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On the existence of solution to a class of nonlinear functional integral equations with two variables

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Abstract

In this article, the existence of a solution for non-linear functional integral equations with two variables is considered in Banach space $C([0, b] \times [0, c])$ by applying Petryshyn's fixed point theorem. Our focus extends to diverse instances of functional integral equations encountered within mathematical analysis. Our study's effectiveness is demonstrated through an example. Furthermore, to confirm the reliability of our proposed approach, we introduce an iterative algorithm via Sinc interpolation, which effectively achieves a precise, approximate solution.

Keywords: Fixed point theorem (FPT), Measure of non-compactness (MNC), Functional integral equation
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1 Introduction

Functional integral equations (FIEs) stand as a pivotal bridge between algebraic systems and continuous functions in the domain of mathematical analysis and its various practical applications. FIEs with two variables, in particular, hold a distinct significance due to their capacity to model complex real-world phenomena that extend across multiple dimensions. These equations offer a flexible framework for addressing problems involving interactions and dependencies between variables, often encountered in physics, engineering, economics, and other disciplines.

This paper delves into the functional integral equations with two variables, unravelling some mathematical foundations, properties, and a solution technique. As these equations find their roots in both functional analysis and integral equations, their study amalgamates essential concepts from both fields, making them a captivating subject of research. Moreover, the applicability of these equations spans a wide spectrum—from describing the behaviour of physical systems regulated by partial differential equations to formulating optimization problems with intricate constraints.

FIEs constitute a captivating field within applied non-linear analysis. This domain garners not only the attention of specialists within its realm but also those whose interests span diverse mathematical avenues, intersecting with

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physics, the theory of gases, radiative transfer theory, engineering, and mathematical biology [6, 7, 8, 18, 19, 22, 16]. In recent times, the concept of Monotone Nonlinear Contraction (MNC) has been adopted by several scholars to prove the existence of solutions for FIEs [1, 4, 5, 11, 14, 15, 17, 20, 29].

This study delves into the area of 2-dimensional functional integral equations (2DFIEs), discussing significant results related to their existence. These equations are crafted using densifying operators within Banach spaces. In the following sections, we present an existence result for a specific class of 2DFIEs.

$$z(\varphi,\tau) = F\left(\varphi,\tau,z(\mu(\varphi,\tau)),q(\varphi,\tau,z(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right), \tag{1.1}$$

where $(\varphi,\tau) \in [0,b]\times[0,c]$. The primary aim of this study is to establish the solvability of Eq. (1.1) and derive its analytic solution using the semi-analytic technique. To accomplish this objective, we employ the PFPT method, which is regarded as a broadening of Darbo's fixed theorem [4]. Various researchers have employed Darbo's condition to address the solvability of FIEs (as seen in [9, 13, 12, 21, 27]). We utilize Petryshyn's fixed point theorem to establish the solvability of Eq. (1.1). We now elucidate the key motivations behind investigating Eq. (1.1) and the resultant findings.

The foremost rationale lies in the simplification of conditions found in numerous papers. Furthermore, this paper serves to unify analogous endeavours within this domain. Lastly, the significance of the bounded condition comes to the fore, highlighting that the sub-linear condition, frequently discussed in the literature, plays a less substantial role.

The paper is structured into five sections, preceded by some preliminaries. In the second section, we delve into preliminary concepts and present the MNC concept. Moving to the third section, we demonstrate the solvability via densifying operators using the PFPT method. Section 4 provides illustrative examples showcasing the effectiveness of our approach concerning FIEs. Lastly, the concluding section introduces an iterative technique utilizing the sinc interpolation method, culminating in a closed-form solution of considerable efficiency.

2 Preliminaries

Suppose that E is a real Banach space and $B_\sigma(z)$ is an open ball with the centre z and with the radius σ.

Definition 2.1. [5] Suppose that $G \subseteq E$, the MNC (Kuratowski) of G defined as

$$\alpha(G) = \inf\left\{\delta > 0 \mid G = \bigcup_{k=1}^{n} G_k \text{ with } \operatorname{diam}(G_k) \le \delta,\ k = 1,2,\dots,n\right\}.$$

Definition 2.2. [4] The MNC (Hausdroff) defined as

$$\mu_1(G) = \inf\{\delta > 0 \mid \text{there exists a finite } \delta \text{ net for } G \subset E\}, \tag{2.1}$$

where the expression a finite δ net for $G \subset E$' means as a set $\{z_1, z_2, ..., z_n\} \subset E$, where $B_\delta(E,z_1)$, $B_\delta(E,z_2)$, $\dots, B_\delta(E,z_n)$ over G. This MNC is similar in the form

$$\mu_1(G) \le \alpha(G) \le 2\mu_1(G),$$

for any $G \subset E$.

Theorem 2.3. [26] If $G, \hat{G} \subset E$ and $\theta \in \mathbb{R}$, then

(i) $\mu_1(G) = 0$ if and only if G is relatively compact;

(ii) $\mu_1(G) \le \mu_1(\hat{G})$, for $G \subseteq \hat{G}$;

(iii) $\mu_1(\overline{G}) = \mu_1(\mathrm{Conv}G) = \mu_1(G)$;

(iv) $\mu_1(G \cup \hat{G}) = \max\{\mu_1(G), \mu_1(\hat{G})\}$;

(v) $\mu_1(\theta G) = |\theta|\mu_1(G)$;

(vi) $\mu_1(G + \hat{G}) \le \mu_1(G) + \mu_1(\hat{G})$.

Here, $C([0,b]\times[0,c])$ is the family of all continuous and real valued functions defined on the $[0,b]\times[0,c]$ with

$$\|z\| = \max\{|z(\varphi,\tau)| : \varphi \in [0,b], \tau \in [0,c]\}.$$

Fix a set $G \in C([0,b]\times[0,c])$, for a given $\delta > 0$, the modulus of continuity for z is

$$\omega(z,\delta) = \sup\{|z(\varphi,\tau) - z(\hat{\varphi},\hat{\tau})| : \varphi,\hat{\varphi} \in [0,b], \tau,\hat{\tau} \in [0,c], |\varphi-\hat{\varphi}|, |\tau-\hat{\tau}| \le \delta\}.$$

Further,

$$\omega(G,\delta) = \sup\{\omega(z,\delta) : z \in G\}, \qquad \omega_0(G) = \lim_{\delta\to 0}\omega(G,\delta).$$

In [4], $\omega_0(G)$ is a regular MNC in $C([0,b]\times[0,c])$.

