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EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS FOR NONSPREADING-TYPE MAPPINGS IN HILBERT SPACE

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ABSTRACT. In this paper by using the idea of mean convergence, we introduce an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the fixed points set of a nonspreading-type mappings in Hilbert space. A strong convergence theorem of the proposed iterative scheme is established under some control conditions. The main result of this paper extend the results obtained by Osilike and Isiogugu (Nonlinear Analysis 74 (2011) 1814-1822) and Kurokawa and Takahashi (Nonlinear Analysis 73 (2010) 1562-1568). We also give an example and numerical results are also given.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert spaced H. Then a mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping F is said to be $firmly\ nonexpansive$ if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$; see, for instance, Browder [3], Goebel and Kirk [6]. It is also known that a firmly nonexpansive mapping F is deduced from an equilibrium problem in a Hilbert space as follows: Let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying

$(A1)$ $f(x, x) = 0,\ \forall x \in C$;
$(A2)$ f is monotone, i.e. $f(x, y) + f(y, x) \leq 0,\ \forall x, y \in C$;
$(A3)$ $\forall x, y, z \in C$, $\lim_{t\to 0^+} f(tz + (1 - t)x, y) \leq f(x, y)$;
$(A4)$ $\forall x \in C, y \mapsto f(x, y)$ is convex and lower semicontinuous.

We know the following theorem; see, for instance, [2, 5].

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert spaced H and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in H$, Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \ for\ all\ y \in C. \tag{1.1}$$

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Further, for any $r > 0$ and $x \in H$, define $T_r : H \to C$ by $z = T_r x$. Then, T_r is firmly nonexpansive, i.e.,

$$\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y\rangle, \ \forall x, y \in H.$$

On the other hand, a mapping $T : C \to C$ is said to be $quasi-nonexpansive$ if $F(T) \neq \emptyset$ and

$$\|Tx - q\| \leq \|x - q\|$$

for all $x \in C$ and $q \in F(T)$, where $F(T)$ is the set of fixed points of T. If T is a nonspreading mapping from C into itself and the set $F(T)$ is nonempty, then T is quasi-nonexpansive. Further, we know that the set of fixed points of a quasi-nonexpansive mapping is closed and convex; see [10]. Then we can define the metric projection of H onto $F(T)$.

In 2010, Kohsaka and Takahashi [11, 12] introduced the following nonlinear mapping: Let E be a smooth, strictly convex and reflexive Banach space, let j be the duality mapping of E and let C be a nonempty closed convex subset of E. Then, a mapping $T : C \to C$ *nonspreading* is said to be if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$, $\forall x, y \in E$. They considered the class of nonspreading mappings to study the resolvents of a maximal monotone operators in the Banach space. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping $T : C \to C$ in a Hilbert space H is defined as follows:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.2}$$

for all $x, y \in C$. Iemoto and Takahashi [8] prove that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \tag{1.3}$$

for all $x, y \in C$. We also know that a nonspreading mapping is deduced from a firmly nonexpansive mapping; see [9, 11, 17]. A strong convergence theorem of the hybrid type for nonspreading mappings have been proved by Matsushita and Takahashi [14].

By using an idea of mean ergodic theorem of Baillon's type, Kurokawa and Takahashi [13] introduced two iterative schemes for finding a fixed point of a nonspreading mapping. Weak and strong convergence theorems of the proposed iterative schemes in Hilbert spaces are proved under some control conditions.

Later in 2010, Osilike and Isiogugu [16] introduce a new class of nonspreading-type of mappings which is more general than the class studied in Kurokawa and Takahashi [13] in Hilbert spaces. By using the idea of mean convergence, they proved a weak mean convergence theorem of Baillons type similar to the ones obtained in [13] for the class of nonspreading-type mappings. Furthermore, using an idea of mean convergence, they also proved a strong convergence theorem similar to the one obtained in [13].

