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Wavelet shrinkage in estimation of regression functions with error in variables

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Abstract

The purpose of this study was to estimate the unknown regression function h in a regression model having errors-in-variables: (Y, X), where $Y = h(U) + E$ and $X = U + T$. We propose a new adaptive estimator through the wavelet shrinkage method to estimate h. In particular, the block thresholding method has been investigated by considering some simple assumptions on E. Finally, using a simulation study, we have compared the proposed estimator with other threshold estimators.

Keywords: Block Thresholding Method, Error in Variable, Nonparametric Regression, Shrinkage Method, Wavelets
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1 Introduction

We are observing n pairs of independent random variables, denoted as $(Y_1, X_1), (Y_2, X_2), ..., (Y_n, X_n)$, in an errors-in-variables regression model. In which value of $\tau \in \{1, 2, .., n\}$, the model is as follows:

$$\begin{aligned} Y_\tau =& h(U_\tau) + \epsilon_\tau \\ X_\tau =& U_\tau + T_\tau \end{aligned} \tag{1.1}$$

Here, we have a scenario where h is an unknown regression function, and we are given n identically and independent i.i.d random variables X_1 through X_n, which follow a uniform distribution of $[0, 1]$. Additionally, T_i and ϵ_i are also i.i.d unobserved random variables, ϵ_i has zero mean. With the known probability density function of T_i denoted as f. We suppose that all these random variables are independent, and ϵ_1 through ϵ_n have finite moments of order 2. Our goal is to estimate the unknown function h based on the observed data $(Y_1, X_1), (Y_2, X_2), ..., (Y_n, Xn)$. A lot of attention has been given to estimate h using model (1.1), as evidenced by sources like [5, 6, 7, 8, 10, 11]. However, this paper focuses on a broader problem: adaptively estimating h. The study also deals with $\epsilon_1, ..., \epsilon_n$: to estimate h, only the finite second order moments of ϵ_1 are assumed. We don't need to understand the distribution of ϵ_1. The assumption in [5], is being made easy, which requires finite moments bigger than 6 for ϵ_1.

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2 Consideration and notations

We suppose that the sequel that h and $f \in L^2_{per}[0,1]$, where

$$L^2_{per}[0,1] = \left\{ h; h \quad \text{is one} - \text{periodic and} \quad ||h||_2 = \left(\int_0^1 h^2(x)dx \right)^{1/2} < \infty \right\}. \tag{2.1}$$

We consider there are a known fixed $U_* > 0$ such that

$$||h||_\infty = \sup_{x\in[0,1]} |h(x)| \le U_* < \infty. \tag{2.2}$$

Every function $h \in L^2_{per}([0,1])$ can be illustrated by its Fourier series

$$h(t) = \sum_{q\in Z} F(h)(q)e^{2i\pi qt}, \quad t \in [0,1] \tag{2.3}$$

in which $F(h)(q)$ refers the Fourier coefficient is offered by

$$F(h)(q) = \int_0^1 h(x)e^{-2i\pi qx}dx, \quad q \in Z \tag{2.4}$$

whenever this integral exists. We take account the ordinary smooth type on f: these are three fixed, $u_f > 0, U_f > 0$ and $\vartheta > 1$, which that, every $q \in Z$, the Fourier coefficient of f, i.e. $F(f)(q)$, is related to

$$\frac{u_f}{(1+q^2)^{\vartheta/2}} \le |F(f)(q) \le \frac{U_f}{(1+q^2)^{\vartheta/2}}. \tag{2.5}$$

3 Function Space and Wavelets

3.1 Periodized Mayer Wavelets

We use an orthonormal wavelet, which is produced by dilation and transmission of a "father" wavelet of Mayer-type φ and a "mother" wavelet of Mayer-type ζ. The principal characteristic of such wavelets are:

1 - Wavelets are smooth and frequency band-limited, i.e., the Fourier transforms φ and ζ have compact supports with

$$\text{supp}(F(\varphi)) \subset [-4\pi 3^{-1}, 4\pi 3^{-1}]$$

and

$$\text{supp}(F(\zeta)) \subset [-8\pi 3^{-1}, -2\pi 3^{-1}] \cup [2\pi 3^{-1}, 8\pi 3^{-1}].$$

where supp denotes the support.