Theorem 2.4. [24] Let $H : E \to E$ be a function that is continuous in E. H is called $\tilde{k}-$set contraction if for every bounded $G \subset E$, $H(G)$ is bounded and $\alpha(HG) \le \tilde{k}\alpha(G), \tilde{k} \in (0,1)$. If

$$\alpha(HG) < \alpha(G), \quad \text{for all } \alpha(G) > 0,$$

then H is called densifying or condensing map.

Theorem 2.5. (Petryshyn's [26]) Let $H : B_\sigma \to E$ be a condensing mapping, such that accomplish the boundary condition

$$\text{If } H(z) = \tilde{k}z, \quad \text{for some } z \in \partial B_\sigma, \quad \text{then } \tilde{k} \le 1,$$

then $\mathbf{F}(H)$, the set of fixed points of H in B_σ, is nonempty.

3 Major Results

Here, we investigate the Eq. (1.1) under these assumptions;

(T_1) $F \in C(I_1 \times \mathbb{R}^2, \mathbb{R})$, $q \in C(I_1, \mathbb{R})$, $h \in C(I_2 \times \mathbb{R}, \mathbb{R})$, where,

$$I_0 = I_b \times I_c, I_1 = \{(\varphi,\tau,z) : 0 \le \varphi \le b, 0 \le \tau \le c, z \in \mathbb{R}\},$$

$$I_2 = \{(\varphi,\tau,\psi,\nu) \in I_0^2 : 0 \le \psi \le \varphi \le b, 0 \le \nu \le \tau \le c\};$$

$$\mu,\phi,\beta : I_0 \to I_0.$$

(T_2) there exist non-negative constants a_1, a_2, a_3 and $a_4, a_1 + a_2a_4 < 1$ such that

$$|F(\varphi,\tau,z_1,z_2,z_3) - F(\varphi,\tau,\hat{z_1},\hat{z_2},\hat{z_3}| \le a_1|z_1-\hat{z_1}| + a_2|z_2-\hat{z_2}| + a_3|z_3-\hat{z_3}|;$$

$$|q(\varphi,\tau,z) - q(\varphi,\tau\hat{z}| \le a_4|z-\hat{z}|;$$

(T_3) there exists a $\sigma > 0$ such that F holds the inequality

$$\sup\{|F(\varphi,\tau,z_1,z_2,z_3)| : (\varphi,\tau) \in I_0, z_1,z_2 \in [-\sigma,\sigma], z_3 \in [-bcN_1, bcN_1]\} \le \sigma,$$

where,

$$N = \sup\{|h(\varphi,\tau,\psi,\nu,z)| : \forall(\varphi,\tau,\psi,\nu) \in I_2 \text{ and } z \in [-\sigma,\sigma]\}.$$

Theorem 3.1. Under the $(T_1)-(T_3)$ with $a_1 + a_2a_4 < 1$, Eq. (1.1) has at least one solution in $E = C(I_0)$.

Proof . Introduce

$$(Hz)(\varphi,\tau) = F\left(\varphi,\tau,z(\mu(\varphi,\tau),q(\varphi,\tau,z(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right).$$

We prove that H is continuous on B_σ. Taking $\delta > 0$ and $z, x \in B_\sigma$ with $\|z - y\| < \delta$,

$$\begin{aligned}
& |(Hz)(\varphi,\tau) - (Hy)(\varphi,\tau)| \\
= & \left|F\left(\varphi,\tau,z(\mu(\varphi,\tau)),q(\varphi,\tau,z(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right. \\
& \left. -F\left(\varphi,\tau,y(\mu(\varphi,\tau)),q(\varphi,\tau,y(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,y(\beta(\psi,\nu)))d\nu d\psi\right)\right| \\
\leq & \left|F\left(\varphi,\tau,z(\mu(\varphi,\tau)),q(\varphi,\tau,z(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right. \\
& \left. -F\left(\varphi,\tau,y(\mu(\varphi,\tau)),q(\varphi,\tau,z(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right| \\
& +\left|F\left(\varphi,\tau,y(\mu(\varphi,\tau)),q(\varphi,\tau,z(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right. \\
& \left. -F\left(\varphi,\tau,y(\mu(\varphi,\tau)),q(\varphi,\tau,y(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right| \\
& +\left|F\left(\varphi,\tau,y(\mu(\varphi,\tau)),q(\varphi,\tau,y(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right. \\
& \left. -F\left(\varphi,\tau,y(\mu(\varphi,\tau)),q(\varphi,\tau,y(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau} h(\varphi,\tau,\psi,\nu,y(\beta(\psi,\nu)))d\nu d\psi\right)\right| \\
\leq & \; a_1|z(\mu(\varphi,\tau) - y(\mu(\varphi,\tau)| + a_2|q(\varphi,\tau,z(\phi(\varphi,\tau)) - q(\varphi,\tau,y(\phi(\varphi,\tau))| \\
& +a_3|\int_0^{\varphi}\int_0^{\tau} |h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu))) - h(\varphi,\tau,\psi,\nu,y(\beta(\psi,\nu)))|d\nu d\psi \\
\leq & \; a_1|z(\mu(\varphi,\tau) - y(\mu(\varphi,\tau)| + a_2a_4|z(\phi(\varphi,\tau)) - y(\phi(\varphi,\tau))| + a_3bc\omega(h,\sigma) \\
\leq & \; (a_1 + a_2a_4)||z - y|| + a_2bc\omega(h,\sigma),
\end{aligned}$$

where

$$\omega(h,\sigma) = sup\{|h(\varphi,\tau,\psi,\nu,z) - h(\varphi,\tau,\psi,\nu,y)| : (\varphi,\tau,\psi,\nu) \in I_2, z,y \in [-\sigma,\sigma], \|z - y\| \leq \delta\}.$$

The function $h(\varphi,\tau,\psi,\nu,z)$ is a uniform continuous function on $I_2 \times [-\sigma,\sigma]$. So $\omega(h,\sigma) \to 0$ as $\delta \to 0$. Therefore, H is continuous on B_σ.