In this paper, we prove a strong convergence theorem of Halpern's type for finding a common element of the set of fixed point, nonspreading-type mapping and the set of solution of an equilibrium problem in a Hilbert space.

2. Preliminaries and lemmas

Throughout this paper, we denote by H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We also denote by $\mathbb{N}$ the set of natural numbers. In a Hilbert space, it is known that

Lemma 2.1. [16] *Let H be a real Hilbert space. Then the following well known results hold:*
(1) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$,
for all $x, y \in H$ and for all $t \in [0,1]$.
(2) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$ for all $x, y \in H$.
(3) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in H which converges weakly to $z \in H$ then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} (\|x_n - z\|^2 + \|z-y\|^2), \ \ \forall y \in H.$$

Let C be a closed convex subset of H and let T be a mapping of C into itself. We denote by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{z \in C : Tz = z\}$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$. We can define the metric projection of H onto C: For each $x \in H$, there exists a unique point $z \in C$ such that

$$\|x - z\| = min\{\|x-y\| : y \in C\}$$

For every point $x \in H$, such that a point z denoted by $P_C(x)$ and P is called the *metric projection* of H onto C. It is known that

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall y \in C$$

for each $x \in H$ and $y \in C$; see [18] for more details.

Lemma 2.2. [19] *Let C be nonempty closed convex subset of a real Hilbert space H. Let $P_C : H \to C$ be the metric projection of H onto C. Let $\{x_n\}_{n=1}^{\infty}$ be sequence in C and let $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all u in C. Then $\{P_C x_n\}_{n=1}^{\infty}$ converges strongly.*

The following lemma was also given in [5]

Lemma 2.3. [5] *Let C be a nonempty closed convex subset of a Hilbert space H and $f : C \times C \to \mathbb{R}$ satisfy $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C\} \ \ for\, all \ \ x \in H. \qquad (2.1)$$

Then the following hold:
(1) T_r is is single-valued;
(2) T_r is firmly nonexpansive, i.e., for all $x, y \in H$,
$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle$;
(3) $F(T_r) = EP(f)$;
(4) $EP(f)$ is closed and convex.

Lemma 2.4. [1, 20] *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the condition*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \beta_n, \ n \geq 0,$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences such that
(i) $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

Let H be a real Hilbert space. Following the terminology of Browder-Petryshyn [4], we say that a mapping $T : D(T) \subseteq H :\to H$ is $k-$*strictly pseudononspreading* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle, \tag{2.2}$$

for all $x, y \in D(T)$. Clearly every nonspreading mapping is k-strictly psedononspreading. The class of k-strictly pseudononspreading mapping is more general than the class of nonspreading mappings (see example [16]).

Observe that if T is k-strictly pseudononspreading and $F(T) \neq \emptyset$, then for all $x \in D(T)$ and for all $p \in F(T)$ we have

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2. \tag{2.3}$$

Thus every k-strictly pseudononspreading map with a nonempty fixed point set $F(T)$ is *demicontractive* (see example [7], [15]).

Lemma 2.5. [16] *Let C be nonempty closed convex subset of a real Hilbert space H. and let $T : C \to C$ be k-strictly pseudononspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Lemma 2.6. [16] *Let C be nonempty closed convex subset of a real Hilbert space H. and let $T : C \to C$ be k-strictly pseudononspreading mapping. Then $(I - T)$ is demiclosed at 0.*

3. Main Results

We first prove a strong convergence theorem.

Theorem 3.1. *Let C be a nonempty closed convex subset of of a real Hilbert space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A2) and let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set and $F(T) \bigcap EP(f) \neq \emptyset$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1)$ and $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ satisfying the conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \liminf_{n\to\infty} r_n > 0.$$

Let $u \in C$ and let $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequence in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \;\; \forall y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)u_n, \;\; n \geq 1, \\ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \;\; n \geq 1. \end{cases} \tag{3.1}$$

Then $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $P_{F(T)\bigcap EP(f)}u$, where $P_{F(T)\bigcap EP(f)} : H \to F(T)\bigcap EP(f)$ is the metric projection of H onto $F(T)\bigcap EP(f)$.

