

A class of bi-univalent functions defined by (p, q) -derivative operator subordinate to (m, n) -Lucas polynomials

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Abstract

We propose a category of normalized analytic functions given by $g(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j \zeta^j$ that are bi-univalent in the disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$ defined by (p, q) -derivative operator, subordinate to (m, n) -Lucas polynomials. For members of this family, we determine estimates for the coefficients $|d_2|$ and $|d_3|$ and the Fekete-Szegő result. New implications of the primary result, as well as pertinent links to previously published findings, are also provided.

Keywords: Bi-univalent function, (p, q) -derivative operator, (m, n) -Lucas polynomial, Fekete-Szegő problem
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1 Preliminaries

Quantum calculus is essential because it is applied in many branches of mathematics, computer science, physics, and other fields. The quantum calculus's extension to the (p, q) -calculus was taken into consideration by the researchers. The (p, q) -calculus, which includes the (p, q) -number, is first examined around the same time (1991) and subsequently on its own by [7, 10, 15, 43]. Fibonacci oscillators were studied with the presentation of the (p, q) -number in [7]. The investigation of the (p, q) -number in [10] allows for the construction of a (p, q) -Harmonic oscillator. In [15], the (p, q) -number was explored to unify or generalize various forms of q -oscillator algebras. The (p, q) -numbers are investigated in [43] to calculate the (p, q) -Stirling numbers. Consequently, many mathematical, computer science, physical, chemical and other related problems require knowledge of (p, q) -calculus. Expanding upon the previously mentioned papers, numerous scientists have studied the (p, q) -calculus in a variety of research fields since 1991. A syntax for embedding the q -series into a (p, q) -series was given by the results in [22]. Additionally, they looked into the (p, q) -hypergeometric series and discovered some outcomes that matched (p, q) -extensions of the well-known q -identities. The q -identities are extended correspondingly to yield the (p, q) -series (see, e.g., [6]). We give some elementary definitions of the terms used in this paper related to (p, q) -calculus. The (p, q) -bracket number is given by $[j]_{p,q} = p^{j-1} + p^{j-2}q + \dots + p^2q^{j-3} + pq^{j-2} + q^{j-1} = \frac{p^j - q^j}{p - q}$ ($p \neq q$), which is an extension of q -number (see [21]), that is $[j]_q = \frac{1 - q^j}{1 - q}$ ($q \neq 1$). Note that $[j]_{p,q}$ is symmetric and if $p=1$, then $[j]_{p,q} = [j]_q$.

Let $\mathbb{N} = \mathbb{N}_0 \setminus \{0\} := \{1, 2, 3, \dots\}$ and \mathbb{R} be the family of real numbers. let $\mathfrak{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, where \mathbb{C} is the complex numbers set.

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Definition 1.1. [41] Consider a function g defined on \mathbb{C} and let $0 < q < p \leq 1$. Then the (p, q) -derivative of g is defined by

$$D_{p,q}g(\zeta) = \frac{g(p\zeta) - g(q\zeta)}{(p - q)\zeta} \quad (\zeta \neq 0),$$

and $D_{p,q}g(0) = g'(0)$, provided $g'(0)$ exists.

We note that $D_{p,q}\zeta^j = [j]_{p,q}\zeta^{j-1}$ and $D_{p,q}\ln(\zeta) = \frac{\ln(p/q)}{(p-q)\zeta}$. Also, $[j]_{p,q} \rightarrow j$, if $p = 1$ and $q \rightarrow 1^-$. Therefore, $D_{p,q}g(\zeta) \rightarrow g'(\zeta)$ as $p = 1$ and $q \rightarrow 1^-$. Any function's (p, q) -derivative is a linear operator. More accurately $D_{p,q}(cg_1(\zeta) + dg_2(\zeta)) = cD_{p,q}g_1(\zeta) + dD_{p,q}g_2(\zeta)$, where c and d are constants. The (p, q) -derivative satisfies the product rules and quotient rules (see [30]). Exponential functions are used to introduce the (p, q) -analogues of many functions, including sine, cosine, and tangent similar to how their Euler expressions. Durani et al. [17] have examined the (p, q) -derivatives of these functions. For further details on (p, q) -calculus, see, among other sources, ([11, 17, 41]).

