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Principally π-Baer rings

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Abstract

We say a ring is right principally π-Baer (or simply right $p.\pi$-Baer) if an idempotent generates the right annihilator of every projection invariant principal left ideal. The class of right $p.\pi$-Baer rings includes the von Neumann regular rings (and hence right p.p-rings) and all π-Baer rings. This class of rings is closed under direct products. The behavior of the right $p.\pi$-Baer condition is investigated concerning various constructions and extensions. Moreover, we extend a theorem of Kist for commutative p.p-rings to right $p.\pi$-Baer rings for which every prime ideal contains a unique minimal prime ideal without using topological arguments.

Keywords: p.p ring, Baer ring, quasi-Baer ring, principally quasi-Baer ring, π-Baer ring, $p.\pi$-Baer ring, semicentral idempotent
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1 Introduction

Throughout this paper, all rings are associative with identity, and all modules are unital. A ring R is called (quasi-)Baer if the right annihilator of each (right ideal) nonempty subset of R is generated by an idempotent. In [12], Kaplansky introduced Baer rings to abstract various properties of AW^*-algebras and von Neumann algebras. In [9], Clark defined a ring to be quasi-Baer if the left annihilator of each ideal is generated, as a left ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra.

A ring R is called right (left) $p.p$ if every principal right (left) ideal is projective (equivalently, if the right (left) annihilator of any element of R is generated (as a right (left) ideal) by an idempotent of R. The ring R is called $p.p$ if it is both right and left $p.p$. Birkenmeier et al. [7] initiated the concept of principally quasi-Baer rings. A ring R is called right principally quasi-Baer (or simply right $p.q$-Baer) if an idempotent generates the right annihilator of a principal right ideal. Equivalently, R is right $p.q$-Baer if R modulo the right annihilator of any principal right ideal is projective. A ring R is called $p.q$-Baer if it is both right and left $p.q$–Baer. The class of $p.q$-Baer rings includes all biregular rings, all quasi-Baer rings and all abelian $p.p$ rings. Further work on quasi-Baer and $p.q$-Baer rings appears in [4, 5, 6, 7, 15].

Recall from [2] that the left (right) ideal I of R is said projection invariant if for each $e = e^2 \in R$, $Ie \subseteq I (eI \subseteq I)$. If R is an abelian ring, then every one-sided ideal is an invariant projection. Note that every ideal of R is projection

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invariant, but not conversely. In [2], Birkenmeier et al. define a ring to be π-Baer if the right annihilator of every projection invariant left ideal (as a right ideal) is generated by an idempotent of R. Like the quasi-Baer case, the $\pi-$Baer condition is left-right symmetric.

Shahidikia et al. in [16] introduced the concept of generalized $\pi-$Baer rings. A ring R is called generalized right $\pi-$Baer, if for any projection invariant left ideal I of R, the right annihilator of I^n is generated by an idempotent, for some positive integer n, depending on I.

In this paper, we introduce the notion of right (left) $p.\pi$-Baer rings. Ring R is right $p.\pi$-Baer if the right annihilator of each projection invariant principal left ideal is generated by an idempotent. Left case may be defined analogously. If ring R is both right and left $p.\pi$-Baer then we say R is $p.\pi$-Baer. The class of right $p.\pi$-Baer is closed with respect to direct products. If R is an abelian ring, then the $p.\pi$-Baer and $p.p$ conditions agree.

A ring R [8] is right π-extending if every projection invariant right ideal of R is essential in a direct summand of R_R. We say that a ring R is right principally $\pi-$extending if every projection invariant principal right ideal of R is essential in a direct summand of R_R. We demonstrate the connection between the principally $\pi-$extending property and the $p.\pi-$Baer property. Recall that an idempotent $e \in R$ is left (right) semicentral if $xe = exe$ ($ex = exe$), for all $x \in R$ and is denoted by $S_l(R)$ ($S_r(R)$). If $xe = ex = exe$ for all $x \in R$, then $e = e^2 \in R$ is called a central idempotent and is denoted by $B(R)$. Let R be a ring, then $Z_l(R)(Z_r(R)), r_R(X)(l_R(X))$, $I(R), P(R)$ and $N(R)$ denote the left singular ideal of R (the right singular ideal of R), right annihilator of X in R (left annihilator of X in R) for a nonempty subset X of R, subring of R is generated by idempotents, the prime radical and the set of nilpotent elements of R, respectively.

2 The Preliminaries and Basic Concepts

Definition 2.1. A principal left (right) ideal I of R is called projection invariant if $Ie \subseteq I$ ($eI \subseteq I$) for all $e = e^2 \in R$.

Definition 2.2. A ring R (with unity) is projection invariant principally right Baer (denoted right principally π-Baer or right p.π-Baer) if for each projection invariant principal left ideal I there exists $c = c^2 \in R$ such that $r_R(I) = cR$

Proposition 2.3. A ring R is right $p.\pi$-Baer if and only if for every projection invariant principal left ideal I of R, there exists $e \in S_l(R)$ such that $r_R(I) = eR$.