2 - If the Fourier transforms of π and ζ are also in C^m for a chosen $m \in N$, then it can be easily shown that φ and ζ obey

$$|\pi(t)| = O((1+|t|)^{-m-1}), \quad |\zeta(t)| = O((1+|t|)^{-m-1}),$$

for every $t \in R$.

3 - The function (φ, ζ) is differentiable for all degree of differentiation. because their Fourier transform has a compact support, and ζ has an infinite number of vanishing moments . that is, for each $v \in N$, $\int_{-\infty}^{\infty} x^v dx = 0$

For the purpose of this paper, we use the periodized Meyer wavelet bases on the unit interval.For any $x \in [0,1]$, any integer j and any $k \in \{0, ..., 2^j - 1\}$. Let

$$\varphi_{j,k} = 2^{j/2}\varphi(2^j x - k), \quad \zeta_{j,k} = 2^{j/2}\zeta(2^j x - k) \tag{3.1}$$

are the elements of the wavelet basis, and

$$\varphi^{per}_{j,k}(x) = \sum_{q\in Z} \varphi_{j,k}(x-q), \quad \zeta^{per}_{j,k}(x) = \sum_{q\in Z} \zeta_{j,k}(x-q), \tag{3.2}$$

where, that is periodic. There is a $j^* \in Z$ in a way that the set $D = \{\varphi^{per}_{j^*,k}, \quad k \in \{0, ..., 2^{j^*} - 1\}; \quad \varphi^{per}_{j,k}, \quad j \in N - \{0, ..., j^* - 1\}, \quad k \in \{0, ..., 2^j - 1\}\}$ is an orthogonal basis of $L^2_{per}([0, 1])$. In the following, index "per" is removed. Consider $j_c \in Z$, which that $j_c > j_*$. A function $h \in L^2_{per}([0, 1])$ it would be developed into a series as

$$h(x) = \sum_{k=0}^{2^{j_c}-1} \alpha_{j_c,k}\varphi_{j_c,k}(x) + \sum_{j=j_c}^{\infty} \sum_{k=0}^{2_j-1} \beta_{j,k}\zeta_{j,k}(x) \tag{3.3}$$

where

$$\alpha_{j,k} = \int_0^1 h(x)\bar{\varphi}_{j,k}dx, \quad \beta_{j,k} = \int_0^1 h(x)\bar{\zeta}_{j,k}dx. \tag{3.4}$$

3.2 Function spaces

We will define the function spaces the function spaces that will be utilized in the formulation of our maxiset.To enhance clarity, we will use the following symbols for notation.

$$\sum_K = \sum_{K\in A_j}, \quad \sum_{(K)} = \sum_{k\in\beta_{j,K}}.$$

3.3 Besov balls

Let $M > 0$, $s > 0$, $p \geq 1$ and $r \geq 1$. Set $\beta_{j_*-1,k} = \alpha_{j_*,k}$. A function h belongs to the Besov balls $B^s_{p,r}(M)$ if and only if there is a constant $M^* > 0$ which that the wavelet coefficients (3.4) is related to

$$\left(\sum_{j=j_*-1}^{\infty} \left(2^{j(s+1/2-1/p)} \left(\sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \right)^{1/p} \right)^r \right)^{1/r} \leq M^*$$

For a particular choice of parameters s, p and r, these sets contain the Holhder and Sobolev balls.