Let us take a normalized regular function g in \mathfrak{D} given by

$$g(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j \zeta^j, \tag{1.1}$$

and let \mathcal{A} be the class of all such functions. Let $\mathcal{S} = \{g \in \mathcal{A} : g \text{ is univalent in } \mathfrak{D}\}$. If $g \in \mathcal{A}$ is of the form (1.1), then

$$D_{p,q}g(\zeta) = 1 + \sum_{j=2}^{\infty} [j]_{p,q} d_j \zeta^{j-1}, \quad (\zeta \in \mathfrak{D}). \tag{1.2}$$

The renowned Koebe theorem (see [18]) states that each function $g \in \mathcal{S}$ has an inverse and is defined as

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \dots \tag{1.3}$$

satisfying $\zeta = g^{-1}(g(\zeta))$ and $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$, $r_0(g) \geq 1/4$, $\zeta, \omega \in \mathfrak{D}$. The notion of bi-univalent functions was first presented by Levin in his work [24]. These are analytic functions, denoted by g , where both g and g^{-1} are univalent in \mathfrak{D} . The set of all bi-univalent functions of the type (1.1) is symbolized by Σ . $\frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right)$, $-\log(1-\zeta)$ and $\frac{\zeta}{1-\zeta}$ are some of the functions in the Σ family. However, $\zeta - \frac{\zeta^2}{2}$, $\frac{\zeta}{1-\zeta^2}$, and the Koebe function do not belong in Σ , even though they are in \mathcal{S} . For a concise analysis and to discover some of the remarkable characteristics of the family Σ , see [8, 9, 40] and the citation provided in these papers. The article by Srivastava et al. [32] gave rise to the recent momentum of studies of the bi-univalent function family. Numerous scholars have looked into several fascinating special families of Σ since this article brought the subject back to life (see [12, 13, 20]).

The (p, q) -calculus was used to study several subclasses of the class \mathcal{S} and the class Σ . In [33], the subordination principle is used to define the (p, q) -starlike and (p, q) -convex functions classes. Novel subclasses of the class Σ associated with (p, q) -differential operators have also been presented and examined in several studies (refer to [3, 5, 16, 28, 27, 42]).

The (m, n) -Lucas polynomials $L_j(\mathcal{x})$ (or $L_j(m(\mathcal{x}), n(\mathcal{x}), \mathcal{x})$) are defined by the below mentioned recurrence relation (see [23]):

$$L_0(\mathcal{x}) = 2, L_1(\mathcal{x}) = m(\mathcal{x}), \quad L_j(\mathcal{x}) = m(\mathcal{x})L_{j-1}(\mathcal{x}) + n(\mathcal{x})L_{j-2}(\mathcal{x}), \tag{1.4}$$

where $j \in \mathbb{N} \setminus \{1\}$, $m(\mathcal{x})$ and $n(\mathcal{x})$ are real polynomials. The generating function (GF) of the (m, n) -Lucas polynomials $L_j(\mathcal{x})$ is given by

$$\mathcal{G}(\mathcal{x}, \zeta) := \sum_{j=0}^{\infty} L_j(\mathcal{x})\zeta^j = \frac{2 - m(\mathcal{x})\zeta}{1 - m(\mathcal{x})\zeta - n(\mathcal{x})\zeta^2}. \tag{1.5}$$

One can easily find from (1.4) that $L_2(\mathcal{x}) = m^2(\mathcal{x}) + 2n(\mathcal{x})$, $L_3(\mathcal{x}) = m^3(\mathcal{x}) + 3m(\mathcal{x})n(\mathcal{x})$ and so on. For specific selections of $m(\mathcal{x})$ and $n(\mathcal{x})$, the (m, n) -Lucas polynomials $L_j(\mathcal{x})$ (or $L_j(m(\mathcal{x}), n(\mathcal{x}), \mathcal{x})$) leads to various known polynomials (see [4]). For members of some subfamilies of Σ linked to (m, n) -Lucas polynomials, fascinating findings about coefficient estimations and Fekete- Szegő result has been found in [1, 25, 29, 36, 37, 38, 39].

For functions g_1 and g_2 regular in \mathfrak{D} , g_1 is said to subordinate g_2 , if there is a Schwarz function ψ in \mathfrak{D} , satisfying $\psi(0) = 0$, $|\psi(\zeta)| < 1$ and $g_1(\zeta) = g_2(\psi(\zeta))$, $\zeta \in \mathfrak{D}$. The notation $g_1 \prec g_2$ indicates this subordination. If $g_2 \in \mathcal{S}$, then $g_1(\zeta) \prec g_2(\zeta)$ is equivalent to $g_1(0) = g_2(0)$ and $g_1(\mathfrak{D}) \subset g_2(\mathfrak{D})$.

