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APPLICATION OF THE KALMAN-BUCY FILTER IN THE STOCHASTIC DIFFERENTIAL EQUATIONS FOR THE MODELING OF RL CIRCUIT

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ABSTRACT. In this paper, we present an application of the stochastic calculus to the problem of modeling electrical networks. The filtering problem have an important role in the theory of stochastic differential equations(SDEs). In this article, we present an application of the continuous Kalman-Bucy filter for a RL circuit. The deterministic model of the circuit is replaced by a stochastic model by adding a noise term in the source. The analytic solution of the resulting stochastic integral equations are found using the Ito formula.

1. INTRODUCTION AND PRELIMINARIES

Modeling of physical systems by ordinary differential equations(ODEs), ignores stochastic effects. By adding random elements into the differential equations obtains what is called a Stochastic differential equations(SDEs), and the term stochastic called noise[5]. Then, a SDE is a differential equation in which one or more of the terms is a stochastic process, and resulting in a solution which is itself a stochastic process.

SDEs play a relevant role in many application areas including environmental modeling, engineering, biological modeling and mostly. One of more important application SDE, is in the modeling electrical networks. Application SDEs in the modeling of electrical circuits has been studied by many researchers. W. kampowsky and et al(1992), described classification and numerical simulation of electrical circuits with white noise[3]. In [7], C. Penski(1999), was presented a new numerical method for SDEs with white noise and its application in circuit simulation. T. Rawat(2008), showed an application of the Ito stochastic calculus to the problem of modeling a series RC Circuit with white noise and colored noise, including numerical solution[8]. However, E. kolarova presented an application of stochastic integral equations to RL circuit in (2008)[4]. R. Narayanan and et al(2009), worked on the automated formal verification of analog/RF circuits.

The filtering problem, an important part have in the theory SDEs. Intuitively, the problem is to filter the noise a way from the observations in an optimal way. In

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1960, Kalman and in 1961, Kalman and Bucy proved what is now Know as the Kalman-Bucy filter. Basically the filter gives a procedure for estimating the state of a system which satisfies a noisy linear differential equation, based on a series of noisy observations.
E. Kolarova(2007), presented an application of the Kalman-Bucy filter to RL circuit. In this paper, we will presented an application of the Kalman-Bucy filter to the problem of modeling RL electrical circuit.
For attention, in the next section, we describe SDE and Ito stochastic calculus. In this part 3, we will presented the Kalman-Bucy filters and in the section 4, with using of Kalman-Bucy filter, we will solving the stochastic RL model.

2. Stochastic differential equation

An SDE, is an ordinary differential equation(ODE) with stochastic process that can model unpredictable real-life behavior of any continuous systems[o]. Given the probability space ω, a stochastic process with state space E is a collection $X_t : t \in T$ of random variablesX_t that take values in E for the parameter set T. If T is a countable then the stochastic process is discrete else continuous. Due to statistical properties, a stochastic process can be used to define the randomness in an uncorrelated white gaussian noise which can be thought of as the derivative of brownian motion(or the wiener process)[6].
A general scalar SDE has the form

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \tag{2.1}$$

where $f(t, X(t)) : [0, T] \times R \to R$ and $g(t, X(t)) : [0, T] \times R \to R$ are drift and diffusion coefficient and $W(t)$ is the so called wiener process, a stochastic process representing the noise[4].
We can represent the SDE in the integral form

$$X(t) = x_0 + \int_0^T f(s, X(s))ds + \int_0^T g(s, X(s))dW(s), \tag{2.2}$$

where the integral with respect to ds is the lebesgue integral and the integrals with respect to $dW(s)$ are stochastic integrals, called the Ito integrals[k]. Although the Ito integral has some very convenient properties, the usual chain rule of classical calculus doesn't hold. Instead, the appropriate stochastic chain rule, known as Ito formula, contains an additional term. This additional term, which roughly speaking is due to the fact that the square of the stochastic differential $(dW(t))^2$ is a equal to dt, in the mean square sense, i.e. $E[(dW(t))^2] = dt$. So the second order term in $dW(t)$ should really appear as a first order term in dt.
Suppose $X(t)$ be a solution of the SDEs (1), for some suitable functions f, g. Let $g(t, x) : (0, \infty) \times R \to R$ be a twice continuously differentiable function. The function