Definition 3.1. (Strong Besov spaces) Let $1 \leq p < \infty$. We say that a function $f \in L^p([0, 1])$ belongs to $B^\alpha_{p,\infty}$ if and only if there are $R > 0$ which that

$$\sup_{\gamma>0} 2^{\gamma\alpha p} \left\| \sum_{j\leq\gamma} \sum_{k\in\Delta_j} \beta_{j,k}\zeta_{j,k} \right\|_p^p \leq R < \infty.$$

Definition 3.2. (W-spaces) Let $0 < r < p < \infty$ and $\sigma = (\sigma_j)_j$ be a positive sequence. We say that a function $f \in L^p([0, 1])$ belongs to $W_\sigma(r, p)$ if and only if there exists $R > 0$ such that

- for $p \geq 2$

$$\sup_{\gamma>0} \gamma^{r-p} \left\| \sum_j \sum_{k\in\Delta_j} \beta_{j,k} I_{\{b_j(p)\leq\gamma\sigma_j\}}\zeta_{j,k} \right\|_p^p \leq R < \infty$$

- for $p \leq 2$

$$\sup_{\gamma>0} \gamma^{r-p} \sum_j \sum_{k\in\Delta_j} |\beta_{j,k}|^p I_{\{b_j(p)\leq\gamma\sigma_j\}} 2^{j(p/2-1)} \leq R < \infty$$

where $b_j(p)$ shows the normalized l_p-norm of wavelet coefficients $(\beta_{j,k})_{k\in\Delta_j}$ i.e:

$$b_j(p) = \left(2^{-j} \sum_{k\in\Delta_j} |\beta_{j,k}|^p \right)^{\frac{1}{p}}.$$

Definition 3.3. ($\bar{W}$-spaces) Let $1 \leq p < \infty$, $k \in R^+$, $n_0 \in N$ and let $\sigma = (\sigma_j)_j$ be a positive sequence. Let us consider the sets $\beta_{j,K}$ with $l_j \asymp \ln(n)^{\frac{p}{2}v_1}$, we say that:

1- A function $f \in L^p([0,1])$ belongs to $\bar{W}_{\sigma,k,n_0}$ if and only if there are $R > 0$ which that for $p \geq 2$,

$$\sup_{n>n_0} n^{\frac{p-r}{2}} \sum_{m\in N} 2^{-mp} \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} I_{\{b_{j,k}(p) \leq k2^m n^{\frac{-1}{2}} \sigma_j\}} \zeta_{j,k} \right\|_p^p \leq R < \infty,$$

for $p \leq 2$

$$\sup_{n>n_0} n^{\frac{p-r}{2}} \sum_{m\in N} 2^{-mp} \sum_j \sum_K \sum_{(K)} |\beta_{j,k}|^p I_{\{b_{j,k}(p) \leq k2^m n^{\frac{-1}{2}} \sigma_j\}} 2^{j(\frac{p}{2}-1)} \leq R < \infty.$$

2- A function $f \in L^p([0,1])$ belongs to $\bar{W}^*_{\sigma,k,n_0}$ if and only if there are $R > 0$ which that :

$$\sup_{n>n_0} n^{\frac{p-r}{2}} \left\| \sum_j \sum_K \sum_{(K)} \beta_{j,k} I_{\{b_{j,k}(p) \leq k2^m n^{\frac{-1}{2}} \sigma_j\}} \zeta_{j,k} \right\| |_p^p \leq R < \infty,$$

where $b_{j,K}(p)$ shows the normalized l_p-norm of wavelet coefficients $(\beta_{j,k})_{k\in B_{j,K}}$ i.e.,

$$b_{j,K} = \left(\frac{1}{\ln(n)^{\frac{p}{2}v_1}} \sum_{(K)} |\beta_{j,k}|^p\right)^{1/p}.$$

Definition 3.4. (Maxisets) Let $1 \leq p < \infty$ and $n_0 \in N$. Let $\hat{f}$ be an estimate of f. The maxiset of $\hat{f}$ of the rate u_n under the L^p risk is the set of functions f such that exists $R > 0$ satisfying

$$\sup_{n\geq n_0} \gamma_n^{-1} E_f^n(||\hat{f} - f||_p^p) \leq R < \infty.$$

Such maxiset will be denoted $M_{n_0}(\hat{f}, p, u_n)$.