Definition 1.2. The (p, q) -analogue of Swamy differential operator for $g \in \mathcal{A}$ is defined as follows:

$$\begin{aligned} W_{p,q}^{\nu,\mu,0}g(\zeta) &= g(\zeta), \\ W_{p,q}^{\nu,\mu,1}g(\zeta) &= \frac{\nu g(\zeta) + \mu z D_{p,q}g(\zeta)}{\nu + \mu}, \\ &\vdots \\ W_{p,q}^{\nu,\mu,k}g(\zeta) &= W_{p,q}^{\nu,\mu}(W_{p,q}^{\nu,\mu,k-1}g(\zeta)), \end{aligned}$$

where $\zeta \in \mathfrak{D}, \mu \geq 0, \nu$ a real number with $\nu + \mu > 0, k \in \mathbb{N}$ and $0 < q < p \leq 1$.

Remark 1.3. i) We observe that $W_{p,q}^{\nu,\mu,k} : \mathcal{A} \rightarrow \mathcal{A}$ is a linear operator. We have

$$W_{p,q}^{\nu,\mu,k}g(\zeta) = \zeta + \sum_{j=2}^{\infty} \left(\frac{\nu + \mu[j]_{p,q}}{\nu + \mu} \right)^k d_j \zeta^j, \tag{1.6}$$

for $g(\zeta)$ given by (1.1).

ii) If we let $\nu = 0$ and $\mu = 1$, then $W_{p,q}^{\nu,\mu,k}g(\zeta)$ reduces to the (p, q) -analogue of Salagean operator discussed in [31].

iii) If we take $\nu = 1 - \mu, \mu \geq 0$, then $A_{p,q}^{\mu,k} (= W_{p,q}^{1-\mu,\mu,k}) : \mathcal{A} \rightarrow \mathcal{A}$ is a linear operator. For $g(\zeta)$ given by (1.1), we have

$$A_{p,q}^{\mu,k}g(\zeta) = \zeta + \sum_{j=2}^{\infty} (1 + \mu([j]_{p,q} - 1))^k d_j \zeta^j, \tag{1.7}$$

which is (p, q) -analogue of Al-Oboudi differential operator.

iv) If we put $\nu = l + 1 - \mu, l > -1, \mu \geq 0$, then $C_{p,q}^{l,\mu,k} (= W_{p,q}^{l+1-\mu,\mu,k}) : \mathcal{A} \rightarrow \mathcal{A}$ is another linear operator. For $g(\zeta)$ given by (1.1), we have

$$C_{p,q}^{l,\mu,k}g(\zeta) = \zeta + \sum_{j=2}^{\infty} \left(\frac{l + 1 + \mu([j]_{p,q} - 1)}{l + 1} \right)^k d_j \zeta^j, \tag{1.8}$$

which is (p, q) -analogue of Catas differential operator.

v) Swamy operator [35], Al-Oboudi operator [2], and Cătaş operator [14] are obtained by taking $q \rightarrow 1^-$ and $p = 1$ in (1.6), (1.7), and (1.8), respectively.

We propose a subfamily of Σ using the (p, q) -analogue of the Swamy derivative operator, subordinate to (m, n) -Lucas polynomials $L_j(\varkappa)$ as in (1.4) with GF (1.5). In this paper, $0 < q < p \leq 1$ is always satisfied by the parameters p and q . This paper also makes use of the inverse functions $g^{-1}(\omega) = f(\omega)$ and $\mathcal{G}(\varkappa, \zeta)$, which are as in (1.3) and (1.5), respectively.

Definition 1.4. Any function $g \in \Sigma$ is said to be in the family $\mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa, \nu, \mu)$, if

$$\frac{1}{2} \left\{ \frac{\zeta(W_{p,q}^{\nu,\mu,k}g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta(W_{p,q}^{\nu,\mu,k}g(\zeta))'}{g(\zeta)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \zeta) - 1, \zeta \in \mathfrak{D}$$

and

$$\frac{1}{2} \left\{ \frac{\omega(W_{p,q}^{\nu,\mu,k}f(\omega))'}{f(\omega)} + \left(\frac{\omega(W_{p,q}^{\nu,\mu,k}f(\omega))'}{f(\omega)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \omega) - 1, \omega \in \mathfrak{D},$$

where $0 < \tau \leq 1, \mu \geq 0, \nu$ a real number with $\nu + \mu > 0$, and $k \in \mathbb{N}$.