Proof . Let R is a right $p.\pi$-Baer and I be a projection invariant principal left ideal of R. By definition there is an idempotent $e \in R$ such that $r_R(I) = eR$. Let $c = c^2 \in R$. Since I is projection invariant principal left ideal, $Ic \subseteq I$ and hence, $eR = r_R(I) \subseteq r_R(Ic)$. Thus $Ice = 0$. Then $ce \in r_R(I) = eR$, hence $ce = ece$. By [10, Proposition 1], $e \in S_l(R)$. The converse is obvious. □

Lemma 2.4. [2, Lemma 2.3] The following conditions are equivalent:

(i) R is abelian.

(ii) Every one-sided ideal is projection invariant.

(iii) Every right ideal generated by an idempotent is projection invariant.

(iv) Every right ideal generated by an idempotent is an ideal.

Recall from [1] that R is said to satisfy the IFP (insertion of factors property) or is semicommutative if right annihilator of any element (subsets) of R is an ideal of R. Note that for every R ring, reduced condition implies IFP and IFP condition implies abelian; but the converse is not hold.

Proposition 2.5. The following conditions are equivalent:

(i) R is domain.

(ii) R is $\pi-$Baer and 0 and 1 are the only idempotents of R.

(iii) R is $p.\pi-$Baer and 0 and 1 are the only idempotents of R.

(iv) R is right $p.\pi-$Baer and 0 and 1 are the only idempotents of R.

Proof . (i)$\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear.

(iv)$\Rightarrow$ (i) Let $xy = 0$ for $x, y \in R$. Then $y \in r_R(Rx)$. Since 0 and 1 are the only idempotents of R, by Lemma 2.4, Rx is projection invariant principal left ideal of R, so $r_R(Rx) = eR$. If $e = 0$, then $y = 0$ and if $e = 1$, then $x = 0$. Thus, R is a domain. □

Proposition 2.6. [7, Proposition 1.7] The following conditions are equivalent:

(i) R is a right p.q-Baer ring.

(ii) The right annihilator of any finitely generated right ideal is generated by an idempotent.

(iii) The right annihilator of any principal ideal is generated by an idempotent.

(iv) The right annihilator of any finitely generated ideal is generated by an idempotent.

Proposition 2.7. If R has the IFP, then the following conditions are equivalent:

(i) R is a right $p.\pi$-Baer ring.

(ii) The right annihilator of any finitely generated projection invariant left ideal is generated by an idempotent.

(iii) The right annihilator of any principal ideal is generated by an idempotent.

(iv) The right annihilator of any finitely generated ideal is generated by an idempotent.

Proof . (i)$\Longleftrightarrow$ (ii) Clearly, (ii) implies (i). Assume that (i) holds and $I = \Sigma_{i=1}^{n} Rx_i$. Since any Rx_i is projection invariant principal left ideal of R, there exists $e_i = e_i^2 \in S_l(R)$ such that $r_R(Rx_i) = e_iR$. Since each e_i is central, there is an idempotent $e \in R$ such that $r_R(I) = r_R(\Sigma_{i=1}^{n} Rx_i) = \bigcap_{i=1}^{n} e_iR = eR$.

The equivalences (i)$\Longleftrightarrow$ (iii) and (ii)$\Longleftrightarrow$ (iv) follow from the fact that $r_R(I) = r_R(IR)$, where I is any projection invariant left ideal of R. □

Proposition 2.8. (i) Assume that $R = I(R)$. If R be right p.q-Baer ring, then R is right $p.\pi$-Baer.

(ii) Let R be a ring with IFP. R is right $p.\pi$-Baer if and only if R is right $p.q$-Baer.

Proof . (i) Assume $R = I(R)$ and I be a projection invariant principal left ideal of R, then I is a principal ideal. By Proposition 2.6, $r_R(I) = eR$.

(ii) Assume R be right $p.\pi$-Baer and I be a principal right ideal of R. Then RI is a principal ideal. By Proposition 2.7, $r_R(I) = r_R(RI) = eR$.

Conversely, assume I be a projection invariant principal left ideal of R, then IR is a principal ideal of R. Proposition 2.6 yields, $r_R(I) = r_R(IR) = eR$. □

The following example shows that there exists a right p.q-Baer ring which is not right $p.\pi$-Baer.

Example 2.9. Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z}, a \equiv d, b \equiv 0 \;\text{ and }\; c \equiv 0 \pmod 2 \right\}.$$

R is a prime ring. Then is right p.q-Baer ring, but R is not domain. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the only idempotents of R. Therefore by Proposition 2.5, R is not right $p.\pi$-Baer ring.

Proposition 2.10. (i) Let R is a right p.p ring. Then R is right $p.\pi$-Baer.

(ii) Let R is an abelian right $p.\pi$-Baer, then R is right p.p.

Proof . (i) Let I be a projection invariant principal left ideal of R. Then there exists $x \in R$ such that $I = Rx$. Since R is right p.p ring, $r_R(I) = r_R(Rx) = r_R(x) = eR$, for some $e = e^2 \in S_l(R)$.

(ii) Let $x \in R$. Then Rx is a projection invariant principal left ideal of R. Thus $r_R(x) = r_R(Rx) = eR$, for some $e \in S_l(R)$. □

Corollary 2.11. Let R is an abelian ring. Then the following conditions are equivalent:

(i) R is right p.p.

(ii) R is p.p.

(iii) R is right $p.\pi$-Baer.

Proof . (i)$\Rightarrow$ (ii) It follows from [7, Proposition 1.14].