Proof. Let $T_\beta x := \beta x+(1-\beta)Tx$. It is clear that $F(T_\beta) = F(T)$ and for all $x, y \in C$, we have

$$\begin{aligned}
\|T_\beta x - T_\beta y\|^2 &= \|\beta(x-y) + (1-\beta)(Tx - Ty)\|^2 \\
&= \beta\|x-y\|^2 + (1-\beta)\|Tx-Ty\|^2 - \beta(1-\beta)\|x - Tx - (y - Ty)\|^2 \\
&\le \beta\|x-y\|^2 - \beta(1-\beta)\|x - Tx - (y - Ty)\|^2 \\
&\quad +(1-\beta)\big[\|x-y\|^2 + k\|x - Tx - (y-Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle\big] \\
&= \|x-y\|^2 - \beta(1-\beta)\|x - Tx - (y - Ty)\|^2 \\
&\quad +k(1-\beta)\|x - Tx - (y-Ty)\|^2 + 2(1-\beta)\langle x - Tx, y - Ty\rangle \\
&= \|x-y\|^2 - (1-\beta)(\beta - k)\|x - Tx - (y - Ty)\|^2 \\
&\quad +2(1-\beta)\langle x - Tx, y - Ty\rangle \\
&\le \|x-y\|^2 + 2(1-\beta)\langle x - Tx, y - Ty\rangle \\
&= \|x-y\|^2 + \frac{2}{(1-\beta)}\langle x - T_\beta x, y - T_\beta y\rangle. \qquad (3.2)
\end{aligned}$$

Let $p \in F(T)\bigcap EP(f)$. Then from $u_n = T_{r_n} x_n$, using (3.2) we obtain

$$\begin{aligned}
\|z_n - p\| &= \|\frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n - p\| \\
&\le \frac{1}{n}\sum_{m=0}^{n-1}\|T_\beta^m x_n - p\| \le \frac{1}{n}\sum_{m=0}^{n-1}\|x_n - p\|| = \|x_n - p\|. \qquad (3.3)
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)u_n - p\| \\
&= \|\alpha_n u + (1-\alpha_n)T_{r_n} z_n - p\| \\
&\le \alpha_n\|u-p\| + (1-\alpha_n)\|T_{r_n} z_n - p\| \\
&\le \alpha_n\|u-p\| + (1-\alpha_n)\|z_n - p\| \\
&\le \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\|. \qquad (3.4)
\end{aligned}$$

By (3.4) and induction, we can conclude that for all $n \in \mathbb{N}$

$$\|x_n - p\| \le \max\{\|u-p\|, \|x_1 - p\|\}.$$

This implies that $\{x_n\}$ and $\{z_n\}$ are bounded. Since $\|T_\beta^n u_n - p\| \le \|u_n - p\|$ and $\|u_n - p\| = \|T_{r_n} z_n - p\| \le \|z_n - p\| \le \|x_n - p\|$, we have that $\{T_\beta^n u_n\}$ and $\{u_n\}$ are also bounded.

Observe that since $\{u_n\}$ is bounded and $\lim_{n\to\infty}\alpha_n = 0$, we obtain

$$\begin{aligned}
\|x_{n+1} - u_n\| &= \|\alpha_n u + (1-\alpha_n)u_n - u_n\| \\
&= \alpha_n\|u - u_n\| \to 0 \quad \text{as} \quad n \to \infty. \qquad (3.5)
\end{aligned}$$

Put $\Omega := F(T)\bigcap EP(f)$. We may assume without loss of generality that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - P_\Omega u, x_n - P_\Omega u\rangle = \lim_{j\to\infty}\langle u - P_\Omega u, x_{n_j} - P_\Omega u\rangle,$$

and $x_{n_j} \rightharpoonup w$ as $j \to \infty$. Since $\|x_{n+1} - u_n\| \to 0$ as $n \to \infty$, it follows that