For particular choices of p, q, τ , and ν , the family $\mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa, \nu, \mu)$ includes many new subfamilies of Σ as mentioned below:

Example 1.1. $S_{\Sigma,p,q}^{\tau,k}(\varkappa, \mu) \equiv \mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa, 1 - \mu, \mu), 0 < \tau \leq 1, \mu \geq 0$, and $k \in \mathbb{N}$ is the set of members $g \in \Sigma$ that satisfy

$$\frac{1}{2} \left\{ \frac{\zeta(A_{p,q}^{\mu,k}g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta(A_{p,q}^{\mu,k}g(\zeta))'}{g(\zeta)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \zeta) - 1, \zeta \in \mathfrak{D}$$

and

$$\frac{1}{2} \left\{ \frac{\omega(A_{p,q}^{\mu,k}f(\omega))'}{f(\omega)} + \left(\frac{\omega(A_{p,q}^{\mu,k}f(\omega))'}{f(\omega)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \omega) - 1, \omega \in \mathfrak{D}.$$

Example 1.2. $T_{\Sigma,p,q}^{\tau,k}(\varkappa, l, \mu) \equiv \mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa, l + 1 - \mu, \mu), \mu \geq 0, 0 < \tau \leq 1, l > -1$, and $k \in \mathbb{N}$ is the set of members $g \in \Sigma$ that satisfy

$$\frac{1}{2} \left\{ \frac{\zeta(C_{p,q}^{\mu,k}g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta(C_{p,q}^{\mu,k}g(\zeta))'}{g(\zeta)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \zeta) - 1, \zeta \in \mathfrak{D}$$

and

$$\frac{1}{2} \left\{ \frac{\omega(C_{p,q}^{\mu,k}f(\omega))'}{f(\omega)} + \left(\frac{\omega(C_{p,q}^{\mu,k}f(\omega))'}{f(\omega)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \omega) - 1, \omega \in \mathfrak{D}.$$

Example 1.3. $\mathfrak{R}_{\Sigma,p,q}^k(x, \nu, \mu) \equiv \mathfrak{S}_{\Sigma,p,q}^{1,k}(\varkappa, \nu, \mu), \mu \geq 0, \nu$ a real number with $\nu + \mu > 0$, and $k \in \mathbb{N}$ is the set of members $g \in \Sigma$ that satisfy

$$\frac{\zeta(W_{p,q}^{\nu,\mu,k}g(\zeta))'}{g(\zeta)} \prec \mathcal{G}(\varkappa, \zeta) - 1, \zeta \in \mathfrak{D}$$

and

$$\frac{\omega(W_{p,q}^{\nu,\mu,k}f(\omega))'}{f(\omega)} \prec \mathcal{G}(\varkappa, \omega) - 1, \omega \in \mathfrak{D}.$$

Example 1.4. If $p = 1$ and $q \rightarrow 1^-$ in the set $\mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa, \nu, \mu)$, then we obtain a subset $\mathfrak{Y}_{\Sigma}^{\tau,k}(\varkappa, \nu, \mu)$, which is the collection of members of $g \in \Sigma$ that satisfy

$$\frac{1}{2} \left\{ \frac{\zeta(\mathfrak{W}^{\nu,\mu,k}g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta(\mathfrak{W}^{\nu,\mu,k}g(\zeta))'}{g(\zeta)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \zeta) - 1, \zeta \in \mathfrak{D}$$

and

$$\frac{1}{2} \left\{ \frac{\omega(\mathfrak{W}^{\nu,\mu,k}f(\omega))'}{f(\omega)} + \left(\frac{\omega(\mathfrak{W}^{\nu,\mu,k}f(\omega))'}{f(\omega)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \omega) - 1, \omega \in \mathfrak{D},$$

where $\mathfrak{W}^{\nu,\mu,k} \equiv W_{p=1,q \rightarrow 1^-}^{\nu,\mu,k}, \mu \geq 0, \nu$ a real number with $\nu + \mu > 0, k \in \mathbb{N}$, and $0 < \tau \leq 1$.

