)$ be two hypergroupoids. We say that $\mathcal{X}_* = (X, *)$ is equal or greater than $\mathcal{X}_\circ = (X, \circ)$ and write $\mathcal{X}_\circ \preceq \mathcal{X}_*$ if for every $x, y \in X$, $x \circ y \subseteq x * y$.

2 Constructing hyperoperations on Metric Spaces

Given a metric space (X, d) and any real number $r \geq 0$, the open ball of radius r and center a is defined as the set $B_r(a) \subseteq X$ such that $B_r(a) = \{x \in X | d(x, a) < r\}$. Additionally, $C_r(a) = \{x \in X | d(x, a) \leq r\}$.

We define four hyperoperations $\circ_{bb}^{rs}$, $\circ_{bc}^{rs}$, $\circ_{cb}^{rs}$ and $\circ_{cc}^{rs}$ as following:

For all $r > 0$, $s > 0$ and $x, y \in X$, $x \circ_{bb}^{rs} y = B_r(x) \cup B_s(y)$,

For all $r > 0$, $s \geq 0$ and $x, y \in X$, $x \circ_{bc}^{rs} y = B_r(x) \cup C_s(y)$, For all $r \geq 0$, $s > 0$ and $x, y \in X$, $x \circ_{cb}^{rs} y = C_r(x) \cup B_s(y)$,

For all $r \geq 0$, $s \geq 0$ and $x, y \in X$, $x \circ_{cc}^{rs} y = C_r(x) \cup C_s(y)$,

Example 2.1. The real numbers with the distance function $d(x, y) = |y - x|$ given by the absolute difference form a metric space. For any $r > 0$ and $x \in \mathbb{X}$, we have $B_r(x) = (x - r, x + r)$ and for every $r \geq 0$, $C_r(x) = [x - r, x + r]$. Then we obtain

For all $r > 0$, $s > 0$ and $x, y \in X$, $x \circ_{bb}^{rs} y = (x - r, x + r) \cup (y - s, y + s)$,

For all $r > 0$, $s \geq 0$ and $x, y \in X$, $x \circ_{bc}^{rs} y = (x - r, x + r) \cup [y - s, y + s]$,

For all $r \geq 0$, $s > 0$ and $x, y \in X$, $x \circ_{cb}^{rs} y = [x - r, x + r] \cup (y - s, y + s)$,

For all $r \geq 0$, $s \geq 0$ and $x, y \in X$, $x \circ_{cc}^{rs} y = [x - r, x + r] \cup [y - s, y + s]$,

In this paper, for the sake of brevity and clarity, $X_{bb}^{rs}, X_{bc}^{rs}, X_{cb}^{rs}$ and X_{cc}^{rs} mean hypergroupoids $(X, \circ_{bb}^{rs})$, $(X, \circ_{bc}^{rs})$, $(X, \circ_{cb}^{rs})$ and $(X, \circ_{cc}^{rs})$, respectively.

In this paper, when we use hyperoperation $\circ$, we mean one of the four hyperoperations defined above.

Theorem 2.2. 1) $X_{bb}^{rs} \preceq X_{cb}^{rs} \preceq X_{cc}^{rs}$ and $X_{bb}^{rs} \preceq X_{bc}^{rs} \preceq X_{cc}^{rs}$.

2) For all $r \leq r'$ and $s \leq s'$, we have $X_{pq}^{rs} \preceq X_{pq}^{r's'}$ when $p, q \in \{b, c\}$.

Lemma 2.3. Let $x, y \in X$. If $r, s > 0$, then $x \circ_{bb}^{rs} y$ is an open set and if $r, s \geq 0$, then $x \circ_{cc}^{rs} y$ is a closed set. Moreover

(1) $x \circ_{bc}^{rs} y$ is an open set if $d(x, y) < r$ and $s < r - d(x, y)$.

(2) $x \circ_{bc}^{rs} y$ is a closed set if $d(x, y) < s$ and $r \leq s - d(x, y)$.

(3) $x \circ_{cb}^{rs} y$ is an open set if $d(x, y) < s$ and $r < s - d(x, y)$.

(4) $x \circ_{cb}^{rs} y$ is a closed set if $d(x, y) < r$ and $s \leq r - d(x, y)$.