$$Y(t) = g(t, X(t)),$$

is a stochastic process, for which

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2, \tag{2.3}$$

where $(dX(t))^2 = dX(t).dX(t)$ is computed according to the rules

$$dt.dt = dt.dW(t) = dW(t).dt = 0, \quad dW(t).dW(t) = dt$$

3. THE KALMAN-BUCY FILTER

Suppose the $X(t)$ at time t of a system is given by a SDE

$$\frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, X(t))W(t), \ t \geq 0, \tag{3.1}$$

where b and σ satisfy conditions

$$|b(t,x)|+|\sigma(t,x)| \leq C(1+|x|); \ |b(t,x)-b(t,y)|+|\sigma(t,x)-\sigma(t,y)| \leq D|x-y|; \ \ x, y \in R, \ t \in [0,T]$$

for some constant C, D and $W(t)$ is white noise. As discussed earlier the Ito interpretation of this equation is

$$(system) \ \ dX(t) = b(t, X(t))dt + \sigma(t, X(t))dU(t), \tag{3.2}$$

where $U(t)$ is Brownian motion. We also assume that the distribution of $X(0)$ is known and independent of $U(t)$.
In the continuous version of the filtering problem we assume that the observations H_t are performed continuously and are of the form

$$H_t = c(t, X(t)) + \gamma(t, X(t)).\tilde{W}(t), \tag{3.3}$$

where c, γ are functions satisfying

$$|c(t,x)| + |\gamma(t,x)| \leq C(1+|x|); \ x \in R, \ \ t \in [0,T]$$

and $\tilde{W}(t)$ denotes white noise, independent of $U(t)$ and X_0.
To obtain a tractable mathematical interpretation of (6) we introduce

$$Z_t = \int_0^t H_s ds$$

and thereby we obtain the stochastic integral representation

$$(observations) \quad dZ_t = c(t, X(t))dt + \gamma(t, X(t))dV(t), \ Z_0 = 0 \tag{3.4}$$

where $V(t)$ is Brownian motion, independent of $U(t)$ and X_0. Note that if H_s is known for $0 \leq s \leq t$, then Z_s is also known for $0 \leq s \leq t$ and conversely. So no information is lost or gained by considering Z_t as our "observations" instead of H_t. But this allows us to obtain a well-defined mathematical model of the situation. Given the observations Z_s satisfying (7) for $0 \leq s \leq t$, what is the best estimate $\hat{X(t)}$ of the state $X(t)$ of the system (5) based on these observations? Then, it is necessary to find a precise mathematical formulation of this problem.

Theorem 3.1. *The solution $\hat{X(t)}$ of the 1-dimensional linear filtering problem*

$$(linear\ system) \quad dX(t) = F(t)X(t)dt + C(t)X(t)dU(t);\ F(t), C(t) \in \boldsymbol{R} \tag{3.5}$$

$$(linear\ observations) \quad dZ_t = G(t)X(t)dt + D(t)dV(t);\quad G(t), D(t) \in \boldsymbol{R} \tag{3.6}$$

(with conditions as stated earlier) satisfies the SDE

$$d\hat{X(t)} = (F(t) - \frac{G^2(t)S(t)}{D^2(t)})\hat{X(t)}dt + \frac{G(t)S(t)}{D^2(t)}dZ(t);\ \hat{X(0)} = E[X(0)] \tag{3.7}$$

where
$S(t) = E[(X(t) - \hat{X(t)})^2]$ *satisfies the (deterministic) Ricati equation*

$$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)S^2(t)}{D^2(t)} + C^2(t),\ S(0) = E[(X(0) - E[X(0)])^2].$$

For precise formulation and for the proof of this theorem see[6], page 81-97.