(ii) $\Rightarrow$ (iii) and (iii)$\Rightarrow$ (i) these implications follow from Proposition 2.10. □

Proposition 2.12. If R be right $p.\pi$-Baer ring, then the following are equivalent:

(i) R is abelian.

(ii) R is semicommutative.

(iii) R is reduced.

Proof . (i)$\Rightarrow$ (ii) Assume that R is an abelian right $p.\pi$-Baer ring. Let $x \in R$, then Rx is a projection invariant principal left ideal of R. Thus $r_R(x) = r_R(Rx) = eR$, where $e \in S_l(R)$. Hence by [7, Lemma 1.1], $r(x)$ is an ideal.

(ii)$\Rightarrow$ (iii) Let $x^2 = 0$ for some $x \in R$. Then $Rx^2 = 0$ and so $x \in r_R(Rx) = eR$, where e is a central idempotent. Then $x = ex = xe = 0$. Hence, R is a reduced ring.

(iii)$\Rightarrow$ (i) The proof is routine. □

Proposition 2.13. An abelian right $p.\pi$-Baer ring is semiprime.

Proof . It follows from Proposition 2.12 and [13, Example 10.17]. □

It is clear that every π-Baer ring is a right $p.\pi$-Baer ring. Examples 2.14 and 2.16 show that there are right $p.\pi$-Baer rings that are not π-Baer.

Example 2.14. Let F be a fild and $F_n = F$ for $n = 1, 2, \cdots$. The ring

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & < \bigoplus_{n=1}^{\infty} F_n, 1 > \end{pmatrix}$$

is a subring of the 2×2 matrix ring over the ring $\prod_{n=1}^{\infty} F_n$, where $< \bigoplus_{n=1}^{\infty} F_n, 1 >$ is the $F-$algebra generated by $\bigoplus_{n=1}^{\infty} F_n$ and $1_{\prod_{n=1}^{\infty} F_n}$. The ring R is a regular ring (hence p.p). Thus R is a right $p.\pi$-Baer ring. But by [7, Example 1.6] the ring R is not right p.q-Baer. Therefore is not quasi–Baer and by [2, Theorem 2.1], R is not π-Baer ring.

Lemma 2.15. [7, Lemma 1.4] Let T be a ring with unity such that $|T| > 1$, and let $S = \prod_{\lambda \in \Lambda} T_\lambda$, where $T_\lambda = T$ and Λ is an infinite indexing set. If R is the subring of S generated by $\bigoplus_{\lambda \in \Lambda} T_\lambda$ and either $1 \in S$ or $\{f : \Lambda \to T \mid \text{f is a constant function}\}$, then R is not quasi-Baer. Moreover if T is a right p.q-Baer ring, then so is R.

Example 2.16. (i) Let T in Lemma 2.15 be a commutative regular ring. Then R is p.p, hence R is right $p.\pi$-Baer ring that is not π-Baer, as is not quasi-Baer.

(ii) Let F be a filed and $F_i = F$ for $i = 1, 2, \cdots$. Let $R = \prod_{i=1}^{\infty} F_i$ and S be the subring of R generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_R, then S is commutative p.p, hence right $p.\pi$-Baer. But S is not quasi-Baer, so is not π-Baer.

Proposition 2.17. Let R be a left Noetherian and right $p.\pi$-Baer ring with IFP. Then R is π-Baer.

Proof . Let I be a projection invariant left ideal of R. Since R is left Noetherian, I is a finitely generated projection invariant left ideal. By Proposition 2.7, $r_R(I) = eR$. Hence R is π-Baer. □

Proposition 2.18. The center of $p.\pi$-Baer ring is a p.p ring.

Proof . Let $Z(R)$ be the center of R and $a \in Z(R)$. Then $r_R(Ra) = eR$ with $e \in S_l(R)$ and $l_R(aR) = Rf$ with $f \in S_r(R)$, as R is a $p.\pi$-Baer ring. Since $r_R(Ra) = l_R(aR)$, we have $e = f$. Hence $e \in B(R)$. Consequently, $r_{Z(R)}(a) = r_R(a) \cap Z(R) = eZ(R)$. Therefore, $Z(R)$ is a p.p ring. □

Corollary 2.19. Let S be a subring of R and $I(R) \subseteq S$. If X is a projection invariant principal left (right) ideal of S, then RX (XR) is a projection invariant principal left (right) ideal of R. Moreover, if R is right (left) $p.\pi$-Baer, then S is right (left) $p.\pi$-Baer.

Proof . Assume $I(R) \subseteq S$ and X is a projection invariant principal left ideal of S. Then $(RX)e = R(Xe) \subseteq RX$ for all $e = e^2 \in R$. So RX is a projection invariant principal left ideal of R. Since R is right $p.\pi$-Baer, $r_R(RX) = eR$ for some $e = e^2 \in R$. It follows that $r_S(X) = eS$, so S is right $p.\pi$-Baer. □

The following example shows that a subring of a right $p.\pi$-Baer ring need not be right $p.\pi$-Baer.