In addition, we show that H satisfied the s densifying map. Choose $\delta > 0$ and any $z \in G$, such that G is bounded subset of E. For $(\varphi_1,\tau_1),(\varphi_2,\tau_2) \in I_0$ with $\varphi_1 \leq \varphi_2, \tau_1 \leq \tau_2$ and $\varphi_1 - \varphi_2, \tau_1 - \tau_2 \leq \delta$,

$$\begin{aligned}
& |(Hz)(\varphi_2,\tau_2) - (Hz)(\varphi_1,\tau_1)| \\
= & \left|F\left(\varphi_2,\tau_2,z(\mu(\varphi_2,\tau_2)),q(\varphi_2,\tau_2,z(\phi(\varphi_2,\tau_2))),\int_0^{\varphi_2}\int_0^{\tau_2} h(\varphi_2,\tau_2,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right. \\
& \left. -F\left(\varphi_1,\tau_1,z(\mu(\varphi_1,\tau_1)),q(\varphi_1,\tau_1,z(\phi(\varphi_1,\tau_1))),\int_0^{\varphi_1}\int_0^{\tau_1} h(\varphi_1,\tau_1,\psi,\nu,z(\beta(\psi,\nu)))d\psi d\nu\right)\right| \\
\leq & \left|F\left(\varphi_2,\tau_2,z(\mu(\varphi_2,\tau_2)),q(\varphi_2,\tau_2,z(\phi(\varphi_2,\tau_2))),\int_0^{\varphi_2}\int_0^{\tau_2} h(\varphi_2,\tau_2,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right. \\
& \left. -F\left(\varphi_2,\tau_2,z(\mu(\varphi_2,\tau_2)),q(\varphi_2,\tau_2,z(\phi(\varphi_2,\tau_2))),\int_0^{\varphi_1}\int_0^{\tau_1} h(\varphi_2,\tau_2,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg|F\left(\varphi_2,\tau_2,z(\mu(\varphi_2,\tau_2)),q(\varphi_2,\tau_2,z(\phi(\varphi_2,\tau_2))),\int_0^{\varphi_1}\int_0^{\tau_1}h(\varphi_1,\tau_1,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\\
&-F\left(\varphi_2,\tau_2,z(\mu(\varphi_2,\tau_2)),q(\varphi_1,\tau_1,z(\phi(\varphi_1,\tau_1))),\int_0^{\varphi_1}\int_0^{\tau_1}h(\varphi_1,\tau_1,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\Bigg|\\
&+\Bigg|F\left(\varphi_2,\tau_2,z(\mu(\varphi_2,\tau_2)),q(\varphi_1,\tau_1,z(\phi(\varphi_1,\tau_1))),\int_0^{\varphi_1}\int_0^{\tau_1}h(\varphi_1,\tau_1,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\\
&-F\left(\varphi_2,\tau_2,z(\mu(\varphi_1,\tau_1)),q(\varphi_1,\tau_1,z(\phi(\varphi_1,\tau_1))),\int_0^{\varphi_1}\int_0^{\tau_1}h(\varphi_1,\tau_1,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\Bigg|\\
&+\Bigg|F\left(\varphi_2,\tau_2,z(\mu(\varphi_1,\tau_1)),q(\varphi_1,\tau_1,z(\phi(\varphi_1,\tau_1))),\int_0^{\varphi_1}\int_0^{\tau_1}h(\varphi_1,\tau_1,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\\
&-F\left(\varphi_1,\tau_1,z(\mu(\varphi_1,\tau_1)),q(\varphi_1,\tau_1,z(\phi(\varphi_1,\tau_1))),\int_0^{\varphi_1}\int_0^{\tau_1}h(\varphi_1,\tau_1,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\Bigg|\\
\leq\;& a_3\left|\int_0^{\varphi_2}\int_0^{\tau_2}h(\varphi_2,\tau_2,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi-\int_0^{\varphi_1}\int_0^{\tau_1}h(\varphi_1,\tau_1,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right|\\
&+a_2|q(\varphi_2,\tau_2,z(\phi(\varphi_2,\tau_2)))-q(\varphi_2,\tau_2,z(\phi(\varphi_1,\tau_1)))|+a_2|q(\varphi_2,\tau_2,z(\phi(\varphi_1,\tau_1)))\\
&-q(\varphi_2,\tau_2,z(\phi(\varphi_2,\tau_2))|+a_1|z(\mu(\varphi_2,\tau_2))-z(\mu(\varphi_2,\tau_2)))|+\omega_1(F,\delta)\\
\leq\;& a_3\int_0^{\varphi_1}\int_0^{\tau_1}|h(\varphi_2,\tau_2,\psi,\nu,z(\beta(\psi,\nu)))-h(\varphi_1,\tau_1,\psi,\nu,z(\beta(\psi,\nu)))|d\nu d\psi\\
&+a_3\int_{\varphi_1}^{\varphi_2}\int_0^{\tau_1}|h(\varphi_2,\tau_2,\psi,\nu,z(\beta(\psi,\nu)))|d\nu d\psi+\omega_1(F,\delta)\\
&+a_3\int_0^{\varphi_1}\int_{\tau_1}^{\tau_2}|h(\varphi_2,\tau_2,\psi,\nu,z(\beta(\psi,\nu)))|d\nu d\psi+a_2a_4|z(\phi(\varphi_2,\tau_2))-z(\phi(\varphi_1,\tau_1))|\\
&+a_1|z(\mu(\varphi_2,\tau_2))-z(\mu(\varphi_2,\tau_2)))|+a_3\int_{\varphi_1}^{\varphi_2}\int_{\nu_1}^{\nu_2}|h(\varphi_2,\tau_2,\psi,\nu,z(\beta(\psi,\nu)))|d\nu d\psi+a_2\omega_1(q,\delta).
\end{aligned}
$$