4 Estimators

In this section, the estimators introduced by [2] are described as follows.

Wavelet coefficient estimators: For any $j \geq j_* \in Z$ and any $k \in \{0, ..., 2^j - 1\}$, we estimate $\alpha_{j,k} = \int_0^1 f(x)\varphi_{j,k}(x)dx$ by

$$\hat{\alpha} = \frac{1}{n} \sum_{\nu=1}^{n} \sum_{s\in C_j} \frac{F(\bar{\varphi_{j,k}})(q)}{F(f)(q)} Y_\nu e^{-2i\pi s x_\nu} \tag{4.1}$$

$H_j = \text{supp}(F(\varphi_j, 0)) = \text{supp}(F(\varphi_{j,k}))$, and $\beta_{j,k} = \int_0^1 f(x)\zeta_{j,k}(x)dx$ by

$$\hat{\beta}_{j,k} = \frac{1}{n} \sum_{\nu=1}^{n} G_\nu I_{\{|G_\nu| \leq \eta_j\}}, \tag{4.2}$$

where

$$G_\nu = \sum_{q\in D_j} \frac{F(\bar{\zeta_{j,k}})(q)}{F(f)(q)} Y_\nu e^{-2i\pi s x_\nu}$$

$H_j = \text{supp}(F(\zeta_j, 0)) = \text{supp}(F(\zeta_{j,k}))$, and threshold η_j is defined by

$$\eta_j = \vartheta 2^{\delta j} \sqrt{\frac{n}{\ln n}}$$

$\vartheta = \sqrt{H_{**}(H_*^2 + E(\xi_1^2))}$, H_* is (3) and $H_{**} = 2^{\delta-1}(2/c_g^2)(8\pi/3)^{2\delta}$. We consider two wavelet estimators for f : a linear estimator and a nonlinear wavelet estimator based on block thresholding method.

Linear estimator: Assuming that $f \in B^*_{p,r}(M)$ with $p \geq 2$, we define the linear estimator $\hat{f}^L$ by

$$\hat{f}^L(x) = \sum_{k=0}^{2^{j_0}-1} \hat{\alpha}_{j_0,k}\varphi_{j_0,k}(x), \tag{4.3}$$

where $\hat{\alpha}_{j,k}$ is explain by (4.1) and $j_0 \in Z$ is related to

$$2^{-1}n^{1/(2q+2\delta+1)} \leq 2^{j_0} \leq 2^{1/(2q+2\delta+1)}.$$

Block thresholding procedures: Let $1 < p < \infty$ and $0 < \nu \leq 2$. Let j_1 be an integer satisfying

$$2^{j_1} \asymp \ln(n)^{\frac{p}{2}\nu_1}$$

and σ be a known increasing positive sequence such that there exists $\nu > 0$ satisfying,

$$\sigma_{j_1} \asymp \ln(n)^{\nu}.$$

Let j_2 be an integer satisfying $2^{j_2} \asymp n^{\frac{\nu}{2}}$. For all $j \in \{j_1, ..., j_2-1\}$, let us divide Δ_j into consecutive nonoverlapping blocks $B_{j,K}$ of length j_1 i.e.,

$$B_{j,K} = \{k \in \Delta_j : (K-1)l_j \leq k \leq Kl_j - 1\}, \quad K \in A_j,$$

where the sets A_j are defined by

$$A_j = \{1, ..., 2^j l_j^{-1}\}.$$

We can express the block thresholding procedure $\hat{f}$ by:

$$\hat{f}(x) = \sum_{j \leq j_1} \hat{\alpha}_{j_1,k}\varphi_{j_1,k}(x) + \sum_{j_1 \leq j < j_2} \sum_{K \in A_j} \sum_{k \in B_{j,K}} \hat{\beta}_{j,k} I_{\{\hat{b}_{j,K}(p) \geq k\sigma_j n^{\frac{-1}{2}}\}} \zeta_{j,k}(x), \quad x \in [0,1], \tag{4.4}$$

where is a positive real number and $\hat{b}_{j,K}(p)$ is the normalized l_p-norm of estimators $(\hat{\beta}_{j,k})_{k \in B_{j,K}}$ i.e.,

$$\hat{b}_{j,K}(p) = (l_j^{-1} \sum_{k \in B_{j,K}} |\hat{\beta}_{j,k}|^p)^{\frac{1}{p}}.$$