4. THE DETERMINISTIC MODEL

Any electrical circuit consists of resistor(R), capacitor(C) and inductor(L). These circuit elements can be combined to form an electrical circuits in four distinct ways: the RC, RL, LC and RLC circuits. Then, a inductor-resistor circuit(RL), is an electric circuit composedof resistor and inductor driven by a voltage or current source. The ODE describing the behavior of the RL circuit is given by Kirchhoff's current law :

$$L\frac{dI}{dt} + RI(t) = V(t),\ I(0) = I_0 \tag{4.1}$$

where the resistance R and the inductance L are constants and $V(t)$ denotes the potential source at time t.
If $V(t)$ is a piecewise continuous function, the solution of the first order linear differential equation (11)is:

$$I(t) = I(0)\exp(\frac{-Rt}{L}) + \frac{1}{L}\int_0^t \exp(-\frac{R(t-s)}{L})V(s)ds$$

Now let us allow some randomness in the potential source. Then voltage may not be deterministic but of the form:

$$V^*(t) = V(t) + "noise" = V(t) + \alpha\xi(t) \tag{4.2}$$

where $\xi(t)$ is a white noise process of mean zero and variance one, and α is nonnegative constant, known as the intensity of noise.
To be able to substitute this into the equation of the circuit we have to describe mathematically the "noise". It is reasonable to look at it as a stochastic process $\xi(t)dt$ by a term $dW(t)$. There $W(t)$ is the Wiener process. Formally the "white noise" is

the time derivative of the Wiener process $W(t)$. *We get a SDE*

$$dI(t) = (\frac{1}{L}V(t) - \frac{R}{L}I(t))dt + \frac{\alpha}{L}dW(t), \tag{4.3}$$

to solve analytical this equation we compute, using the Ito formula (6), the derivative of the function

$$g(t, I(t)) = e^{\frac{Rt}{L}} I(t),$$
$$dg(t, I(t)) = e^{\frac{Rt}{L}} \frac{V(t)}{L} dt + e^{\frac{Rt}{L}} \frac{\alpha}{L}\ dW(t).$$

From this we get the solution

$$I(t) = e^{\frac{Rt}{L}} I(0) + \frac{1}{L}\int_0^t e^{\frac{R(s-t)}{L}} V(s)ds + \frac{\alpha}{L}\int_0^t e^{\frac{R(s-t)}{L}} dW(s). \tag{4.4}$$

the solution $I(t)$ *is a random process and for it's expectation we have for every* $t > 0$,

$$m(t) = E[I(t)] = e^{\frac{-Rt}{L}} E[I_0] + \frac{1}{L}\int_0^t V(s) e^{\frac{R(s-t)}{L}} ds. \tag{4.5}$$

The second moment $D(t) = E[I(t)^2]$ *can be computed as a solution of the ordinary differential equation*

$$\frac{dD(t)}{dt} = (-\frac{2R}{L})D(t) + 2m(t)\frac{V(t)}{L} + \frac{\alpha^2}{L^2}, \tag{4.6}$$

The solution $I(t)$ *is a Gaussian process, that* $I(t)$ *is distributed* $N(m(t), \sigma^2(t))$, *where* $\sigma^2(t) = E[I(t)^2] - m^2(t)$. *Based on the properties of the normal distribution, we can compute in any* t, *that*

$$P(|I(t) - m(t)| < 1.96\sigma(t)) = 2\phi(1.96) - 1 = 0.95, \tag{4.7}$$

where

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds$$

let us provide some measurement of the current continuously up to time $s \leq t$, $t > 0$. *As described above, we can get from this measurement the observation equation*

$$dZ(t) = Q(t)dt + dU(t), \tag{4.8}$$

Now, we face to the filtering problem: To find the best estimate of the current $\hat{Q}(t)$, *under observations (9), while the equation (1) holds. According to the Kalman-Bucy filter, we have*

$$d\hat{I}(t) = (\frac{-R}{L} - S(t))\hat{I}(t)dt + S(t)dZ(t),\ \hat{I}(0) = E(I_0) = 0 \tag{4.9}$$

where $S(t) = E[(I(t) - \hat{I}(t))^2]$ *satisfies the deterministic Riccati equation*

$$S'(t) = \frac{-2R}{L}S(t) - S^2(t) + \frac{\sigma^2}{L^2},\ S(0) = E((I_0)^2) = A^2 \tag{4.10}$$