Proof . (1) If $d(x, y) < r$ and $s < r - d(x, y)$. We show that $C_s(y) \subseteq B_r(x)$. If $z \in C_s(y)$ then $d(x, z) \leq d(x, y) + d(y, z) \leq r - s + s = r$. So $z \in B_r(x)$. Therefore $x \circ_{bc}^{rs} y = B_r(x)$ and so $x \circ_{bc}^{rs} y$ is open.

Proofs of (2), (3) and (4) are by the similar to way. □

Theorem 2.4. For each $(x, y) \in H^2$:

(1) $x \circ_{pq}^{rs} y = x \circ_{pp}^{rr} x \cup y \circ_{qq}^{ss} y$, where $p, q \in \{b, c\}$.

(2) For every $\circ \in \{\circ_{bb}^{rs}, \circ_{bc}^{rs}, \circ_{cb}^{rs}, \circ_{cc}^{rs}\}$, we have $x \in x \circ x$.

(3) For every $\circ \in \{\circ_{bb}^{rs}, \circ_{bc}^{rs}, \circ_{cb}^{rs}, \circ_{cc}^{rs}\}$, we have $y \in x \circ x \Leftrightarrow x \in y \circ y$;

Proof . It is straightforward. □

Theorem 2.5. For every $\circ \in \{\circ_{bb}^{rs}, \circ_{bc}^{rs}, \circ_{cb}^{rs}, \circ_{cc}^{rs}\}$,

(4) $\{x, y\} \subseteq x \circ y$,

(5) $x \circ y \cap y \circ x \neq \emptyset$,

(6) $x \circ X = X = X \circ x$,

(7) $(x \circ_{pp}^{rs} x) \circ_{pp}^{rs} x = \cup_{z \in x \circ_{pp}^{rs} x} z \circ_{pp}^{rs} z$, when $p \in \{b, c\}$.

Proof . It is straightforward. □

By (5) and (6) of Theorem 2.5 we obtain:

Corollary 2.6. A hypergroupoid $X \in \{X^{rs}_{bb}, X^{rs}_{bc}, X^{rs}_{cb}, X^{rs}_{cc}\}$ is a weak commutative quasihypergroup.

Proof . Weak commutativity obtain from Theorem 2.5 part (5) and reproduce law obtain from Theorem 2.5 part (6). $\square$

Corollary 2.7. A hypergroupoid $X \in \{X^{rs}_{bb}, X^{rs}_{bc}, X^{rs}_{cb}, X^{rs}_{cc}\}$ is a weak commutative H_v-group.

Proof . Let $\circ \in \{\circ^{rs}_{bb}, \circ^{rs}_{bc}, \circ^{rs}_{cb}, \circ^{rs}_{cc}\}$. By Theorem 2.5 part (4), for every $x, y, z \in X$, we have

$$\{x, y, z\} \subseteq (x \circ y) \circ z \cap x \circ (y \circ z).$$

This show that proof is complete. $\square$

Theorem 2.8. [4] Let $(X, \circ)$ be a hypergroupoid satisfying

(1) $x \circ y = x \circ x \cup y \circ y$;

(2) $x \in x \circ x$;

(3) $y \in x \circ x \Leftrightarrow x \in y \circ y$.

Then $(X, \circ)$ is a hypergroup if and only if

$$\forall (a, c) \in X^2, \quad c \circ c \circ c - c \circ c \subseteq a \circ a \circ a.$$

where $c \circ c \circ c = (c \circ c) \circ c \cup c \circ (c \circ c)$.

Hamidizadeh, Mirvakili and Manaviyat [7] by using Theorem 2.8 show that:

Theorem 2.9. Let (X, d) be a metric space.

(1) A hypergroupoid X^{rr}_{bb} is a hypergroup if and only if for all $x, y \in X$ we have

$$B^2_r(x) - B_r(x) \subseteq B^2_r(y),$$

where $B^2_r(x) = \bigcup_{z \in B_r(x)} B_r(z)$.

(2) A hypergroupoid X^{rr}_{cc} is a hypergroup if and only if for all $x, y \in X$ we have

$$C^2_r(x) - C_r(x) \subseteq C^2_r(y),$$

where $C^2_r(x) = \bigcup_{z \in C_r(x)} C_r(z)$.

Lemma 2.10. We have:

(1) $x \circ^{rs}_{bb} x = B_{r \vee s}(x)$.