Starting from this general definition, we distinguish two kinds of procedures:

1. The global thresholding procedure which corresponds to the procedure $\hat{f}$ described by (4.4) with

$$l_j = 2^j$$

such procedure will be denote $\hat{f}^g$.

2. the blockShrink procedure which corresponds to the procedure $\hat{f}$ described by (4.4) with

$$l_j \asymp \ln(n)^{\frac{p}{2}\nu_1}$$

such procedure will be denote $\hat{f}^o$.

5 Results

Let us first set the assumptions we shall need to prove our Theorems.Let us recall that the $\hat{\alpha}_{j,k}$ and $\hat{\beta}_{j,k}$ are estimate of the wavelet coefficients $\alpha_{j,k}$ and $\beta_{j,k}$ and that σ_j is a known positive sequence.

Assumption 1: There is a fixed number $C > 0$ which that the following moments condition hold:

$$E^n_f(|\hat{\alpha}_{j_1,k} - \alpha_{j_1,k}|^p) \leq C\sigma^p_{j_1} n^{\frac{-p}{2}}.$$

Assumption 2: There is a fixed number $C > 0$ which that the following moments condition hold:

$$E^n_f(|\hat{\beta}_{j,k} - \alpha_{j,k}|^{2p}) \leq C\sigma^{2p}_{j_1} n^{-p} \quad j \in \{j_1, ..., j_2 - 1\}.$$

Assumption 3: There is a fixed number $C > 0$ such that the following moments condition hold:

$$P_f^n((l_j^{-1}\sum_{(K)}|\hat{\beta}_{j,k}-\alpha_{j,k}|^p)^{\frac{1}{p}} \geq \lambda\sigma_j n^{\frac{-1}{2}}) \leq Cn^{-h(\lambda)}, \quad K \in A_j, \quad j \in \{j_1, ..., j_2-1\},$$

where λ is a large enough real number and h is a positive function such that $\lim_{x\to\infty} h(x) = \infty$.

Assumption 4: There is a fixed number $C > 0$ which that the sequence σ and the integer j_2 satisfy the following weighted inequality:

$$\sum_{j<j_2}\sum_{k\in\Delta_j} \sigma_j^p||\zeta_{j,k}||_p^p \leq Cn^{\frac{p}{2}}.$$

Theorems below investigate the maxiset properties property of the block thresholding procedure $\hat{f}^g$ and $\hat{f}^o$ measured under the L^p risk for the rate of convergence $n^{\frac{-\alpha p}{2}}$.

Theorem 5.1. Let $1 < p < \infty$, $\sigma = (\sigma_j)_j$ be a known positive increasing sequence and let $\hat{f}$ be the block thresholding procedure described by (4.4). Then under the assumption 1-4 , for any $\alpha \in (0, 1)$, any $k > k_0$ with k_0 large enough, the maxiset associated to the global thresholding procedure satisfies:

$$M_{n_0}(\hat{f}^g, p, n^{\frac{-\alpha p}{2}}) \subseteq B_{p,\infty}^{\frac{\alpha}{\nu}} \bigcap W_\sigma((1-\alpha)p, p)$$

and the maxiset associated to the BlockShrink procedure satisfies:

$$M_{n_0}(\hat{f}^o, p, n^{\frac{-\alpha p}{2}}) \subseteq B_{p,\infty}^{\frac{\alpha}{\nu}} \bigcap \bar{W}^*{}_{\sigma,k,n_0}((1-\alpha)p, p).$$