To clarify, we have

$$\omega_1(q,\delta)=\sup\{|q(\varphi,\tau,z)-q(\hat{\varphi},\hat{\tau},z)|:|\varphi-\hat{\varphi}|\leq\delta,|\tau-\hat{\tau}|\leq\delta,z\in[-\sigma,\sigma]\},$$

$$\omega_1(h,\delta)=\sup\{|h(\varphi,\tau,\psi,\nu,z)-h(\hat{\varphi},\hat{\tau},\psi,\nu,z)|:|\varphi-\hat{\varphi}|\leq\delta,|\tau-\hat{\tau}|\leq\delta,(\varphi,\tau,\psi,\nu)\in I_2,z\in[-\sigma,\sigma]\},$$

$$\omega_1(F,\delta)=\sup\{|F(\varphi,\tau,z_1,z_2,z_3)-q(\hat{\varphi},\hat{\tau},z_1,z_2,z_3)|:|\varphi-\hat{\varphi}|\leq\delta,|\tau-\hat{\tau}|\leq\delta,z_1,z_2\in[-\sigma,\sigma],z_3\in[-bcN_1,bcN_1]\}.$$

From above quantities, we have

$$
\begin{aligned}
|(Hz)(\varphi_2,\tau_2)-(Hz)(\varphi_1,\tau_1)| \leq\;& (a_1+a_2a_4)\|z(\varphi_2,\tau_2)-z(\varphi_1,\tau_1)\|+a_2\omega_1(q,\delta)+\omega_1(F,\delta)\\
&+a_3bc\omega_1(F,\delta)+\delta a_3cN+\delta a_3bN+\delta^2a_3N.
\end{aligned}
$$

Taking $\delta\to 0$, we have

$$\omega(Hz,\delta)\leq(a_1+a_2a_4)\omega(z,\delta).$$

This yields to

$$\mu_1(HG)\leq(a_1+a_2a_4)\mu_1(G),$$

Consequently H is a condensing map. At this time, let $z\in\partial B_\sigma$ and if $Hz=\tilde{k}z$, then, $||Hz||=\tilde{k}||z||=\tilde{k}\sigma$ and from(T_3), we get

$$|Hz(\varphi,\tau)|=\left|F\left(\varphi,\tau,z(\mu(z(\varphi,\tau)),q(\varphi,\tau,z(\phi(\varphi,\tau))),\int_0^{\varphi}\int_0^{\tau}h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\right)\right|\leq\sigma$$

for all $(\varphi,\tau)\in I_0$, hence $||Hz||\leq\sigma$, this means that $\tilde{k}\leq 1$. $\square$

Corollary 3.2. Let

(c_1) $F_1 \in C(I_1 \times \mathbb{R}, \mathbb{R})$, $q \in C(I_1, \mathbb{R})$, $h \in C(I_2 \times \mathbb{R}, \mathbb{R})$,

(c_2) there exist non-negative constants $a_1, a_2, a_3, a_1 a_3 < 1$ such that

$$|F_1(\varphi, \tau, z_1, z_2) - F_1(\varphi, \tau, \hat{z_1}, \hat{z_2}| \leq a_1|z_1 - \hat{z_1}| + a_2|z_2 - \hat{z_2}|;$$

$$|q(\varphi, \tau, z) - q(\varphi, \tau, \hat{z}| \leq a_3|z - \hat{z}|;$$

(c_3) there exists a $\sigma > 0$ such that F_1 does the following inequality

$$\sup\{|F_1(\varphi, \tau, z_1, z_2)| : (\varphi, \tau) \in I_0, z_1 \in [-\sigma, \sigma], z_2 \in [-bcN_1, bcN_1]\} \leq \sigma,$$

where,

$$N = sup\{|h(\varphi, \tau, \psi, \nu, z)| : \forall(\varphi, \tau, \psi, \nu) \in I_2 \ \ and \ \ z \in [-\sigma, \sigma]\}.$$

Then,

$$z(\varphi, \tau) = F_1\left(\varphi, \tau, q(\varphi, \tau, z(\phi(\varphi, \tau))), \int_0^{\varphi}\int_0^{\tau} h(\varphi, \tau, \psi, \nu, z(\beta(\psi, \nu)))d\nu d\psi\right), \tag{3.1}$$

has at least one solution in $C(I_0)$.

The proof of this corollary is linked to the Theorem 3.1.

Corollary 3.3. Let

(S_1) $F_2 \in C(I_1 \times \mathbb{R}, \mathbb{R})$, $q \in C(I_1, \mathbb{R})$, $h \in C(I_2 \times \mathbb{R}, \mathbb{R})$,

(S_2) there exist non-negative constants f_1, f_2 such that

$$|q(\varphi, \tau, 0)| \leq f_1, \qquad |F_2(\varphi, \tau, 0, 0)| \leq f_2;$$

(S_3) there exist non-negative constants $0 < a_1, a_2 < 1$ such that

$$|F_2(\varphi, \tau, z_1, z_2) - F_2(\varphi, \tau, \hat{z_1}, \hat{z_2}| \leq a_1|z_1 - \hat{z_1}| + a_2|z_2 - \hat{z_2}|;$$

$$|q(\varphi, \tau, z) - q(\varphi, \tau, \hat{z}| \leq a_3|z - \hat{z}|;$$

(S_4) there exist non-negative constants d_1, d_2 such that

$$|h(\varphi, \tau, \psi, \nu, z)| \leq d_1 + d_2|z|;$$

(S_5) $a_1 + a_3 + a_2 bcd_2 < 1$.

Then,

$$z(\varphi, \tau) = q(\varphi, \tau, z(\phi(\varphi, \tau))) + F_2\left(\varphi, \tau, z(\mu(\varphi, \tau)), \int_0^{\varphi}\int_0^{\tau} h(\varphi, \tau, \psi, \nu, z(\beta(\psi, \nu)))d\nu d\psi\right), \tag{3.2}$$

at least has one solution in $C(I_0)$.