(2) $x \circ^{rs}_{bc} x = \begin{cases} B_r(x) & r > s \\ C_s(x) & r \leq s \end{cases}$

(3) $x \circ^{rs}_{cc} x = C_{r \vee s}(x)$

(4) $x \circ^{rs}_{cb} x = \begin{cases} C_r(x) & r \geq s \\ B_s(x) & r < s \end{cases}$

Example 2.11. Consider G is an undirected connected graph, then the set V of vertices of G can be turned into a metric space by defining $d(x,y)$ to be the length of the shortest path connecting the vertices x and y.

In [3], Corsini presents a commutative quasihypergroup H_G associated with a given hypergraph G. Necessary and sufficient conditions for H_G to be associative are found. For certain classes of hypergraphs that include finite hypergraphs, a sequence of hypergraphs is described such that the corresponding quasi-hypergroups form a joint space. Moreover, Nikkhah, Davvax and Mirvakili [9] defined a hyperoperation ${}_m\circ_n$ for all $n, m \in \mathbb{N} \cup \{0\}$ on H as follows:

$$\forall (x,y) \in H^2, x \;{}_m\circ_n\; y = E^n(x) \cup E^m(y),$$

where $E^0(x) = x$ and $E(x)$ is the neighborhood of vertex x in graph G and $E^n(x) = E(E^{n-1}(x)) = \cup_{z \in E^{n-1}(x)} E(z)$. When $n = m = 1$, the hyperoperation ${}_m\circ_n$ introduced by Corsini [3].

The hypergroupoid $H_G = (H, {}_m\circ_n)$ is called a hypergraph hypergroupoid or a h.g. hypergroupoid.

The r-ball $B_r(x)$ of center x and radius $r \geq 0$ consists of all vertices of G at distance at most r^- from x: Suppose that $r \geq 0$. Let $\lfloor r \rfloor$ be the function that takes as input a real number r, and gives as output the greatest integer less than or equal to r and let $\lceil r \rceil$ be the ceiling function maps r to the smallest integer greater than or equal to r.

Then $B_r(x) = E^{\lceil r \rceil}(x)$ and $C_r(x) = E^{\lfloor r \rfloor}(x)$. Now we have

$$\circ_{bb}^{rs} = {}_{\lfloor r \rfloor}\circ_{\lfloor s \rfloor}, \; \circ_{bc}^{rs} = {}_{\lfloor r \rfloor}\circ_{\lceil s \rceil}, \; \circ_{cb}^{rs} = {}_{\lceil r \rceil}\circ_{\lfloor s \rfloor}, \; \circ_{cc}^{rs} = {}_{\lceil r \rceil}\circ_{\lceil s \rceil}.$$

Moreover, for all $m, n \in \mathbb{N}$, we have

$$\circ_{bb}^{mn} = {}_m\circ_n, \; \circ_{bc}^{mn} = {}_m\circ_{n+1}, \; \circ_{cb}^{mn} = {}_{m+1}\circ_n, \; \circ_{cc}^{mn} = {}_{m+1}\circ_{n+1}.$$

Therefore, the hyperoperation $\circ_{bb}^{rr}$ is to coincide with the hyperoperation $\circ$ in [3].

Theorem 2.12. The H_v-group $(X, \circ_{pp}^{rr})$ is a transposition H_v-group, where $p \in \{b, c\}$.

Proof . Let $a/b = \{x \mid a \in x \circ_{pp}^{rr} b\}$ and $b \setminus a = \{x \mid a \in b \circ_{pp}^{rr} x\}$. Suppose that $a/b \bigcap d \setminus c \neq \emptyset$ then there exists $x \in X$ such that $a \in x \circ_{pp}^{rr} b$ and $c \in d \circ_{pp}^{rr} x$. First assume that $p = b$. So $a \in B_r(x) \cup B_r(b)$ and $c \in B_r(x) \cup B_r(d)$. We have one of the four following cases:

(1) $a \in B_r(x)$ and $c \in B_r(x)$, then $x \in B_r(a)$ and $x \in B_r(c)$ and so $x \in a \circ d \cap b \circ c$.

(2) $a \in B_r(x)$ and $c \in B_r(d)$, then $c \in a \circ d \cap b \circ c$.

(3) $a \in B_r(b)$, then $a \in a \circ d \cap b \circ c$.