Theorem 5.2. Let $1 < p < \infty$, $\sigma = (\sigma_j)_j$ be a known positive increasing sequence and let $\hat{f}$ be the block thresholding procedure described by (4.4).Then , under the assumption 1-4, for any $\alpha \in (0, 1)$, any $k > k_0$ with k_0 large enough, the maxiset associated to the global thresholding procedure satisfies:

$$B_{p,\infty}^{\frac{\alpha}{\nu}} \bigcap \bar{W}^*{}_\sigma((1-\alpha)p, p) \subseteq M_{n_0}(\hat{f}^g, p, n^{\frac{-\alpha p}{2}}),$$

and the maxiset associated to the BlockShrink procedure satisfies:

$$B_{p,\infty}^{\frac{\alpha}{\nu}} \bigcap \bar{W}_{\sigma,k,n_0}((1-\alpha)p, p) \subseteq M_{n_0}(\hat{f}^o, p, n^{\frac{-\alpha p}{2}}).$$

Theorem 5.3. Let $1 < p < \infty$,$R > 0$, $\delta \geq 0$, $\sigma = (2^{\delta_j})_j$ and $\hat{f}$ be the Block Shrink procedure taken with $\nu = \frac{2}{1+2\delta}$. Under the assumptions 1-4, for $s > 0$, $r \geq 1$, $\pi \geq p$, any $k > k_0$ with k_0 large enough, there is a fixed $C > 0$ such that:

$$\sup_{f\in B_{\pi,r}^s(R)} E_f^n(||\hat{f}-f||_p^p) \leq Cn^{\frac{-sp}{1+2s+2\delta}}, \quad n \geq n_0$$

Remark 5.4. According to [1], the threshold value $\varpi = 4.50524$ is obtained by solving $\varpi - Log\varpi - 3 = 0$. This threshold value is obtained so that the wavelet estimation is optimized.

Remark 5.5. Based on the optimal properties of local and global optimization, the optimal block length is $L = Logn$. Considering the length of the block and the threshold value introduced in the local and general modes of the obtained wavelet estimator, it is simultaneously adaptive.

To prove Theorems 5.1, 5.2, and 5.3, we need to prove assumptions 1 to 4 for regression model error-in-variables. So, to prove the theorems, first, we prove assumptions 1 to 4 in the estimation model of the regression function with error-in-variable. Author in [2] has proved assumptions 1 and 2 in the case of $p = 2$. To prove assumption 3, we consider the following lemma.

Lemma 5.6. (**Cirelson, Ibragimov, Sudakov's inequality**) Let $(\varrho_t, t \in T)$ be a Gaussian process. Let defined $N = E[\sup_{t\in T}\varrho_t]$ and $W = \sup_{t\in T} Var(\varrho_t)$. Then, for all $c > 0$ we have,

$$p(\sup_{t\in T}\varrho_t \geq c + N) \leq \exp(\frac{-c^2}{2W}).$$

In estimating the regression function with the error-in-variable using the wavelet method, we have the following relationship for the wavelet coefficients:

$$e_{j,k,n} = \hat{\beta}_{j,k} - \beta_{j,k} = \frac{1}{n}\sum_{\nu=1}^{n}\sum_{q}\frac{\overline{F(\zeta_{j,k})(q)}}{F(f)(q)}\xi_\nu e^{-2i\pi q x_v}$$

Let us set $C_q = \{a = (a_{j,k}); \sum_{(K)} |a_{j,k}|^q \leq 1\}$ in which $q \in R$ is related to $q^{-1} + p^{-1} = 1$ and $\{Z(a); a \in C_q\}$ the centered Gaussian process can state as:

$$Z(a) = \sum_{(K)} a_{j,k} e_{j,k,n}.$$

So we can say:

$$\sup Z(a) = (\sum_{(K)} |e_{j,k,n}|^p)^{1/p}.$$

Using Holder's inequality and the assumption 2, we consider

$$N = E_f^n(\sup_{a\in C_q} Z(a)) \leq \sum_{(K)}(E_f^n|e_{j,k,n}|^p)^{1/p} \leq (Cn^{-2}2^{2\delta_j}l_j)^{1/p} \tag{5.1}$$