Example 2.20. Let $R = \mathbb{Z} \oplus \mathbb{Z}$, then by Theorem 2.24, R is a right $p.\pi$-Baer ring. Consider the subring $S = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b(mod p)\}$, where p is a prime number. The only idempotents of S are $(0, 0)$ and $(1, 1)$. For projection invariant principal left ideal $S(0, p)$, we have $r_S(S(0, p)) = (p, 0)S$, that dose not contain a nonzero idempotent of S. Hence S is not right $p.\pi$-Baer ring.

Proposition 2.21. A ring R is right $p.\pi$-Baer if and only if whenever I is a projection invariant principal left ideal of R there exists $e \in S_r(R)$ such that $I \subseteq Re$ and $r_R(I) \cap Re = (1 - e)Re$.

Proof . Assume R is right $p.\pi$-Baer and I is a projection invariant principal left ideal of R. Then there exists $c \in S_l(R)$ such that $r_R(I) = cR$. Hence $I \subseteq R(1 - c)$. Let $e = 1 - c$, then $I \subseteq Re$ and $e \in S_r(R)$. Now $r_R(I) \cap Re = (1 - e)R \cap Re = (1 - e)Re$. Conversely, let I be a projection invariant principal left ideal of R and assume there exists $e \in S_r(R)$ such that $I \subseteq Re$ and $r_R(I) \cap Re = (1 - e)Re$. Let $\alpha \in r_R(I)$. Then $\alpha = e\alpha + (1 - e)\alpha$. So $\alpha e = e\alpha e + (1 - e)\alpha e$. But $\alpha e \in r_R(I) \cap Re$, hence $\alpha e = (1 - e)\alpha e$ and so $e\alpha e = 0$. Since $e \in S_r(R)$, $0 = e\alpha e = e\alpha$ and $\alpha = (1 - e)\alpha \in (1 - e)R$. Hence $r_R(I) \subseteq (1 - e)R$. Thus R is right $p.\pi$-Baer. □

Observe that Proposition 2.21 is the right $p.\pi$Baer analogue of characterization of a right p.q-Baer ring in [7, Proposition 1.9]. However, contrary to the right p.q-Baer case in [7, Corollary 1.10], following example shows that $I + (1 - e)Re$ is not essential in Re.

Example 2.22. Let R be the free ring $\mathbb{Z} < x, y >$. Since R is domain, R is right $p.\pi$-Baer. Take $I = Rx$. Since 0 and 1 are the only idempotents of R, I is a projection invariant principal left ideal of R. Then $r_R(I) \cap Re = 0 = (1 - e)Re$, and $I + (1 - e)Re = I$ is not essential in R.

Proposition 2.23. Let R be a right $p.\pi$-Baer ring. If every essential left ideal of R is an essential extension of a projection invariant principal left ideal of R, then R is left nonsingular.

Proof . Let $0 \neq x \in Z_l(R)$. Then $l_R(x) \leq^{ess} R$. By hypothesis, there exists a projection invariant principal left ideal I of R such that $I \leq^{ess} l_R(x)$. So $I \leq^{ess} R$. Hence $Ix = 0$ implies that $x \in r_R(I) = eR$, because R is right $p.\pi$-Baer. It is clear that $I \subseteq l_R(r_R(I)) = R(1-e)$. Since I is essential in R, $e = 0$ and hence $x = 0$, a contradiction. □

If R be an abelian right $p.\pi$-Baer ring, then R is right nonsingular, since every right p.p ring is right nonsingular.

Theorem 2.24. Let $R = \prod_{i \in K} R_i$. Then R is a right $p.\pi$-Baer ring if and only if R_i is a right $p.\pi$-Baer ring for each $i \in K$.

Proof . Let π_i, be ith projection homomorphism of $R = \prod_{i \in K} R_i$. Assume that each R_i is a right $p.\pi$-Baer ring and I is a projection invariant principal left ideal of R. Then there exists $e_i = e_i^2 \in R_i$ such that $r_{R_i}(\pi_i(I)) = e_iR_i$. Now put $e = (e_i)_{i \in K} \in R$. Clearly, $\pi_i(e) = e_i$ for each $i \in K$. Thus $r_R(I) = eR$ and hence R is a right $p.\pi$-Baer ring.

Conversely, assume R is a right $p.\pi$-Baer ring, then each projection invariant principal left ideal I_i of R_i can consider as a projection invariant principal left ideal of R, since R is a right $p.\pi$-Baer ring, $r_R(I_i) = eR$. If put $e_i = \pi_i(e)$, then we have $r_{R_i}(I_i) = e_iR_i$. Therefore R_i is a right $p.\pi$-Baer ring. □

Definition 2.25. A ring R is called principally left (right) π-extending if every projection invariant principal left (right) ideal of R is essential in a left (right) ideal is generated by an idempotent of R.

Theorem 2.26. Let R be a left nonsingular ring. Then R is principally left π-extending if and only if R is right $p.\pi$-Baer and ${}_RI \leq^{ess} l_R(r_R(I))$ for every projection invariant principal left ideal I of R.