Fekete-Szegö inequality [19] and estimates for $|d_2|$ and $|d_3|$ are found in Section 2 for functions in $\mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa, \nu, \mu)$. Along with relevant links to the earlier results, there are also a few intriguing implications of the main result.

2 Main Results

Initially, we compute the Fekete-Szegö inequality for functions in $\mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa, \nu, \mu)$, as well as the bounds for $|d_2|$ and $|d_3|$.

Theorem 2.1. Let $0 < \tau \leq 1, \mu \geq 0, \nu$ a real number such that $\nu + \mu > 0$ and $k \in \mathbb{N}$. If a function $g \in \mathfrak{S}_{\Sigma,p,q}^{\tau,k}(\varkappa, \nu, \mu)$, then

(i)

$$|d_2| \leq \frac{2\tau\sqrt{|m^3(\varkappa)|}}{\sqrt{|(2\tau(\tau + 1)(N - M) - \tau(\tau + 3)M^2)m^2(\varkappa) - 2n(\varkappa)(1 + \tau)^2|}}, \tag{2.1}$$

(ii)

$$|d_3| \leq \frac{4\tau^2 m^2(\varkappa)}{(1+\tau)^2 M^2} + \frac{2\tau |m(\varkappa)|}{(1+\tau)N}, \tag{2.2}$$

and for $\xi \in \mathbb{R}$,

(iii)

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{2\tau |m(\varkappa)|}{(1+\tau)N} & ; |1 - \xi| \leq \mathbb{J} \\ \frac{4\tau^2 |m^3(\varkappa)| |1-\xi|}{|(2\tau(\tau+1)(N-M) - \tau(\tau+3)M^2)m^2(\varkappa) - 2n(\varkappa)(1+\tau)^2|} & ; |1 - \xi| \geq \mathbb{J}, \end{cases} \tag{2.3}$$

where

$$\mathbb{J} = \frac{|(2\tau(\tau+1)(N-M) - \tau(\tau+3)M^2)m^2(\varkappa) - 2n(\varkappa)(1+\tau)^2|}{4\tau^2 m^2(\varkappa)}, \tag{2.4}$$

$$M = \left(2 \left(\frac{\nu + \mu [2]_{p,q}}{\nu + \mu} \right)^k - 1 \right), \tag{2.5}$$

and

$$N = \left(3 \left(\frac{\nu + \mu [3]_{p,q}}{\nu + \mu} \right)^k - 1 \right). \tag{2.6}$$

Proof . Let $g \in \mathfrak{S}_{\Sigma, p, q}^{\tau, k}(\varkappa, \nu, \mu)$. Then, we can write because of Definition 1.4:

$$\frac{1}{2} \left\{ \frac{\zeta (W_{p,q}^{\nu, \mu, k} g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta (W_{p,q}^{\nu, \mu, k} g(\zeta))'}{g(\zeta)} \right)^{\frac{1}{\delta}} \right\} = \mathcal{G}(\varkappa, r(\zeta)) - 1$$

and

$$\frac{1}{2} \left\{ \frac{\omega (W_{p,q}^{\nu, \mu, k} f(\omega))'}{f(\omega)} + \left(\frac{\omega (W_{p,q}^{\nu, \mu, k} f(\omega))'}{f(\omega)} \right)^{\frac{1}{\delta}} \right\} = \mathcal{G}(\varkappa, s(\omega)) - 1,$$

where r and s are functions regular in \mathfrak{D} , with $r(0) = 0, |r(\zeta)| < 1, s(0) = 0, |s(\omega)| < 1$. Or, equivalently

$$\frac{1}{2} \left\{ \frac{\zeta (W_{p,q}^{\nu, \mu, k} g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta (W_{p,q}^{\nu, \mu, k} g(\zeta))'}{g(\zeta)} \right)^{\frac{1}{\delta}} \right\} = -1 + L_0(\varkappa) + L_1(\varkappa)r(\zeta) + L_2(\varkappa)r^2(\zeta) + \dots, \tag{2.7}$$

and

$$\frac{1}{2} \left\{ \frac{\omega (W_{p,q}^{\nu, \mu, k} f(\omega))'}{f(\omega)} + \left(\frac{\omega (W_{p,q}^{\nu, \mu, k} f(\omega))'}{f(\omega)} \right)^{\frac{1}{\delta}} \right\} = -1 + L_0(\varkappa) + L_1(\varkappa)s(\omega) + L_2(\varkappa)s^2(\omega) + \dots. \tag{2.8}$$