and

$$\begin{aligned}
W &= \sup_{a\in C_q} Var(Z(a)) = \sup_{a\in C_q} E[\sum_{k\in B_{j,k}}\sum_{\acute{k}\in B_{j,k}} a_{j,k}a_{j,\acute{k}} e_{j,k,n} e_{j,\acute{k},n}] \qquad (5.2)\\
&= n^{-2}\sup_{a\in C_q} E[\sum_{k\in B_{j,k}}\sum_{\acute{k}\in B_{j,k}} a_{j,k}a_{j,\acute{k}}[\sum_{\nu=1}^{n}\sum_{q}\sum_{\acute{q}}\frac{\overline{F(\zeta_{j,k})(q)}}{F(f)(q)}\xi_\nu e^{-2i\pi q x_\nu}\frac{\overline{F(\zeta_{j,k})(\acute{q})}}{F(f)(\acute{q})}\xi_\nu e^{-2i\pi \acute{q} x_\nu}]]\\
&= n^{-2}\sup_{a\in C_q} [\sum_{k\in B_{j,k}}\sum_{\acute{k}\in B_{j,k}} a_{j,k}a_{j,\acute{k}}[\sum_{\nu=1}^{n}\sum_{q}\sum_{\acute{q}}\frac{\overline{F(\zeta_{j,k})(q)}}{F(f)(q)}E(\xi_\nu)^2 E(e^{-2i\pi(q-\acute{q})x_\nu})\frac{\overline{F(\zeta_{j,k})(\acute{q})}}{F(q)(\acute{q})}]]\\
&= n^{-2}\sup_{a\in C_q} E[\sum_{k\in B_{j,k}}\sum_{\acute{k}\in B_{j,k}} a_{j,k}a_{j,\acute{k}}[\sum_{\nu=1}^{n}\sum_{q}\sum_{\acute{q}}\frac{\overline{F(\zeta_{j,k})(q)}}{F(f)(q)}\xi_\nu e^{-2i\pi q x_\nu}\frac{\overline{F(\zeta_{j,k})(\acute{q})}}{F(f)(\acute{q})}\xi_\nu e^{-2i\pi \acute{q} x_\nu}]]\\
&= n^{-2}\sup_{a\in C_q} [\sum_{k\in B_{j,k}}\sum_{\acute{k}\in B_{j,k}} a_{j,k}a_{j,\acute{k}}[\sum_{\nu=1}^{n}\sum_{q}\sum_{\acute{q}}\frac{\overline{F(\zeta_{j,k})(q)}}{F(f)(q)})\frac{\overline{F(\zeta_{j,k})(\acute{q})}}{F(f)(\acute{q})}d]]
\end{aligned}$$

where d is a constant as follows:

$$d = E(\xi_\nu)^2 E(e^{-2i\pi(q-\acute{q})x_\nu}).$$

So relation (5.2) is:

$$\begin{aligned}
&= n^{-2}\sup_{a\in C_q} [\sum_{k\in B_{j,k}}\sum_{\acute{k}\in B_{j,k}} a_{j,k}a_{j,\acute{k}}[\sum_{\nu=1}^{n}\sum_{q}\frac{\overline{F(\zeta_{j,k})(q)}F(\zeta_{j,k})(q)}{|F(f)(q)|^2}\\
&= n^{-1}\sup_{a\in C_q} [\sum_{k\in B_{j,k}}\sum_{\acute{k}\in B_{j,k}} a_{j,k}a_{j,\acute{k}}[\int_0^1 \zeta_{j,k}(x)\overline{\zeta_{j,k}}(x)dx|F(f)(q)|^{-2}\\
&\leq n^{-1}\frac{C_g}{(1+q^2)^{\delta/2}}\sup_{a\in C_q}\sum_{k\in B_{j,k}}\sum_{\acute{k}\in B_{j,k}} a_{j,k}a_{j,\acute{k}}\int_0^1 \zeta_{j,k}(x)\overline{\zeta_{j,k}}(x)dx\\
&= n^{-1}\frac{C_g}{(1+s^2)^{\delta/2}}\sup_{a\in C_q}(\sum_{k\in B_{j,k}}|a_{j,k}|^2) \leq n^{-1}C.
\end{aligned}$$