$u_{n_j} \rightharpoonup w$ as $j \to \infty$. Next we will show that $w \in F(T)$. Using (3.2) we obtain for all $m = 0, 1, 2, ..., n-1$ and for arbitrary $y \in C$

$$\begin{aligned}
\|T_\beta^{m+1}x_n - T_\beta y\|^2 &= \|T_\beta(T_\beta^m x_n) - T_\beta y\|^2 \\
&\leq \|T_\beta^m x_n - y\|^2 + \frac{2}{1-\beta}\langle T_\beta^m x_n - T_\beta^{m+1}x_n, y - T_\beta y\rangle \\
&= \|T_\beta^m x_n - T_\beta y + T_\beta y - y\|^2 + \frac{2}{1-\beta}\langle T_\beta^m x_n - T_\beta^{m+1}x_n, y - T_\beta y\rangle \\
&= \|T_\beta^m x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle T_\beta^m x_n - T_\beta y, T_\beta y - y\rangle \\
&\quad + \frac{2}{1-\beta}\langle T_\beta^m x_n - T_\beta^{m+1}x_n, y - T_\beta y\rangle. \qquad (3.6)
\end{aligned}$$

Summing (3.6) from $m = 0$ to $n-1$ and dividing by n we obtain

$$\begin{aligned}
\frac{1}{n}\|T_\beta^n x_n - T_\beta y\|^2 &\leq \frac{1}{n}\|x_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_n - T_\beta y, T_\beta y - y\rangle \\
&\quad + \frac{2}{n(1-\beta)}\langle x_n - T_\beta^n x_n, y - T_\beta y\rangle. \qquad (3.7)
\end{aligned}$$

Replacing n by n_j in (3.7) we obtain

$$\begin{aligned}
\frac{1}{n_j}\|T_\beta^{n_j}x_{n_j} - T_\beta y\|^2 &\leq \frac{1}{n_j}\|x_{n_j} - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_{n_j} - T_\beta y, T_\beta y - y\rangle \\
&\quad \frac{2}{n_j(1-\beta)}\langle x_{n_j} - T_\beta^{n_j}x_{n_j}, y - T_\beta y\rangle. \qquad (3.8)
\end{aligned}$$

Since $\{x_n\}$ and $\{T_\beta^n x_n\}$ are bounded, letting $j \to \infty$ in (3.8) yields

$$0 \leq \|T_\beta y - y\|^2 + 2\langle w - T_\beta y, T_\beta y - y\rangle. \qquad (3.9)$$

Since $y \in C$ was arbitrary, if we set $y = w$ in (3.9) we obtain

$$0 \leq \|T_\beta w - w\|^2 - 2\|T_\beta w - w\|^2,$$

from which it follows that $w \in F(T_\beta) = F(T)$.
Since $P_\Omega : H \to \Omega$ is the metric projection, we have

$$\lim_{j\to\infty}\langle u - P_\Omega u, x_{n_j} - P_\Omega u\rangle = \langle u - P_\Omega u, w - P_\Omega u\rangle \leq 0.$$

Hence we have $\limsup_{n\to\infty}\langle u - P_\Omega u, x_n - P_\Omega u\rangle \leq 0$. Using Lemma 2.1 (ii) and (3.3) we have

$$\begin{aligned}
\|x_{n+1} - P_\Omega u\|^2 &= \|\alpha_n u + (1-\alpha_n)u_n - P_\Omega u\|^2 \\
&= \|\alpha_n u + (1-\alpha_n)T_{r_n}z_n - P_\Omega u\|^2 \\
&= \|\alpha_n u - \alpha_n P_\Omega u + (1-\alpha_n)T_{r_n}z_n - (1-\alpha_n)P_\Omega u\|^2 \\
&= \|\alpha_n(u - P_\Omega u) + (1-\alpha_n)(T_{r_n}z_n - P_\Omega u)\|^2 \\
&\leq (1-\alpha_n)^2\|T_{r_n}z_n - P_\Omega u\|^2 + 2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\leq (1-\alpha_n)^2\|z_n - P_\Omega u\|^2 + 2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \\
&\leq (1-\alpha_n)^2\|x_n - P_\Omega u\|^2 + 2\alpha_n\langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle.
\end{aligned}$$
$$(3.10)$$