Proof . Assume ${}_RR$ is principally left π-extending and I be a projection invariant principal left ideal of R. Then there is $e = e^2 \in R$ such that ${}_RI \leq^{ess} Re$. Since $Z({}_RR) = 0$, $r_R(I) = r_R(Re) = (1-e)R$. So R is right $p.\pi$-Baer. Hence $l_R(r_R(I)) = l_R((1-e)R) = Re$. Thus, ${}_RI \leq^{ess} l_R(r_R(I))$. Conversely, assume R is right $p.\pi$-Baer and ${}_RI \leq^{ess} l_R(r_R(I))$ for every projection invariant principal left ideal I of R. Since R is right $p.\pi$-Baer, $r_R(I) = eR$. Now, $l_R(r_R(I)) = l_R(eR) = R(1-e) = Rc$, for some $c = c^2 \in R$. Therefore ${}_RI \leq^{ess} Rc$. Then ${}_RR$ is principally left π-extending. □

Lemma 2.27. [2, Lemma 2.4] Assume R_R is nonsingular. Then the following conditions are equivalent:

(i) Every closed right ideal of R is a right annihilator.

(ii) R is right cononsingular.

(iii) For each $I_R \leq R_R, I_R \leq^{ess} r_R(l_R(I))$.

Corollary 2.28. Let R is right nonsingular and right cononsingular. Then R is left $p.\pi$-Baer if and only if R_R is principally right π-extending.

Proof . This follows from Lemma 2.27 and Theorem 2.26. □

Note: If R be a right $p.\pi$-Baer ring with $S_l(R) = B(R)$, then R have no nonzero nilpotent projection invariant principal left ideal. Assume $I \neq 0$ be nilpotent projection invariant principal left ideal. Then $I^n = 0$ and $I^{n-1} \neq 0$ for some positive integer n. Since R is right $p.\pi$-Baer, $r_R(I) = eR$ for some $e = e^2 \in R$. But $I^{n-1} \subseteq r_R(I) = eR$. Then $eI^{n-1} = I^{n-1}$, hence we obtain $I^{n-1} = eI^{n-1} = I^{n-1}e = 0$, a contradiction.

For a ring R and (R, R)-bimodule M, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the addition componentwise and the following multiplication:

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2),$$

where $a_1, a_2 \in R$ and $m_1, m_2 \in M$. Ring T is isomorphism to the ring of matrices $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}$, where $a \in R$ and $m \in M$, see [11, 15].

Theorem 2.29. Let R be an abelian ring and define

$$R_n := \{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \},$$

with n a positive integer ≥ 2. If R_n be a right $p.\pi$-Baer ring, then R is right $p.\pi$-Baer.

Proof . Let $I = Ra$ be projection invariant principal left ideal of R, where $a \in R$. Consider

$$A = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n.$$

Since R is an abelian ring, by [11, Lemma 2], R_n is an abelian ring, hence R_nA is a projection invariant principal left ideal of R_n. Then, $r_{R_n}(R_nA) = eR_n$ for some $e = e^2 \in R_n$. So there exists $f = f^2 \in R$ such that $e = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}$. It is clear that $\begin{pmatrix} af & 0 & \cdots & 0 \\ 0 & af & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & af \end{pmatrix} = 0$, so $fR \subseteq r_R(Ra)$. Next let $b \in r_R(Ra)$, then $Rab = 0$. Clearly, $\begin{pmatrix} b & 0 & \cdots & 0 \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix} \in R_n$ is contained in $r_{R_n}(R_nA) = eR_n$, so $b = fb \in fR$. Therefore R is right $p.\pi$-Baer. □

The following example shows that the converse assertion fails.

Example 2.30. Let $R=\mathbb{Z}$. Since R is domain, R is right $p.\pi$-Baer. Let $R_2\begin{pmatrix}0&1\\0&0\end{pmatrix}$ be a projection invariant principal left ideal of R_2, then for $a,b\in R$,

$$r_{R_2}\left(R_2\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)=r_{R_2}\left(\begin{pmatrix}a&b\\0&a\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)=r_{R_2}\left(\begin{pmatrix}0&a\\0&0\end{pmatrix}\right)=\left\{\begin{pmatrix}0&d\\0&0\end{pmatrix}\mid d\in\mathbb{Z}\right\},$$

is not generated by an idempotent. Let $\begin{pmatrix}0&d\\0&0\end{pmatrix}\in\begin{pmatrix}f&0\\0&f\end{pmatrix}R_2$ for $f=f^2\in R$. Then $\begin{pmatrix}0&d\\0&0\end{pmatrix}=\begin{pmatrix}fa&fb\\0&fa\end{pmatrix}$. Hence $d=0$, a contradiction. Thus R_2 is not right $p.\pi-$Baer.

3 $p.\pi$-Baer Matrix Rings

In this section, we characterize right $p.\pi$-Baer 2-by-2 generalized triangular matrix rings. Throughout this section, $R=\begin{pmatrix}A&M\\0&B\end{pmatrix}$ denotes a 2-by-2 generalized upper triangular matrix ring where A and B are rings and M is a (A,B)-bimodule.

Theorem 3.1. Let $R=\begin{pmatrix}A&M\\0&B\end{pmatrix}$ be an abelian 2-by-2 generalized upper triangular matrix ring where A and B are rings and M is a (A,B)-bimodule. Then R is right $p.\pi$-Baer if and only if the following conditions are satisfied:

(i) A and B are right $p.\pi$-Baer rings.

(ii) For every $a\in A$, $m\in M$ and $b\in B$ there exist $e=e^2\in A$, $k\in M$ and $f=f^2\in B$ such that

(a) $r_A(Aa)=eA$ and $ek+kf=k$.