For $p = c$, the proof is similar. □

Let X be a metric space and $D = \sup\{d(x,y) | x, y \in X\}$.

Theorem 2.13. The hypergroupoid $(H, \circ_{pp}^{rs})$ is commutative, $p \in \{b, c\}$, if one of the following conditions is satisfied:

(1) $r = s$,

(2) $r > D$ or $s > D$.

Proof .

(1) If $r = s$ then in reference [7] the authors prove that $(H, \circ_{pp}^{rs})$ is commutative.

(2) If $r > D$ then for every $x \in H$, we have $B_r(x) = H$ and $C_r(x) = H$. Moreover, if $r > D$, then for every $x \in H$ we have $B_s(x) = H$ and $C_s(x) = H$. Then for all $x, y \in H$, we have $x \circ_{pp}^{rs} y = H = y \circ_{pp}^{rs} x$.

□

Theorem 2.14. Let (X, d) be a metric space with maximum distance D. If for every $x, y \in X$, there exists $z \in X$ such that $d(x,z) \leq D/2$ and $d(y,z) \leq D/2$, then

(1) If $r > D/2$ and $s > D/2$, then $x \circ_{bb}^{rs} (y \circ_{bb}^{rs} z) = X$ and $(x \circ_{bb}^{rs} y) \circ_{bb}^{rs} z = X$, for all $x, y, z \in X$.

(2) If $r > D/2$ and $s \geq D/2$, then $x \circ_{bc}^{rs} (y \circ_{bc}^{rs} z) = X$ and $(x \circ_{bc}^{rs} y) \circ_{bc}^{rs} z = X$, for all $x, y, z \in X$.

(3) If $r \geq D/2$ and $s > D/2$, then $x \circ_{cb}^{rs} (y \circ_{cb}^{rs} z) = X$ and $(x \circ_{cb}^{rs} y) \circ_{cb}^{rs} z = X$, for all $x, y, z \in X$.

(4) If $r \geq D/2$ and $s \geq D/2$, then $x \circ_{cc}^{rs} (y \circ_{cc}^{rs} z) = X$ and $(x \circ_{cc}^{rs} y) \circ_{cc}^{rs} z = X$, for all $x, y, z \in X$.

Proof .

(1) Assume that $r \geq D/2$, $s > D/2$ and $r < s$. Since $(x \circ_{bb}^{rr} x) \circ_{bb}^{rr} x \subseteq (x \circ_{bb}^{rs} y) \circ_{bb}^{rs} z$, it is sufficient to show that $(x \circ_{bb}^{rr} x) \circ_{bb}^{rr} x = X$. Consider an arbitrary element $y \in X$. Thus there is $z \in X$ such that $d(x, z) \leq D/2$ and $d(y, z) \leq D/2$ and so $z \in x \circ_{bb}^{rr} x$ and $y \in z \circ_{bb}^{rr} x \subseteq (x \circ_{bb}^{rr} x) \circ_{bb}^{rr} x$. Therefore $X \subseteq (x \circ_{bb}^{rr} x) \circ_{bb}^{rr} x$ and we are done.

The proof of (2), (3) and (4) is similar to that of (1).

□

The following Example demonstrates that the existence of such a "z" for every pair of elements in the Theorem 2.14 is not a removable condition.

Example 2.15. Let (X, d) be the metric space where d is discrete metric, that $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise. Set $r = s = \frac{2}{3}$. Then for all $x, y, z \in X$, we have $x \circ_{pq}^{rs} (y \circ_{pq}^{rs} z) = (x \circ_{pq}^{rs} y) \circ_{pq}^{rs} z = \{x, y, z\}$.

In the next Examples some classes of metric spaces satisfying conditions of the last Theorem are introduced.

Example 2.16. For a natural number k, let G be a graph of diameter $2k$. Similar to Example 2.11, (G, d) is a metric space. Clearly, for all $x, y \in G$, there exists a path of length at most $2k$ from x to y such as $x, z_1, \ldots, z_{2k-1}, y$. Then $d(x, z_k) \leq D/2 = k$ and $d(y, z_k) \leq D/2 = k$ and so such metric spaces satisfy the condition of Theorem 2.14.