Proof . Let $\sigma = \frac{N_2}{1-N_1}$, where $N_1 = a_1 + a_3 + a_2 bcd_2, N_2 = \mu + a_2 bcd_1 + \nu$, and

$$F(\varphi, \tau, z_1, z_2, z_3) = z_1 + F_2(\varphi, \tau, z_2, z_3),$$

where,

$$z_1 = q(\varphi, \tau, z(\phi(\varphi, \tau))), z_3 = \int_0^{\varphi}\int_0^{\tau} h(\varphi, \tau, \psi, \nu, z(\beta(\psi, \nu)))d\nu d\psi.$$

It is note that (T_2) is handled by (S_2). Now, we show that (S_3) is fulfill, we have

$$|z(\varphi, \tau)| \ = \ \left|q(\varphi, \tau, z(\phi(\varphi, \tau))) + F_2\left(\varphi, \tau, z(\mu(\varphi, \tau)), \int_0^{\varphi}\int_0^{\tau} h(\varphi, \tau, \psi, \nu, z(\beta(\psi, \nu)))d\nu d\psi\right)\right|,$$

$$
\begin{aligned}
&\leq |q(\varphi,\tau,z(\phi(\varphi,\tau)))-q(\varphi,\tau,0)|+|q(\varphi,\tau,0)| \\
&\quad +a_1|z(\mu(\varphi,\tau))|+a_2\Big|\int_0^{\varphi}\int_0^{\tau}h(\varphi,\tau,\psi,\nu,z(\beta(\psi,\nu)))d\nu d\psi\Big|+|F_2(\varphi,\tau,0,0)|, \\
&\leq a_3\|z\|+a_1\|z\|+f_1+a_2bc(d_1+d_2\|z\|)+f_2, \\
&\leq (a_1+a_3+a_2bcd_2)\|z\|+f_1+a_2bcd_1+f_2,
\end{aligned}
$$

for all $(\varphi,\tau)\in I_0$, hence

$$\sup|F(\varphi,\tau,z_1,z_2,z_3)|\leq N_1\sigma+N_2=N_1\frac{N_2}{1-N_1}+N_2=\sigma.$$

□

Corollary 3.4. [10] Let

(E_1) $A\in C(I_0,\mathbb{R}), F_3\in C(I_1\times\mathbb{R},\mathbb{R})$, $h\in C(I_2\times\mathbb{R},\mathbb{R})$,

(E_2) there exist non-negative constants l_1,l_2 such that

$$|A(\varphi,\tau)|\leq l_1,\qquad |F_3(\varphi,\tau,0,0)|\leq l_2;$$

(E_3) there exist non-negative constants $0<a_1,a_2<1$ such that

$$|F_3(\varphi,\tau,z_1,z_2)-F_3(\varphi,\tau,\hat{z_1},\hat{z_2}|\leq a_1|z_1-\hat{z_1}|+a_2|z_2-\hat{z_2}|;$$

(E_4) such that non-negative constants d_1,d_2 such that

$$|h(\varphi,\tau,z_1,z_2,z_3)|\leq b_1+b_2|z|;$$

(E_5) $a_1+a_2bcb_2<1$.

Then,

$$z(\varphi,\tau)=A(\varphi,\tau)+F_3\left(\varphi,\tau,z(\varphi,\tau),\int_0^{\varphi}\int_0^{\tau}h(\varphi,\tau,\psi,\nu,z(\psi,\nu))d\nu d\psi\right),\tag{3.3}$$

at least has one solution in $C(I)=C([0,1]\times[0,1])$.

Proof . Let $\hat{r}=\frac{P_2}{1-P_1}$, where $P_1=a_1+a_2bcb_2, P_2=a_2bcb_1+l_2+l_1$, and

$$F(\varphi,\tau,z_1,z_2,z_3)=A(\varphi,\tau)+F_3(\varphi,\tau,z_1,z_3),$$

where,

$$z_3=\int_0^{\varphi}\int_0^{\tau}h(\varphi,\tau,\psi,\nu,z(\psi,\nu))d\nu d\psi.$$

It is note that (T_2) is handled by (E_2). Condition (E_3) is fulfill, we get

$$
\begin{aligned}
|z(\varphi,\tau)| &= \Big|A(\varphi,\tau)+F_3\left(\varphi,\tau,z(\varphi,\tau),\int_0^{\varphi}\int_0^{\tau}h(\varphi,\tau,\psi,\nu,z(\psi,\nu))d\nu d\psi\right)\Big| \\
&\leq \Big|q_3\left(\varphi,\tau,z(\varphi,\tau),\int_0^{\varphi}\int_0^{\tau}h(\varphi,\tau,\psi,\nu,z(\psi,\nu))d\nu d\psi\right)-q_3(\varphi,\tau,0,0)\Big|+|F_3(\varphi,\tau,0,0)|+|A(\varphi,\tau)|, \\
&\leq a_1|z(\varphi,\tau)|+a_2\Big|\int_0^{\varphi}\int_0^{\tau}h(\varphi,\tau,\psi,\nu,z(\psi,\nu))d\nu d\psi\Big|+|F_3(\varphi,\tau,0,0)|+|A(\varphi,\tau)|, \\
&\leq a_1||z||+a_2bc(b_1+b_2|z|)+l_2+l_1, \\
&\leq (a_1+a_2bcb_2)||z||+a_2bcb_1+l_2+l_1,
\end{aligned}
$$

for all $(\varphi,\tau)\in I_0$, hence

$$\sup|F(\varphi,\tau,z_1,z_2,z_3)|\leq P_1\sigma+P_2=P_1\frac{P_2}{1-P_1}+P_2=\sigma.$$

□

4 Applications

In this part two examples for two-dimensional functional integral equations are considered.