(b) $f\in r_B(Bb)\cap r_B(Aak+Am+Mb)$.

(c) If $\delta\in B$ and $y\in M$ such that $Bb\delta=0$ and $Aay+(Am\delta+Mb\delta)=0$, then $\delta\in fB$ and $y\in eM+kB$.

Proof . Let I be projection invariant principal left ideal of A, then there exist $a\in A$, such that $I=Aa$. Clearly, $K=\begin{pmatrix}Aa&0\\0&0\end{pmatrix}$ is projection invariant principal left ideal of R. Since R is right $p.\pi$-Baer,

$$r_R\left(\begin{pmatrix}Aa&0\\0&0\end{pmatrix}\right)=\begin{pmatrix}e&k\\0&f\end{pmatrix}\begin{pmatrix}A&M\\0&B\end{pmatrix}=\begin{pmatrix}eA&eM+kB\\0&fB\end{pmatrix},$$

where $e=e^2, f=f^2$ and $ek+kf=k$. Thus $r_A(Aa)=eA$ and hence A is right $p.\pi$-Baer. Similarly, B is right $p.\pi$-Baer. For each $a\in A, m\in M$ and $b\in B$, ideal $K=R\begin{pmatrix}a&m\\0&b\end{pmatrix}=\{\begin{pmatrix}\alpha a&\alpha m+xb\\0&\beta b\end{pmatrix}\mid\alpha\in A,x\in M,\beta\in B\}$ is a projection invariant principal left ideal of R. Let $\begin{pmatrix}\alpha a&\alpha m+xb\\0&\beta b\end{pmatrix}\in R\begin{pmatrix}a&m\\0&b\end{pmatrix}$, then

$$\begin{pmatrix}\alpha a&\alpha m+xb\\0&\beta b\end{pmatrix}\begin{pmatrix}e&k\\0&f\end{pmatrix}=\begin{pmatrix}\alpha ae&\alpha ak+(\alpha m+xb)f\\0&\beta bf\end{pmatrix}=0.$$

Thus, $r_A(Aa)=eA$, $f\in r_B(Bb)$ and $f\in r_B(Am+Mb+Aak)$, because $k=ek+kf$, so $Aak=Aaek+Aakf$, then $Aak=Aakf$. Let $\delta\in B$ and $y\in M$ such that $Bb\delta=0$ and $Aay+(Am\delta+Mb\delta)=0$. Then

$$\begin{pmatrix}Aa&Am+Mb\\0&Bb\end{pmatrix}\begin{pmatrix}0&y\\0&\delta\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

Hence $\begin{pmatrix}0&y\\0&\delta\end{pmatrix}\in r_R(R\begin{pmatrix}a&m\\0&b\end{pmatrix})=\begin{pmatrix}eA&eM+kB\\0&fB\end{pmatrix}$, then $\delta\in fB$ and $y\in eM+kB$.

Conversely, assume A and B are right $p.\pi$-Baer rings and there exist $e=e^2\in A,k\in M$ and $f=f^2\in B$ such that conditions $(a),(b)$ and (c) hold. From condition (a) follows that $\begin{pmatrix}e&k\\0&f\end{pmatrix}$ is an idempotent. Also, it is clear that

$\begin{pmatrix} e & k \\ 0 & f \end{pmatrix} R \subseteq r_R(R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix})$. Now assume $\begin{pmatrix} \gamma & y \\ 0 & \delta \end{pmatrix} \in r_R(R \begin{pmatrix} a & m \\ 0 & b \end{pmatrix})$, then for $\alpha \in A, x \in M$ and $\beta \in B$, we have $\begin{pmatrix} \alpha a & \alpha m + xb \\ 0 & \beta b \end{pmatrix} \begin{pmatrix} \gamma & y \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore $Aa\gamma = 0, Aay + (Am + Mb)\delta = 0$ and $\beta b\delta = 0$. By condition (c), $\delta \in fB$ and $y \in eM + kB$. Therefore $\begin{pmatrix} \gamma & y \\ 0 & \delta \end{pmatrix} \in \begin{pmatrix} e & k \\ 0 & f \end{pmatrix} R$. Then R is right $p.\pi$-Baer. □

4 $p.\pi$-Baer Polynomial Extensions

In this section, we investigate the behavior of the $p.\pi-$Baer condition with respect to polynomial extension $R[x](R[[x]])$.

Lemma 4.1. [2, Lemma 4.1] Let $e(x) \in R[x](R[[x]])$, where $e(x) = e_0 + e_1x + \cdots + e_nx^n(e(x) = e_0 + e_1x + \cdots)$.

(i) $e(x) = (e(x))^2$ if and only if $e_k = \sum_{i+j=k} e_ie_j$, for $0 \leqslant k$ and $0 \leqslant i, j \leqslant k$. Hence, $e(x) = 0$ if and only if $e_0 = 0$.

(ii) If $e(x) = e_0 + e_1x + \cdots + e_nx^n = (e(x))^2$, then $\sum_{i=0}^{n} e_i = (\sum_{i=0}^{n} e_i)^2 \in R$.

(iii) Assume that $e(x) = (e(x))^2$. Then,

(a) $e_i \in I(R)$, for all $i \geqslant 0$. Hence, $I(R[x]) \subseteq I(R)[x]$ and $I(R[[x]]) \subseteq I(R)[[x]]$.