(see theorem 1). To solve the Riccati equation, we substitute in (10),

$$S(t) = M(t) - \frac{R}{L},\ S'(t) = M'(t),\ S^2(t) = M^2(t) - \frac{2R}{L}M(t) + \frac{R^2}{L^2},$$

we get a separable ordinary differential equation for the function $M(t)$,

$$M'(t) = \frac{R^2+\sigma^2}{L^2} - M^2(t) \Rightarrow \frac{dM(t)}{dt} = \nu^2 - M^2(t) \Rightarrow \frac{dt}{dM(t)} = \frac{1}{\nu^2 - M^2(t)},$$

where $\nu = \frac{\sqrt{R^2+\sigma^2}}{L}$.
That then of solve this equation, we get the implicit solution

$$\frac{-1}{2} ln|\frac{M(t)-\nu}{M(t)+\nu}| = t + k \Rightarrow M(t) = \nu\frac{e^{-2(t+k)}+1}{e^{-2(t+k)}-1}$$

then:

$$S(t) = M(t) - \frac{R}{L} = \nu\frac{1+e^{-2(t+k)}}{e^{-2(t+k)-1}} - \frac{R}{L},$$

However, $k = \frac{1}{2}Ln|\frac{S(0)+\frac{R}{L}+\nu}{S(0)+\frac{R}{L}-\nu}|$ and for large t we have $M(t) \approx \nu$ and $S(t) = \nu - \frac{R}{L}$. We substitute this to the equation (17) and get the following SDE for the filter

$$d\hat{I}(t) = (\frac{-R}{L} - \nu + \frac{R}{L})\hat{I}(t)dt + (\nu - \frac{R}{L})dZ(t),\ \hat{I}(0) = E(I_0) = 0$$

so

$$d\hat{I}(t) = -\nu\hat{I}(t)dt + (\nu - \frac{R}{L})dZ(t). \tag{4.11}$$

This equation can be solved using the Ito formula for the function $h(t,x) = e^{\nu t}x$. We have

$$d(e^{\nu t}\hat{I}(t)) = \nu e^{\nu t}\hat{I}(t)dt + e^{\nu t}d\hat{I}(t) = e^{\nu t}(\nu - \frac{R}{L})dZ(t)$$
$$\Rightarrow e^{\nu t}\hat{I}(t) = \int_0^t (\nu - \frac{R}{L})e^{\nu s}dZ(s)$$

Thus

$$\hat{I}(t) = \nu(1 - \frac{R}{L})\int_0^t e^{\nu(s-t)}dZ(s)$$

is the solution of the filtering problem for the RL circuit with stochastic source.

References

[1] *L. Arnold, Stochastic Differential Equations, Theory and Applications, Johan Wiley Sons, (1974).*
[2] *D. Halliday, R. Resnick, J. Walker, Fundamentals of Physics, Johan wiley, Sons, (1997).*
[3] *W. Kampowsky, P. Rentrop, W. Schmidt, Classification and numerical simulation of electric circuits, Surveys Math. Indust, 2(1992)23-65.*
[4] *E. Kolarova, An application of Stochastic Integral Equations to electrical networks, Acta Electrotechnica et informatica. Vol. 8. No. 3. (2008), pp. 14-17.*
[5] *P.E. Kloeden, E. Platen. Numerical solution of Stochastic Differential Equations, Springer, Berlin, (1995).*
[6] *B. Oksendal, Stochastic Diferential Equations, an introduction with applications, Springer-Verlage, (2000).*
[7] *C. Penski, A new numerical method for SDEs and its application in circuit simulation, Com. and Appl. Math, 115(2000), 461-470.*
[8] *T. K. Rawat, H. Parthasarathy, Modeling of an RC circuit using a SDEs, Int. J. SC. Tech, vol. 13, No. 2. April-june(2008).*

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