On account of (1.4), (2.7) and (2.8), we obtain

$$\frac{1}{2} \left\{ \frac{\zeta (W_{p,q}^{\nu, \mu, k} g(\zeta))'}{g(\zeta)} + \left(\frac{\zeta (W_{p,q}^{\nu, \mu, k} g(\zeta))'}{g(\zeta)} \right)^{\frac{1}{\delta}} \right\} = 1 + L_1(\varkappa)r_1\zeta + [L_1(\varkappa)r_2 + L_2(\varkappa)r_1^2]\zeta^2 + \dots, \tag{2.9}$$

and

$$\frac{1}{2} \left\{ \frac{\omega (W_{p,q}^{\nu, \mu, k} f(\omega))'}{f(\omega)} + \left(\frac{\omega (W_{p,q}^{\nu, \mu, k} f(\omega))'}{f(\omega)} \right)^{\frac{1}{\delta}} \right\} = 1 + L_1(\varkappa)s_1\omega + [L_1(\varkappa)s_2 + L_2(\varkappa)s_1^2]\omega^2 + \dots. \tag{2.10}$$

If $|r(\zeta)| = |r_1\zeta + r_2\zeta^2 + r_3\zeta^3 + \dots| < 1, \zeta \in \mathfrak{D}$, and if $|s(\omega)| = |s_1\omega + s_2\omega^2 + s_3\omega^3 + \dots| < 1, \omega \in \mathfrak{D}$, then we know that

$$|r_i| \leq 1 \quad \text{and} \quad |s_i| \leq 1 \quad (i \in \mathbb{N}). \tag{2.11}$$

From (2.9) and (2.10), it can be inferred that

$$\left(\frac{1+\tau}{2\tau} \right) M d_2 = L_1(\varkappa)r_1, \tag{2.12}$$

$$\left(\frac{1+\tau}{2\tau}\right)(Nd_3 - Md_2^2) + \left(\frac{1-\tau}{4\tau^2}\right)M^2d_2^2 = L_1(\varkappa)r_2 + L_2(\varkappa)r_1^2, \quad (2.13)$$

$$-\left(\frac{1+\tau}{2\tau}\right)Md_2 = L_1(\varkappa)s_1, \quad (2.14)$$

and

$$\left(\frac{1+\tau}{2\tau}\right)(N(2d_2^2 - d_3) - Md_2^2) + \left(\frac{1-\tau}{4\tau^2}\right)M^2d_2^2 = L_1(\varkappa)s_2 + L_2(\varkappa)s_1^2. \quad (2.15)$$

From (2.12) and (2.14), we have

$$r_1 = -s_1 \quad (2.16)$$

and also

$$\left(\frac{(1+\tau)^2}{2\tau^2}\right)M^2d_2^2 = (r_1^2 + s_1^2)(L_1(\varkappa))^2. \quad (2.17)$$

If we add (2.13) and (2.15), then we obtain

$$2\left[\left(\frac{1+\tau}{\tau}\right)(N-M) + \left(\frac{1-\tau}{2\tau^2}\right)M^2\right]d_2^2 = L_1(\varkappa)(r_2 + s_2) + L_2(\varkappa)(r_1^2 + s_1^2). \quad (2.18)$$

Substituting the value of $(r_1^2 + s_1^2)$ from (2.17) in (2.18), we get

$$d_2^2 = \frac{2\tau^2(L_1(\varkappa))^3(r_2 + s_2)}{2[(2\tau(\tau+1)(N-M) + (1-\tau)M^2)(L_1(\varkappa))^2 - (1+\tau)^2M^2L_2(\varkappa)]}, \quad (2.19)$$

which produces (2.1), when applied (2.11). After deducting (2.15) from (2.13) and then applying (2.16), we get

$$d_3 = d_2^2 + \frac{\tau L_1(\varkappa)(r_2 - s_2)}{(1+\tau)N}. \quad (2.20)$$

Then in view of (2.17), (2.20) becomes

$$d_3 = \frac{2\tau^2(L_1(\varkappa))^2(r_1^2 + s_1^2)}{(1+\tau)^2M^2} + \frac{\tau L_1(\varkappa)(r_2 - s_2)}{(1+\tau)N},$$