Example 4.1. In this example we consider the following functional integral equation

$$z(\varphi,\tau)=q(\varphi,\tau)+\int_0^{\varphi}\int_0^{\tau}P(\varphi,\tau,\psi,\nu)Q(\psi,\nu,z(\psi,\nu))d\nu d\psi,$$

where $h(\varphi,\tau,\psi,\nu,z(\psi,\nu))=P(\varphi,\tau,\psi,\nu)Q(\psi,\nu,z(\psi,\nu))$, which is a generalization for the two-dimensional Hammerstein integral equation [25]

$$z(\varphi,\tau)=q(\varphi,\tau)+\int_0^{1}\int_0^{1}h(\varphi,\tau,\nu,\psi,z(\nu,\psi))d\nu d\psi.$$

It is the well-known 2D Fredholm integral equation which treated by different authors; for instance see [3].

Example 4.2. In this example, we take into account to the following 2DFIE

$$z(\varphi,\tau)=\frac{e^{-\varphi\tau}}{2(1+\varphi^2\tau^2)}\cos\big(\frac{\varphi\tau}{3}\big)+\frac{1+\varphi^2\tau^2}{4(2+3\varphi^4\tau^2)}\sin z(\varphi,\tau)+\frac{1}{2}\int_0^{\varphi}\int_0^{\tau}\psi\nu\sin z(\psi,\nu)d\nu d\psi \tag{4.1}$$

for $(\varphi,\tau)\in[0,1]\times[0,1]=I$. Here, we get

$$\phi(\varphi,\tau)=\beta(\varphi,\tau)=(\varphi,\tau),$$

$$F(\varphi,\tau,z_1,z_2,z_3)=\frac{e^{-\varphi\tau}}{2(1+\varphi^2\tau^2)}\cos(\frac{\varphi\tau}{3})+\frac{1}{4}z_2+\frac{1}{2}z_3,$$

$$q(\varphi,\tau,z)=\frac{1+\varphi^2\tau^2}{2+3\varphi^4\tau^2}\sin z(\varphi,\tau),$$

$$h(\varphi,\tau,\psi,\nu,z)=\psi\nu\sin z(\psi,\nu).$$

It can simply mark that F,q are continuous functions, and

$$|F(\varphi,\tau,z_1,z_2,z_3)-F(\varphi,\tau,\hat{z_1},\hat{z_2},\hat{z_3})|\le\frac{1}{4}|z_2-\hat{z_2}|+\frac{1}{2}|z_3-\hat{z_3}|,$$

$$|q(\varphi,\tau,z)-q(\varphi,\tau,\hat{z}|\le\frac{1}{2}|z-\hat{z}|.$$

Here $a_2=\frac{1}{4}$, $a_3=\frac{1}{2}$ and $a_4=\frac{1}{2}$. It is clear that the functions hold the (T_1) and (T_2). Now one may control that the (T_3) also holds. Take $\sigma=\frac{4}{3}$, then we get $N\le 1$ and for $\varphi,\tau\in[0,1],\ z_2\in[-\frac{4}{3},\frac{4}{3}],\ z_3\in[-1,1]\}$

$$\sup\{|F(\varphi,\tau,z_1,z_2,z_3)|\le\sup\left\{|\frac{e^{-\varphi\tau}}{2(1+\varphi^2\tau^2)}\cos(\frac{\varphi\tau}{3})+\frac{1}{4}z_2+\frac{1}{2}z_3|\right\}\le\frac{4}{3}.$$

Hence, $(T_1)-(T_3)$ are fulfill. Thus, by Theorem 3.1, the Eq. (1.1) has a solution in $C(I)$.

5 An iterative technique

Deriving the analytic solution for Eq. (4.1) proves to be a complex undertaking. Consequently, resorting to numerical techniques offers a viable approach to approximate its solution. Several numerical methods, rooted in collocation, expansion, and projection techniques, have been documented in literature [23, 28, 31, 32]. These methods typically transform the non-linear complexities into an algebraic system with indeterminate coefficients. However, our approach diverges by employing 2D sinc interpolation to devise an iterative method for approximating the solution.

The distinctive facet of our technique lies in its divergence from the conventional practice of expressing the nonlinear problems as algebraic systems through the expansion of $z(s,t)$ using Sinc functions featuring unknown coefficients. In contrast, our technique streamlines computations by obviating the need for such conversions. Additionally, our method boasts exponential precision, akin to the exponential accuracy observed in [30]. This translates to significantly enhanced computational efficiency, a characteristic that sets our approach apart. Let us to consider sinc function [30]:

$$sinc(\tau) = \begin{cases} \dfrac{\sin(\pi\tau)}{\pi\tau}, \tau \neq 0 \\ 1, \qquad \tau = 0. \end{cases} \tag{5.1}$$

For step size $\hat{h} > 0$, the k'th sinc function is introduced by

$$S(k,\hat{h})(\tau) = sinc(\frac{\tau - k\hat{h}}{\hat{h}}), \quad k = 0, \pm 1, \pm 2, \ldots \tag{5.2}$$

and therefore, easily we conclude that

$$S(k,\hat{h})(j\hat{h}) = \delta_{kj} = \begin{cases} 1, k = j \\ 0, k \neq j. \end{cases} \tag{5.3}$$

Assume v is a function defined over $\mathbb{R}$, then for $\hat{h} > 0$ the series,

$$C(v,\hat{h})(\tau) = \sum_{k=-\infty}^{\infty} v(k\hat{h}) S(k,\hat{h})(\tau), \tag{5.4}$$

is called the Whittaker cardinal expansion of function v, which is convergent (see[30]). Obviously by (5.3-5.4) cardinal function interpolates function v at the points $\{k\hat{h}\}_{k=-\infty}^{\infty}$. Because the subintervals for integrating in (4.1) are $[0, \varphi]$ and $[0, \tau]$ with $\varphi, \tau \in [0, 1]$, then for applying Whittaker cardinal expansion we present the following conformal map:

$$\begin{aligned} &\phi : [0,1] \longrightarrow (-\infty, \infty) \\ &\quad t \longrightarrow \ln(\tfrac{\tau}{1-\tau}) \end{aligned} \tag{5.5}$$

easily, we have $\lim\limits_{\tau\to 0} \phi(\tau) = -\infty$ and $\lim\limits_{\tau\to 1} \phi(\tau) = \infty$. By (5.2) and (5.5) a two dimensional combination of the functions $S(k,\hat{h})$ and ϕ is given by $S(k,\hat{h})o\phi(\varphi)S(k',\hat{h})o\phi(\tau)$ with domain $[0,1] \times [0,1]$. Thus the integrand function $z(\psi,\nu)$ in (4.1) may be approximated by interpolation $S(k,\hat{h})o\phi(\psi)S(k',\hat{h})o\phi(\nu)$ as follows,

$$z_n(\psi,\nu) = \sum_{k=-N}^{N} \sum_{k'=-N}^{N} z(k\hat{h}, k'\hat{h}) S(k,\hat{h})o\phi(\psi) S(k',\hat{h})o\phi(\nu). \tag{5.6}$$