Since $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \langle u - P_\Omega u, x_{n+1} - P_\Omega u\rangle \le 0$, it follows from Lemma 2.4 that $\lim_{n\to\infty} \|x_n - P_\Omega u\| = 0$.

$$0 \le \|u_n - P_\Omega u\| \le \|u_n - x_{n+1}\| + \|x_{n+1} - P_\Omega u\| \to 0 \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty} \|u_n - P_\Omega u\| = 0$.
Since $\|x_n - P_\Omega u\| \to 0$, we have $\|x_{n+1} - x_n\| \to 0$. In sequence, we show that $\|z_n - u_n\| \to 0$, as $n \to \infty$. For $p \in F(T) \bigcap EP(f)$, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|T_{r_n} z_n - T_{r_n} p\|^2 \\
&\le \langle T_{r_n} z_n - T_{r_n} p, z_n - p\rangle \\
&= \langle u_n - p, z_n - p\rangle \\
&= \frac{1}{2}(\|u_n - p\|^2 + \|z_n - p\|^2 - \|z_n - u_n\|^2) \qquad (3.11)
\end{aligned}$$

and hence $\|u_n - p\|^2 \le \|z_n - p\|^2 - \|z_n - u_n\|^2$.
Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n u + (1-\alpha_n) u_n - p\|^2 \\
&= \|\alpha_n u + (1-\alpha_n) T_{r_n} z_n - p\|^2 \\
&\le \alpha_n \|u - p\|^2 + (1-\alpha_n)\|T_{r_n} z_n - p\|^2 \\
&\le \alpha_n \|u - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\
&\le \alpha_n \|u - p\|^2 + (1-\alpha_n)(\|z_n - p\|^2 - \|z_n - u_n\|^2) \\
&\le \alpha_n \|u - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 - \|z_n - u_n\|^2) \\
&\le \alpha_n \|u - p\|^2 + \|x_n - p\|^2 - (1-\alpha_n)\|x_n - u_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n)\|z_n - u_n\|^2 &\le \alpha_n \|u - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\le \alpha_n \|u - p\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}$$

So, we have $\|z_n - u_n\| \to 0$, and $\|z_n - P_\Omega u\| \le \|z_n - u_n\| + \|u_n - P_\Omega u\| \to 0 \ as \ n \to \infty$. This implies $u_{n_j} \to w$ as $j \to \infty$. Finally, we prove that $w \in EP(f)$. By $u_n = T_{r_n} z_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \ge 0, \ \ \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \ge f(y, u_n), \ \ \forall y \in C,$$

and hence

$$\langle y - u_{n_j}, \frac{u_{n_j} - z_{n_j}}{r_{n_j}}\rangle \ge f(y, u_{n_j}), \ \ \forall y \in C.$$

Since $\frac{u_{n_j} - z_{n_j}}{r_{n_j}} \to 0$ and $u_{n_j} \rightharpoonup w$, from (A4) we have $0 \ge f(y, w)$ for all $y \in C$. For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $f(y_t, w) \le 0$. So, from (A1) and (A4) we have

$$\begin{aligned}
0 &= f(y_t, y_t) \\
&\le t f(y_t, y) + (1-t) f(y_t, w) \\
&\le t f(y_t, y)
\end{aligned}$$

and hence $0 \leq f(y_t, y)$. From (A3), we have $0 \leq f(w, y)$ for all $y \in C$ and hence $w \in EP(F)$. Therefore $w \in F(T) \bigcap EP(f)$. $\square$

If $f(x, y) = 0$, $\forall (x, y) \in C \times C$, we have that $u_n = z_n$ for all $n \in \mathbb{N}$. Hence the following Corollary is directly obtained by Theorem 3.1