Example 2.17. Let S be a convex subspace of an Euclidean vector space such as a Cube with side length k of $\mathbb{R}^n$. Since S is convex, the shortest path from x to y is contained in S, for all $x, y \in S$ and so clearly the desired point, z, exists in the middle of the path. Thus a wide class of metric spaces may satisfy the condition of Theorem 2.14.

Definition 2.18. A hypergroupoid X is called n-total hypergroupoid if every products of n elements is equal to X. 2-total hypergroupoid means $x \circ y = X$, for all $x, y \in X$. 3-total hypergroupoid means $x \circ (y \circ z) = X$ and $(x \circ y) \circ z = X$, for all $x, y, z \in X$.

Theorem 2.19. Every 3-total hypergroupoid is a hypergroup.

Proof . For every $x, y, z \in X$, we have $x \circ (y \circ z) = X = (x \circ y) \circ z$ and so the associative law holds. □

Theorem 2.20. (1) $r > D$ or $s > D$ if and only if the hypergroupoid $(X, \circ_{bb}^{rs})$ is a total hypergroup.

(2) If $r > D$ or $s \geq D$ then The hypergroupoid $(X, \circ_{bc}^{rs})$ is a total hypergroup.

(3) If $r \geq D$ or $s > D$ then The hypergroupoid $(X, \circ_{cb}^{rs})$ is a total hypergroup.

(4) If $r \geq D$ or $s \geq D$ then The hypergroupoid $(X, \circ_{cc}^{rs})$ is a total hypergroup.

Proof . For all $x, y \in X$ and $p, q \in \{b, c\}$ we have $x \circ_{pq}^{rs} y = X = y \circ_{pq}^{rs} x$. So $(X, \circ_{bb}^{rs})$ is a total hypergroup. □

By using Lemma 2.14 and Theorem 2.19 we obtain:

Corollary 2.21. Let (X, d) be a metric space with maximum distance D. If for all $x, y \in X$, there exists $z \in X$ such that $d(x, z) \leq D/2$ and $d(y, z) \leq D/2$, then

(1) If $r > D/2$ or $s > D/2$ then the hypergroupoid $(X, \circ_{bb}^{rs})$ is 3-total hypergroup.

(2) If $r > D/2$ or $s \geq D/2$ then the hypergroupoid $(X, \circ_{bc}^{rs})$ is a total hypergroup.

(3) If $r \geq D/2$ or $s > D/2$ then the hypergroupoid $(X, \circ_{cb}^{rs})$ is a total hypergroup.

(4) If $r \geq D/2$ or $s \geq D/2$ then the hypergroupoid $(X, \circ_{cc}^{rs})$ is a total hypergroup.

Theorem 2.22. Let $r < D/2$ and $s < D/2$. Then the hypergroupoid $(X, \circ_{pq}^{rs})$ is a hypergroup if and only if for all $a, b, c \in X$, one of the following conditions is valid:

(1) if $B_{2r}(a) \cup C_s(c) \subset B_r(a) \cup C_{2s}(c)$, then $B_r(a) \cup C_{2s}(c) - B_{2r}(a) \cup C_s(c) \subset B_{r+s}(b)$.

(2) if $B_r(a) \cup C_{2s}(c) \subset B_{2r}(a) \cup C_s(c)$, then $B_{2r}(a) \cup C_s(c) - B_r(a) \cup C_{2s}(c) \subset B_{r+s}(b)$.

(3) if $B_{2r}(a) \cup C_s(c) \nsubseteq B_r(a) \cup C_{2s}(c)$ and $B_r(a) \cup C_{2s}(c) \nsubseteq B_{2r}(a) \cup C_s(c)$, then $(B_r(a) \cup C_{2s}(c) - B_{2r}(a) \cup C_s(c)) \cup (B_{2r}(a) \cup C_s(c) - B_r(a) \cup C_{2s}(c)) \subset B_{r+s}(b)$.

Proof . (1) Let $(X, \circ_{pq}^{rs})$ is a hypergroup and $a, b, c \in X$. Then $a \circ_{pq}^{rs} (b \circ_{pq}^{rs} c) = (a \circ_{pq}^{rs} (b \circ_{pq}^{rs} c)$. So

$$\begin{aligned} B_r(a) \cup C_c(B_r(b) \cup C_s(c)) &= B_r(a) \cup B_{r+s}(b) \cup C_{2s}(c) \\ &= B_r(B_r(a) \cup C_s(b)) \cup C_s(c) \\ &= B_{2r}(a) \cup B_{r+s}(b) \cup C_s(c). \end{aligned}$$

Since $B_{2r}(a) \cup C_s(c) \subseteq B_r(a) \cup C_{2s}(c)$,

$$B_r(a) \cup C_{2s}(c) - B_{2r}(a) \cup C_s(c) \subseteq B_{s+r}(b).$$

The converse is clear.