Considering (5.6) and (5.3) if $\phi(\psi) = k\hat{h}$ and $\phi(\nu) = k'\hat{h}$, for $k, k' = -N, \ldots, N$, then $z_n(k\hat{h}, k'\hat{h}) = z(k\hat{h}, k'\hat{h})$. That is (5.6) is interpolation of the function z which the interpolating points are,

$$\begin{aligned} u_k &= \phi^{-1}(k\hat{h}) = \frac{e^{k\hat{h}}}{1+e^{k\hat{h}}}, \quad k = -N+1, \ldots, N, \quad \varphi_{-N} = 0 \\ s_{k'} &= \phi^{-1}(k'\hat{h}) = \frac{e^{k'\hat{h}}}{1+e^{k'\hat{h}}}, \quad k' = -N+1, \ldots, N, \quad t_{-N} = 0. \end{aligned} \tag{5.7}$$

Similar to [30], by using (5.4-5.7), we compute the integral on $[0,\varphi] \times [0,\tau]$, for $\varphi, \tau \in [0,1]$ as

$$\int_0^{\varphi} \int_0^{\tau} z(\psi,\nu) d\nu d\psi \approx \hat{h}^2 \sum_{k=-N}^{N} \sum_{k'=-N}^{N} \frac{z(\psi_k, \nu_{k'})}{\phi'(\psi_k)\phi'(\nu_{k'})}, \tag{5.8}$$

where, $\phi'(\varsigma_k) = \dfrac{1}{\varsigma_k(1-\varsigma_k)}, \varsigma = \psi, \nu$. Thus, (5.8) is an extended results of [2] in two dimensional.

Now, with the help of (5.6-5.8) an iterative algorithm to solve E.q (4.1) is given.

Algorithm

$$
\begin{aligned}
&z_0(\varphi,\tau)=0,\\
&z_{n+1}(\varphi,\tau)=\frac{e^{-\varphi\tau}}{2(1+\varphi^2\tau^2)}\cos\left(\frac{\varphi\tau}{3}\right)+\frac{1+\varphi^2\tau^2}{4(2+3\varphi^4\tau^2)}\sin(z_n(\varphi,\tau))\\
&\qquad+\frac{\hat{h}^2}{2}\sum_{k=-N}^{N}\sum_{k'=-N}^{N}(\psi_k\nu_{k'})^2(1-\psi_k)(1-\nu_{k'})z_n(\psi_k,\nu_{k'}),\quad n=0,1,2,\ldots
\end{aligned}
\tag{5.9}
$$

where the points ψ_k and $\nu_{k'}$ for $k,k^{'}=-N,\ldots,N$ are given by (5.7). For some values $N=5,10$ and $\hat{h}=\frac{\pi}{200}$, we achieve a sequence of approximate solutions $z_i(\varphi,\tau)$ for $i=0,1,\ldots$ by (5.9). Because the solution is in Banach space $C(I)=C([0,1]\times[0,1])$, it's enough that we show the above sequence is a Cauchy sequence. To this end consider to the terms of the sequence as follows,

$$
\begin{aligned}
&z_0(\varphi,\tau)=0\\
&z_1(\varphi,\tau)=\frac{e^{-\varphi\tau}\cos\frac{\varphi\tau}{3}}{2+2\varphi^2\tau^2}\\
&z_2(\varphi,\tau)=0.0000706468+z_1(\varphi,\tau)+\frac{1+\varphi^2\tau^2}{8+12\varphi^4\tau^2}\sin(z_1(\varphi,\tau))\\
&z_3(\varphi,\tau)=0.0000796307+z_1(\varphi,\tau)+\frac{1+\varphi^2\tau^2}{8+12\varphi^4\tau^2}\sin(z_2(\varphi,\tau))\\
&z_4(\varphi,\tau)=0.0000807123+z_1(\varphi,\tau)+\frac{1+\varphi^2\tau^2}{8+12\varphi^4\tau^2}\sin(z_3(\varphi,\tau))\\
&z_5(\varphi,\tau)=0.0000808411+z_1(\varphi,\tau)+\frac{1+\varphi^2\tau^2}{8+12\varphi^4\tau^2}\sin(z_4(\varphi,\tau))\\
&\textit{etc.}
\end{aligned}
\tag{5.10}
$$

Since $\max(\frac{1+\varphi^2\tau^2}{8+12\varphi^4\tau^2})\approx 0.14$ and $\max(\sin(z_1(\varphi,\tau)))\approx 0.47$, for all $(\varphi,\tau)\in I$, (5.10) is a Cauchy sequence. Therefore $\|z_5(\varphi,\tau)-z_4(\varphi,\tau)\|\le 8.7\times 10^{-5}$, and hence we can get $z_5(\varphi,\tau)$ as an approximate solution of (4.1).

6 Conclusion

We established the existence result for non-linear functional integral equations with two variables in $C(I_0)$. Moreover, an example is exhibited to prove the proficiency of our results, and we have created an iterative algorithm by Sinc interpolation to find an approximate solution for the preceding problem with acceptable accuracy.

References

[1] A. Alsaadi, M. Kazemi, and M.M.A. Metwali, *On generalization of Petryshyn's fixed point theorem and its application to the product of n-nonlinear integral equations*, AIMS Math. **8** (2023), no. 12, 30562–30573.