Corollary 3.2. *([16], Theorem 3.2) Let C be a nonempty closed convex subset of of a real Hilbert space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A2) and let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set and $F(T) \bigcap EP(f) \neq \emptyset$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1 - \beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1)$ satisfying the conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty$$

and let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, \ n \geq 1, \\ z_n = \frac{1}{n} \sum_{m=0}^{n-1} T_\beta^m x_n, \ n \geq 1, \end{cases} \tag{3.12}$$

Then $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $P_{F(T)\bigcap EP(f)} u$, where $P_{F(T)\bigcap EP(f)} : H \to F(T) \bigcap EP(f)$ is the metric projection of H onto $F(T) \bigcap EP(f)$.

Remark 3.3. If T is nonspreading, then T is 0-strictly pseudononspreading. By putting $\beta = 0$, then $T_0 = T$. By Theorem 3.1, we obtain the result of Kurokawa and Takahashi ([13], Theorem 4.1).

4. Example and numerical results

In this section, we give examples and numerical results for our main theorem.

Example 4.1. Let $T : [-9, 3] \to [-9, 3]$ be define by

$$Tx = \begin{cases} x, & \text{[-9, 0)}; \\ -3x, & \text{[0, 3]}. \end{cases}$$

Let $H = \mathbb{R}$ and $C = [-9, 3]$, and let $f(x, y) = y^2 + xy - 2x^2$. Find $\hat{x} \in [-9, 3]$ such that

$$F(\hat{x}, y) + \frac{1}{r} \langle y - \hat{x}, \hat{x} - z \rangle \geq 0, \ \ \forall y \in [-9, 3].$$

Solution. To see that T is $k-$strictly pseudononspreading, if $x, y \in [-9, 0)$, then

$$|Tx - Ty|^2 = |x - y|^2 + k|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty \rangle \ \ \forall k \in [0, 1),$$

since $|Tx - Ty|^2 = |x - y|^2$, and $k|x - Tx - (y - Ty)|^2 = 2\langle x - Tx, y - Ty \rangle = 0$. For all $x, y \in [0, 3]$, we have $|Tx - Ty|^2 = 9|x - y|^2$, $|x - Tx - (y - Ty)|^2 = 16|x - y|^2$ and $2\langle x - Tx, y - Ty \rangle = 32xy \geq 0$. Thus

$$\begin{aligned} |Tx - Ty|^2 = 9|x - y|^2 &= |x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2 \\ &\leq |x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty \rangle. \end{aligned}$$

Finally for all $x \in [-9, 0), y \in [0, 3]$ we have $|Tx - Ty|^2 = |x + 3y|^2 = x^2 + 6xy + 9y^2$, $2\langle x - Tx, y - Ty\rangle = 0$, and $\frac{1}{2}|x - Tx - (y - Ty)|^2 = 8y^2$. Hence

$$\begin{aligned}
&|x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\
&= x^2 - 2xy + 9y^2 \\
&= x^2 + 6xy + 9y^2 - 8xy \\
&\geq x^2 + 6xy + 9y^2 \ \ (\text{since} - 8xy \geq 0) \\
&= (x + 3y)^2 = |x + 3y|^2 = |Tx - Ty|^2.
\end{aligned}$$

Hence, for all $x, y \in [-9, 3]$, we obtain

$$|Tx - Ty|^2 \leq |x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle.$$

Thus T is $\frac{1}{2}-$strictly pseudononspreading, observe that $F(T) = [-9, 0]$. We observe that if $x = 1$, $y = 0$,

$$|Tx - Ty|^2 = 9|x - y|^2 = 9 > 1 = |x - y|^2 + 2\langle x - Tx, y - Ty\rangle$$