(2) follows similar to (1).

(3) Let $(X, \circ_{pq}^{rs})$ is a hypergroup and let $a, b, c \in X$. Then $a \circ_{pq}^{rs} (b \circ_{pq}^{rs} c) = (a \circ_{pq}^{rs} (b \circ_{pq}^{rs} c)$. So

$$B_r(a) \cup B_{r+s}(b) \cup C_{2s}(c) = B_{2r}(a) \cup B_{r+s}(b) \cup C_s(c).$$

Since $B_{2r}(a) \cup C_s(c) \nsubseteq B_r(a) \cup C_{2s}(c)$ and $B_{2r}(a) \cup C_s(c) \nsupseteq B_r(a) \cup C_{2s}(c)$,

$$(B_r(a) \cup C_{2s}(c) - B_{2r}(a)) \cup (C_s(c) \cup (B_{2r}(a) \cup C_s(c) - B_r(a) \cup C_{2s}(c)) \subseteq B_{s+r}(b).$$

The converse is clear. □

Example 2.23. The Example 2.15 is an Example of metric spaces such that $(X, \circ_{pq}^{rs})$ is a hypergroup. In Example 2.11 we can show by choosing the appropriate r and s that $(X, \circ_{pq}^{rs})$ is not a hypergroup, as a result of these conditions the Theorem 2.22 does not hold.

Definition 2.24. Let A be a family of hyperoperations on hypergroupoid H. We say that A is mutually weak associative if for all $\circ, \bullet \in A$ and for every $x, y, z \in H$ we have

$$x \circ (y \bullet z) \cap (x \bullet y) \circ y \neq \emptyset.$$

Theorem 2.25. Set $A = \{\circ_{pq}^{rs} | r, s > 0, p, q \in \{b, c\}\}$. Then A is a mutually weak associative.

Proof . For every$x, y, z \in X$ and $p, q, p', q' \in \{b, c\}$ and $r, s, r', s' \geq 0$ we have

$$\{x, y, z\} \subseteq x \circ_{pq}^{rs} (y \circ_{p'q'}^{r's'} z) \cap (x \circ_{p'q'}^{r's'} y) \circ_{pq}^{rs} z.$$

Then A is a mutually weak associative. □

Definition 2.26. Let X and Γ be two non-empty sets. Then X is called a Γ-H_v-group if each $\gamma \in \Gamma$ be a hyperoperation on X such that (X, γ) be an H_v-group and for all $\alpha, \beta \in \Gamma$ and $x, y, z \in X$ we have

$$x\alpha(y\beta z) \cap (x\alpha y)\beta z \neq \emptyset.$$

In fact, Γ has a mutually weak associative law.

Theorem 2.27. Let $\Gamma \subseteq \{\circ_{pq}^{rs} | r, s \in [0, \infty], p, q \in \{b, c\}\}$. Then (X, Γ) is a Γ-H_v-group.

Proof . For all $x, y, z \in X$ and $p, q, p', q' \in \{b, c\}$ and $r, s, r', s' \geq 0$ we have

$$\{x, y, z\} \subseteq x \circ_{pq}^{rs} (y \circ_{p'q'}^{r's'} z) \cap (x \circ_{pq}^{rs} y) \circ_{p'q'}^{r's'} z.$$

By Corollary 2.6, we have $(X, \circ_{pq}^{rs})$ is an H_v-group. Therefore (X, Γ) is a Γ-H_v-group. □

3 Conclusion

Mirvakili et al. obtained hyperstructures on an arbitrary metric space according to the concept of neighbourhood in metric space for the first time. In this article, we continued this method and identified another category of these hyperstructures with a new definition. We also provided the necessary and sufficient conditions for the obtained hypergroups to participate. For new research in this field, it is possible to mention the construction of non-commutative hypergroups from metric spaces and the construction of hypergroups from topological spaces, which are generalizations of metric spaces.

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