(b) $e_i, e_0e_i, e_ie_0 \in N(R)$, for all $i \geqslant 1$ where $N(R)$ denotes the subring of R (not necessarily with unity) generated by the nilpotent elements of R.

(iv) Let I be a right ideal of R. Then I is (principal) projection invariant if and only if $I[x]$ (respectively, $I[[x]]$) is a (principal) projection invariant right ideal in $R[x]$ (respectively, $R[[x]]$).

Theorem 4.2. Let $R[x](R[[X]])$ be a right $p.\pi$-Baer ring, then R is a right $p.\pi$-Baer.

Proof . Assume that $I = Ra$ is a projection invariant principal left ideal of R. By Lemma 4.1 $I[x] = R[x]a$ is a projection invariant principal left ideal of $R[x]$. Thus $r_{R[x]}(R[x]a) = e(x)R[x]$. We claim that $r_R(Ra) = e_0R$. By hypothesis $R[x]ae(x) = 0$, then $ae_0 = 0$ and hence $e_0R \subseteq r_R(Ra)$. Now let $b \in r_R(Ra)$, then $R[x]ab = 0$ and so $b \in r_{R[x]}(R[x]a) = e(x)R[x]$. Hence $b = e(x)g(x)$, where $g(x) \in R[x]$. So $e(x)b = b$, hence $b = e_0b \in e_0R$. Therefore, $r_R(Ra) = e_0R$, hence R is a right $p.\pi-$Baer ring. □

Theorem 4.3. Let R be a ring with $S_l(R) = B(R)$. If R be right $p.\pi$-Baer ring, then $R[x]$ is right $p.\pi$-Baer.

Proof . Assume $I[x] = R[x]p(x)$ be a projection invariant principal left ideal of $R[x]$, where $p(x) = c_0 + c_1x + \cdots + c_nx^n \in R[x]$. By Lemma 4.1, Rc_i is a projection invariant principal left ideal of R. Then there exists $e_i \in S_l(R)$ such that $r_R(Rc_i) = e_iR$, for $i = 0, 1, \cdots, n$. Let $e = e_0e_1 \cdots e_n$. Then $e \in S_l(R)$ and $eR = \bigcap_{i=0}^{n} r_R(Rc_i)$. Hence $eR[x] \subseteq r_{R[x]}(R[x]p(x))$.

It is clear that $r_{R[x]}(R[x]p(x)) \subseteq r_{R[x]}(Rp(x))$. Now, let $\alpha(x) \in r_{R[x]}(Rp(x))$ and $g(x) = b_0 + \cdots + b_kx^k \in R[x]$. Then

$$g(x)p(x)\alpha(x) = b_0p(x)\alpha(x) + \cdots + b_kx^kp(x)\alpha(x) = b_0p(x)\alpha(x) + \cdots + b_kp(x)\alpha(x)x^k = 0.$$

Hence $\alpha(x) \in r_{R[x]}(R[x]p(x))$. So $r_{R[x]}(R[x]p(x)) = r_{R[x]}(Rp(x))$. Now let $\alpha(x) = a_0 + a_1x + \cdots + a_mx^m \in r_{R[x]}(Rp(x))$, then $Rp(x)\alpha(x) = 0$. For any $b \in R$, we have the following system of equations:

(0) $bc_0a_0 = 0$

(1) $bc_0a_1 + bc_1a_0 = 0$

(2) $bc_0a_2 + bc_1a_1 + bc_2a_0 = 0$

(3) $bc_0a_3 + bc_1a_2 + bc_2a_1 + bc_3a_0 = 0$

⋮

(k) $\Sigma_{i+j=k} bc_ia_j = 0$

$(n + m)$ $bc_na_m = 0$.

From equation (0), $a_0 \in r_R(Rc_0) = e_0R$, where $e_0 \in S_l(R)$. Take $b = se_0$ in equation (1), where $s \in R$, then $se_0c_0a_1 + se_0c_1a_0 = 0$. But $se_0c_0a_1 = 0$, as e_0 is central idempotent. So $bc_1a_0 = 0$. Hence $a_0 \in r_R(Rc_1) = e_1R$

where $e_1 \in S_l(R)$. Thus $a_0 \in e_0e_1R$. So $bc_0a_1 = 0$, and $a_1 \in r_R(Rc_0) = e_0R$. In equation (2), take $b = se_0e_1$. Then $se_0e_1c_0a_2 + se_0e_1c_1a_1 + se_0e_1c_2a_0 = 0$. But $se_0e_1c_0a_2 = se_0e_1c_1a_1 = 0$. Hence $se_0e_1c_2a_0 = bc_2a_0 = 0$. So $a_0 \in e_0e_1e_2R$. Equation (2) simplifies to

$$(2)' \; bc_0a_2 + bc_1a_1 = 0.$$

Take $b = se_0$ in equation $(2)'$. Then $se_0c_0a_2 + se_0c_1a_1 = 0$ and so, $bc_1a_1 = se_0c_1a_1 = 0$, as $se_0c_0a_2 = 0$. Thus $a_1 \in r_R(Rc_1) = e_1R$. Then $a_1 \in e_0e_1R$, and $bc_0a_2 = 0$, so $a_2 \in r_R(Rc_0) = e_0R$. We have $a_0 \in e_0e_1e_2R$, $a_1 \in e_0e_1R$ and $a_2 \in e_0R$. Now in equation (3), take $b = se_0e_1e_2$. Continuing this procedure yields $a_i \in eR$ for all $i = 0, 1, \cdots, n$. Hence $\alpha(x) \in eR[x]$. Thus $R[x]$ is right $p.\pi$-Baer. □