which produces (2.2), when applied (2.11). For $\xi \in \mathbb{R}$, we obtain in view of (1.4) from (2.19) and (2.20):

$$|d_3 - \xi d_2^2| = |L_1(\varkappa)| \left| \left(B(\xi, \varkappa) + \frac{\tau}{(1+\tau)N} \right) r_2 + \left(B(\xi, \varkappa) - \frac{\tau}{(1+\tau)N} \right) s_2 \right|,$$

where

$$B(\xi, x) = \frac{2\tau^2(1-\xi)L_1^2(\varkappa)}{[(2\tau(\tau+1)(N-M) + M^2(1-\tau))L_1^2(\varkappa) - M^2(1+\tau)^2L_2(\varkappa)]}.$$

Clearly,

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{2\tau|L_1(\varkappa)|}{(1+\tau)N}; & 0 \leq |B(\xi, \varkappa)| \leq \frac{\tau}{(1+\tau)N} \\ 2|B(\xi, \varkappa)||L_1(\varkappa)|; & |B(\xi, \varkappa)| \geq \frac{\tau}{(1+\tau)N}. \end{cases}$$

This produces (2.3), where J is the same as in (2.4) on using $L_1(\varkappa) = m(\varkappa)$, $L_2(\varkappa) = m^2(\varkappa) + 2n(\varkappa)$. Thus, Theorem 2.1 has been demonstrated. \square

Remark 2.2. The results of Altınkaya and Yalçın [3, Theorem 2.1 and Theorem 3.1] are obtained by allowing $\mu = 1, \nu = 0, n(x) = 1$, and $m(x) = x$ in Theorem 2.1.

Corollary 2.3. Let us assume that $\nu = 1 - \mu$ in Theorem 2.1. Then the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for any function $g \in S_{\Sigma, p, q}^{\tau, k}(\varkappa, \mu)$ are given by (2.1), (2.2), and (2.3), respectively, with $M = M_1 = 2(1 + \mu([2]_{p, q} - 1)^k - 1)$, and $N = N_1 = 3(1 + \mu([3]_{p, q} - 1)^k - 1)$. For J in (2.4), M , and N are to be substituted with M_1 , and N_1 , respectively.

Corollary 2.4. Let us assume that $\nu = l + 1 - \mu$ in Theorem 2.1. Then the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for any function $g \in T_{\Sigma, p, q}^{\tau, k}(\varkappa, l, \mu)$ are given by (2.1), (2.2), and (2.3), respectively, with

$$M = M_2 = \left(2 \left(\frac{l + 1 + \mu([2]_{p, q} - 1)}{l + 1} \right)^k - 1 \right), \quad \text{and} \quad N = N_2 = \left(3 \left(\frac{l + 1 + \mu([3]_{p, q} - 1)}{l + 1} \right)^k - 1 \right).$$

For J in (2.4), M and N are to be substituted with M_2 and N_2 , respectively.

When $\tau = 1$, the following would result from Theorem 2.1.

Corollary 2.5. Let $\mu \geq 0, \nu$ a real number satisfying $\nu + \mu > 0$ and $k \in \mathbb{N}$. If a function $g \in \mathfrak{R}_{\Sigma, p, q}^k(\varkappa, \nu, \mu)$, then

(i)

$$|d_2| \leq \frac{\sqrt{|m^3(\varkappa)|}}{\sqrt{|(N - M - M^2)m^2(\varkappa) - 2M^2n(\varkappa)|}},$$

(ii)

$$|d_3| \leq \frac{m^2(\varkappa)}{M^2} + \frac{|m(\varkappa)|}{N},$$

and for $\xi \in \mathbb{R}$

(iii)

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|m(\varkappa)|}{N}; & |1 - \xi| \leq \mathbb{J}_1 \\ \frac{|m^3(\varkappa)||1 - \xi|}{|(N - M - M^2)m^2(\varkappa) - 2M^2n(\varkappa)|}; & |1 - \xi| \geq \mathbb{J}_1, \end{cases}$$

where (2.5) provides M , (2.6) provides N , and

$$\mathbb{J}_1 = \left| \frac{(N - M - M^2)m^2(\varkappa) - 2M^2n(\varkappa)}{4m^2(\varkappa)} \right|.$$

Remark 2.6. We acquire the results of Altinkaya and Yalçın [3, Corollary 2.1 and Corollary 3.1] by taking $n(x) = 1, m(x) = x, \nu = 0$, and $\mu = 1$ in Corollary 2.5.