[2] R. Arab, M. Rabbani, and R. Mollapourasl, *On solution of a nonlinear integral equation with deviating argument based the on fixed point technique*, Appl. Comput. Math. **14** (2015), no. 1, 38–49.

[3] A. Babaaghaie and K. Maleknejad, *A new approach for numerical solution of two-dimensional nonlinear Fredholm integral equations in the most general kind of kernel, based on Bernstein polynomials and its convergence analysis*, J. Comput. Appl. Math. **344** (2018), 482–494.

[4] J. Banas and K. Goebel, *Measures of Non-Compactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[5] J. Banas and K. Sadarangani, *Solutions of some functional-integral equations in Banach algebra*, Math. Comput. Modell. **38** (2003), no. 3, 245–250.

[6] A. BenAmar, A. Jeribi, and M. Mnif, *Some fixed point theorems and application to biological model*, Numer. Funct. Anal. Optim. **29** (2008), no. 1–2, 1–23.

[7] J. Caballero, A.B. Mingarelli, and K. Sadarangani, *Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer*, Electron. J. Differ. Equa. **2006** (2006), Paper No. 57, 11.

[8] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, 1991.

[9] M.A. Darwish and S.K. Ntouyas, *On a quadratic fractional Hammerstein–Volterra integral equation with linear modification of the argument*, Nonlinear Anal.: Theory Meth. Appl. **74** (2011), no. 11, 3510–3517.

[10] A. Das, B. Hazarika, and P. Kumam, *Some new generalization of Darbo's fixed point theorem and its application on integral equations*, Mathematics **7** (2019), no. 3, 214.

[11] S. Deb, H. Jafari, A. Das, and V. Parvaneh, *New fixed point theorems via measure of noncompactness and its application on fractional integral equation involving an operator with iterative relations*, J. Inequal. Appl. **2003** (2023), Article number: 106.

[12] A. Deep, Deepmala, and J. Rezaei Roshan, *Solvability for generalized nonlinear functional integral equations in Banach spaces with applications*, J. Integral Equ. Appl. **33** (2021), no. 1, 19–30.

[13] A. Deep, Deepmala, and M. Rabbani, *A numerical method for solvability of some non-linear functional integral equations*, Appl. Math. Comput. **402** (2021), 125637.

[14] A. Deep, Deepmala, and R. Ezzati, *Application of Petryshyn's fixed point theorem to solvability for functional integral equations*, Appl. Math. Comput. **395** (2021), 125878.

[15] A. Deep, A. Kumar, S. Abbas, and B. Hazarika, *An existence result for functional integral equations via Petryshyn's fixed point theorem*, J. Integral Equ. Appl. **34** (2022), no. 2, 165–181.

[16] M. Eshaghi, B. Hayati, M. Kamyar, and H. Khodaei, *On stability and nonstability of systems of functional equations*, Q. Math. **44** (2021), no. 4, 557–567.

[17] B. Hazarika, H.M. Srivastava, R. Arab, and M. Rabbani, *Application of simulation function and measure of noncompactness for solvability of nonlinear functional integral equations and introduction to an iteration algorithm to find solution*, Appl. Math. Comput. **360** (2019), 131–146.

[18] S. Hu, M. Khavanin, and W.A.N. Zhuang, *Integral equations arising in the kinetic theory of gases*, Appl. Anal. **34** (1989), no. 3–4, 261–266.

[19] A. Jeribi, *A nonlinear problem arising in growing cell populations*, Nonlinear Anal.: Real World Appl. **3** (2002), 85–105.

[20] M. Kazemi and R. Ezzati, *Existence of solutions for some nonlinear Volterra integral equations via Petryshyn's fixed point theorem*, Int. J. Nonlinear Anal. Appl. **9** (2018), no. 1, 1—12.

[21] M. Kazemi and M.R. Doostdar, *Existence results for some weakly singular integral equations via measures of non-compactness*, Int. J. Nonlinear Anal. Appl. **15** (2024), no. 2, 301–308.

[22] M. Kunze, *On a special class of nonlinear integral equations*, J. Integral Equ. Appl. **7** (1995), no. 3, 329–350.

[23] H.R. Marzban, H.R. Tabrizidooz, and M. Razzaghi, *A composite collocation method for the nonlinear mixed Volterra–Fredholm–Hammerstein integral equations*, Commun. Nonlinear Sci. Numer. Simul. **16** (2011), no. 3, 1186–1194.

[24] R.D. Nussbaum, *The fixed point index and fixed point theorems for k-set contractions*, PhD thesis, University of Chicago, 1969.

[25] B.G. Pachpatte, *Multidimensional Integral Equations and Inequalities*, Vol. 9, Springer Science & Business Media, 2011.

[26] W.V. Petryshyn, *Structure of the fixed points sets of k-set-contractions*, Arch. Ration. Mech. Anal. **40** (1971), 312–328.

[27] M. Rabbani, A. Deep, and Deepmala, *On some generalized non-linear functional integral equations of two variables via measures of noncompactness and numerical method to solve it*, Math. Sci. **15** (2021), 317–324.

[28] M. Rabbani and S.H. Kiasoltani, *Solving of nonlinear system of Fredholm-Volterra integro-differential equations by using discrete collocation method*, J. Math. Comput. Sci. **3** (2011), no. 4, 382–389.

[29] P. Saini, Ü. Çakan, and A. Deep, *Existence of solutions for 2D nonlinear fractional Volterra integral equations in Banach Space*, Rocky Mountain J. Math. **53** (2022), no. 6, 1965–1981.

[30] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Vol. 20, Springer Science and Business Media, 2012.

[31] K. Wang and Q. Wang, *Lagrange collocation method for solving Volterra–Fredholm integral equations*, Appl. Math. Comput. **219** (2013), no. 21, 10434–10440.

[32] Y.H. Youssri and R.M. Hafez, *Chebyshev collocation treatment of Volterra–Fredholm integral equation with error analysis*, Arab. J. Math. **9** (2020), 471–480.