If $p = 2$ in relation (5.1), then we have from relation (5.2) and Sudokov-Ibragimov lemma:

$$\begin{aligned}
V_n = P((q_j^{-1}\sum_{K}|\hat{\beta}_{j,k} - \beta_{j,k}|^p)^{1/p} \geq \lambda n^{-1/2}) &\leq P((q_j^{-1}\sum_{(K)}|e_{j,k,n}|^p)^{1/p} \geq (\lambda - C)n^{-1/2}))\\
&\leq P(\sup \quad Z(a) \geq C + N) \leq \exp(-q_j h(\lambda)).
\end{aligned}$$

So, assumption (3) was prove. For prove assumption (4), because $2^{j_2} \asymp 2^{\nu/2}$, then

$$\sum_{j<j_2} \sum_{k\in\Delta_j} \sigma_j^p \|\zeta_{j,k}\|_p^p = \sum_{j<j_2} 2^{j(p\nu/2)} \leq C2^{j_2(p\nu/2)} \leq Cn^{p/2}. \tag{5.3}$$

So, using assumptions 1-4 and Theorems 5.1 and 5.2, theorem 5.3 can be used to obtain the convergence rate of the block thresholding estimator. Theorem 5.1, 5.2 and 5.3 was proved in [3].

6 Simulation

The model (1.1) is considered. In this model, variable U is unobserved, and variable Y is observed. Instead of the variable U, the observable variable X, which has a uniform distribution of 0 and 1, is used. The error term ϵ is from the standard normal distribution and the variable T is also considered from the standard normal distribution. To simulate a sample, 1024 of the mentioned distributions are considered. The MSE criterion was used to measure the regression function estimator and the simulation results are presented in the chart and table below.

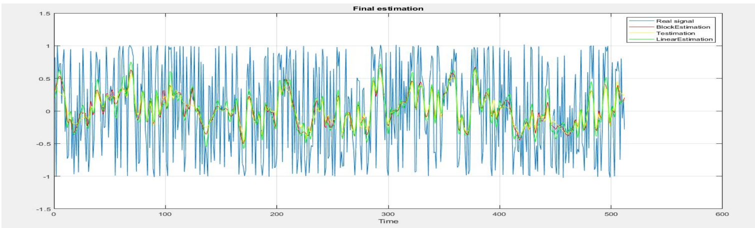

Figure 1: Estimation of the function by the method Linear, Hard Thresholding and Block Thresholding

Table 1: The MSE values for a linear and two nonlinear wavelet estimators

Linear Estimate	Tresholding Estimate	Block Tresholding Estimate
0.4528	0.3948	0.2856

7 Conclusion

The conventional wavelet method achieves adaptivity by employing term-by-term thresholding of the empirical wavelet coefficients. This involves comparing each individual empirical wavelet coefficient to a predetermined threshold, and retaining the coefficient if its magnitude surpasses the threshold level, or discarding it otherwise. This method is spatially adaptive and the estimator is within a logarithmic of the optimal convergence rate across a broad range of Besov classes. This achieves a degree of tradeoff between variance and bias contributions to the mean squared error. However, the tradeoff is not optimal. The block thresholding for wavelet function estimation thresholds empirical wavelet coefficients in groups rather than individually. This method increases estimation precision by utilizing information about neighbouring wavelet coefficients.

Many parametric and non-parametric methods exist for estimating the regression function with variable errors. The wavelet method is one of the non-parametric methods. Of course, the wavelet method can also be used as a linear, thresholding, or block thresholding method. Both theoretically and through simulation, it can be concluded that the block threshold method has a lower MSE value than other wavelet methods in estimating the regression function with error.

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