Corollary 2.7. Let us assume that $q \rightarrow 1^-$ and $p = 1$ in Theorem 2.1. Then the upper bounds of $|d_2|, |d_3|$, and $|d_3 - \xi d_2^2|, \xi \in \mathbb{R}$, for any function $g \in \mathfrak{Q}_{\Sigma}^{\tau, k}(\varkappa, \nu, \mu)$, are given by (2.1), (2.2), and (2.3), respectively, with $M = M_3 = \left(2 \left(\frac{\nu + 2\mu}{\nu + \mu} \right)^k - 1 \right)$, and $N = N_3 = \left(3 \left(\frac{\nu + 3\mu}{\nu + \mu} \right)^k - 1 \right)$. For J in (2.4), M and N are to be substituted with M_3 and N_3 , respectively.

Remark 2.8. If $k = 0$ in the set $\mathfrak{Q}_{\Sigma}^{\tau, k}(\varkappa, \nu, \mu)$, then we obtain a subset $\mathfrak{Q}_{\Sigma}^{\tau}(\varkappa), 0 < \tau \leq 1$, which is the collection of members of $g \in \Sigma$ that satisfy

$$\frac{1}{2} \left\{ \frac{\zeta g'(\zeta)}{g(\zeta)} + \left(\frac{\zeta g'(\zeta)}{g(\zeta)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \zeta) - 1, \zeta \in \mathfrak{D}$$

and

$$\frac{1}{2} \left\{ \frac{\omega f'(\omega)}{f(\omega)} + \left(\frac{\omega f'(\omega)}{f(\omega)} \right)^{\frac{1}{\tau}} \right\} \prec \mathcal{G}(\varkappa, \omega) - 1, \omega \in \mathfrak{D}.$$

Corollary 2.9. Let $0 < \tau \leq 1$. If a function $g \in \mathfrak{Q}_{\Sigma}^{\tau}(\varkappa)$, then

(i)

$$|d_2| \leq \frac{2\tau \sqrt{|m^3(\varkappa)|}}{\sqrt{|\tau(\tau - 1)m^2(\varkappa) - 2(1 + \tau)^2n(\varkappa)|}},$$

(ii)

$$|d_3| \leq m^2(\varkappa) + \frac{|m(\varkappa)|}{2},$$

and for $\xi \in \mathbb{R}$

(iii)

$$|d_3 - \xi d_2^2| \leq \begin{cases} \frac{|m(\varkappa)|}{2}; & |1 - \xi| \leq \left| \frac{\tau(\tau - 1)m^2(\varkappa) - 2(1 + \tau)^2 n(\varkappa)}{4m^2(\varkappa)} \right| \\ \frac{|m^3(\varkappa)| |1 - \xi|}{|\tau(\tau - 1)m^2(\varkappa) - 2(1 + \tau)^2 n(\varkappa)|}; & |1 - \xi| \geq \left| \frac{\tau(\tau - 1)m^2(\varkappa) - 2(1 + \tau)^2 n(\varkappa)}{4m^2(\varkappa)} \right|. \end{cases}$$

Remark 2.10. We obtain the results of Altinkaya and Yalçın [4, Corollaries 1 and 3] by taking $\tau = 1$ in Corollary 2.9. These results are also stated in [29].

3 Conclusions

This study establishes upper bounds on $|d_2|$ and $|d_3|$ for functions in subfamily of Σ related to (m, n) -Lucas polynomials. Moreover, the Fekete-Szegő functional $|d_3 - \xi d_2^2|$, $\xi \in \mathbb{R}$ has been identified for functions in these subfamilies. By adjusting the parameters in Theorem 2.1, a few implications have been brought to light. Relevant connections to the current research are also discovered. Nevertheless, this paper does not address all of the significant subclasses of Σ that exist in the literature. For example, authors [26, 28, 42] have examined various subclasses involving (p, q) -operators introduced in (p, q) -calculus. Functions from this class may be studied with interesting results due to their symmetry properties. Topics about fuzzy differential subordination, and fuzzy differential subordination may be added in future research. By defining the (p, q) -analogue of the Swamy operator defined for p -valent functions [35], the results obtained in this paper could be extended. It is recommended that the interested reader review these papers and the associated references.

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