So T is not nonspreading. For $r > 0$ and $z \in [-9, 3]$, by Lemma 2.3, there exists $x \in [-9, 3]$ such that for each $y \in [-9, 3]$

$$\begin{aligned}
&f(x, y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0 \\
&\Leftrightarrow y^2 + xy - 2x^2 + \frac{1}{r}(y - x)(x - z) \geq 0 \\
&\Leftrightarrow ry^2 + rxy - 2rx^2 + xy - x^2 - yz + xz \geq 0 \\
&\Leftrightarrow ry^2 + (rx + x - z)y - (2rx^2 + x^2 - xz) \geq 0.
\end{aligned}$$

Put $G(y) = ry^2 + (rx + x - z)y - (2rx^2 + x^2 - xz)$. Then G is a quadratic function of y with coefficient $a = r, b = rx + x - z$ and $c = -(2rx^2 + x^2 - xz)$. We next compute the discriminant Δ of G as follows:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (rx + x - z)^2 + 4r(2rx^2 + x^2 - xz) \\
&= z^2 - 2(rx + x)z + (rx + x)^2 + 8rx^2 + 4rx^2 - 4rxz \\
&= z^2 - 2rxz - 2xz + r^2x^2 + 2rx^2 + x^2 + 8r^2x^2 + 4rx^2 - 4rxz \\
&= z^2 - 6rxz - 2xz + 9r^2x^2 + 6rx^2 + x^2 \\
&= z^2 - 2(3rx + x) + (9r^2 + 6r + 1)x^2 \\
&= [z - (3r + 1)x]^2
\end{aligned}$$

We know that $G(y) \geq 0$ for all $y \in [-9, 3]$. If it has most one solution in $[-9, 3]$, so $\Delta \leq 0$ and hence $z = 3rx + x$. Now we have $x = T_r z = \frac{z}{3r+1}$.
Since $T_\beta := \beta I + (1 - \beta)T$, we obtain

$$T_\beta x = \begin{cases} x, & x \in [-9, 0); \\ (4\beta - 3)x, & x \in [0, 3]. \end{cases}$$

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by $x_1 = x \in [-9, 3]$ and

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \ \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ \ n \geq 1, \end{cases} \tag{4.1}$$

We next give two numerical results for algorithm (4.1).

Let $\alpha_n = \frac{1}{200n}$ and $r_n = \frac{n}{n+1}$. Choose $\beta = \frac{5}{6}$ and $x_1 = u = 1$. Then algorithm (4.1) becomes

$$\begin{cases} x_{n+1} = \frac{1}{200n} + (1 - \frac{1}{200n})\left(\frac{z_n}{3r_n+1}\right) \ \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ \ n \geq 1. \end{cases} \tag{4.2}$$

n	x_n	z_n
1	1.000000	1.000000
2	0.403000	0.268667
3	0.091832	0.044215
4	0.015249	0.005648
5	0.002909	0.000869
$\vdots$	$\vdots$	$\vdots$
123	0.000041	0.000001
124	0.000041	0.000000

Table 1:

Let $\alpha_n = \frac{1}{200n}$ and $r_n = \frac{n}{n+1}$. Choose $\beta = \frac{5}{6}$ and $x_1 = u = -1$. Then algorithm (4.1) becomes

$$\begin{cases} x_{n+1} = -\frac{1}{200n} + (1 - \frac{1}{200n})\left(\frac{Z_n}{3r_n+1}\right) \ \ n \geq 1, \\ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, \ \ n \geq 1. \end{cases} \tag{4.3}$$

n	x_n	z_n
1	-1.00000	-1.00000
2	-0.40300	-0.40300
3	-0.13650	-0.13650
4	-0.04360	-0.04360
5	-0.01406	-0.01406
$\vdots$	$\vdots$	$\vdots$
68	-0.00001	-0.00001
69	0.00000	0.00000

Table 2:

Conclusion. Table 1 and Table 2 show that the sequence $\{x_n\}$ and $\{z_n\}$ converge to 0 which solves both the equilibrium problem of f and the fixed point problem of T. On the other hand, using Lemma 2.3 (3) , we can check that $F(T_r) = EP(f) = \{0\}$.

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