5 Minimal Prime Ideals of $p.\pi$-Baer Rings

In this section, we investigate the condition, every prime ideal contains a unique minimal prime ideal, in a right $p.\pi-$Baer ring. Let P be a prime ideal of a ring R. $O(P)$ and $\overline{O}(P)$ be defined as

$$O(P) = \{a \in R | aRs = 0, \text{ for some } s \in R \setminus P\} \text{ and } \overline{O}(P) = \{x \in R | x^n \in O(P), \text{ for some } n \in \mathbb{N}\}.$$

In [6] Birkenmeier et al. show that if $\overline{O}(P) = P$ for every minimal prime ideal P of a right p.q-Baer ring R, then every prime ideal of R contains a unique minimal prime ideal. In this section, we derive some conditions which ensure that a prime ideal $P = \overline{O}(P)$ in the class of ablian right $p.\pi-$Baer rings. Let $N(R)$ denote the set of nilpotent elements of a ring R. A ring is called 2-primal if $P(R) = N(R)$. Every commutative ring is $2-$primal. It is clear that $O(P) \subseteq P$ and $O(P) \subseteq \overline{O}(P)$. Since $\overline{O}(P)/O(P)$ is the set of all nilpotent elements in the ring $R/O(P)$, the condition that $P = \overline{O}(P)$ gives us important information about the ring $R/O(P)$. For any subset A of R, let $supp(A)$ be the set of all prime ideals P such that $A \nsubseteq P$. If $A = \{a\}$, we write $supp(a)$. For any $P \in Spec(R)$, one can show that $\{supp(s) | s \in R\}$ forms a base on $Spec(R)$. This induced topology on $Spec(R)$ is called the hull-kernel topology on $Spec(R)$. From Dauns and Hofmann, $Spec(R)$ is a compact space. Let $\kappa(R) = \cup R/O(P)$ be the disjoint union of rings $R/O(P)$ with $P \in Spec(R)$. For any $a \in R$, let $\widehat{a} : Spec(R) \to \kappa(R)$ be defined by $\widehat{a}(P) = a + O(P)$. Then $\kappa(R)$ is a sheaf of rings over $Spec(R)$ with the topology on $\kappa(R)$ generated by a base $\{\widehat{a}(Supp(s)) | a, s \in R\}$, and $\widehat{a}$ is a global section for $a \in R$. We mean a sheaf representation whose base space is $Spec(R)$ and whose stalks are the quotients $R/O(P)$.

Lemma 5.1. Let P be a prime ideal of a ring R with IFP. Then $O(P) = \bigcup_{s \in R \setminus P} l_R(sR)$.

Proof . Assume $x \in O(P)$. Then there is $s \in R \setminus P$ such that $xRs = 0$. Thus $xs = 0$ and hence $xsR = 0$. Then $x \in l_R(sR) \subseteq \cup_{s \in R \setminus P} l_R(sR)$. Now let $x \in \cup_{s \in R \setminus P} l_R(sR)$, then there is $s \in R \setminus P$ such that $x \in l_R(sR)$. So $xsR = 0$, hence $xs = 0$. Since R satisfies IFP, $xRs = 0$, which implies that $x \in O(P)$. □

Proposition 5.2. Let P be a prime ideal of ring R and $B_P = \{e \in R |$ e is a central idempotent such that $e \notin P\}$. Then B_P is a denominator set.

Proof . It is clear that $1 \in B_P$. B_P is multiplicatively closed. Assume $e, f \in B_P$ and $ef \in P$. Observe that $(ef)^2 = ef$. So $eRf \subseteq P$, a contradiction. Hence $ef \notin P$. Now let $e \in B_P$ and $a \in R$ such that $ea = 0$. Then $ae = ea = 0$. □

Lemma 5.3. Let P be a prime ideal of an abelian right $p.\pi$-Baer ring R. Then $O(P) = \{a \in R | aRe = 0, e \in R \backslash P\} = T$

Proof . Let $x \in O(P)$. Then $xRs = 0$ for some $s \in R \setminus P$. Since $Rxs = 0$, $s \in r_R(Rx) = eR$. So $s = es$ and thus $e \notin P$. Now we have $xeR = 0$ and since e is central, then $xRe = 0$, hence $x \in T$. The reverse containment is obvious. □

Lemma 5.4. Let P be a prime ideal of an abelian right $p.\pi$-Baer ring R. Then $O(P) = \sum Rf$, where $f \in P$.

Proof . Let $a \in O(P)$. Then by Lemma 5.3 there exists idempotent $e \in R \setminus P$, such that $aRe = 0$. So $ae = 0$ and hence $a = a(1-e) \in R(1-e) \subseteq \sum R(1-e)$. Therefore $O(P) \subseteq \sum R(1-e)$. Now let e be idempotent of R such that $e \notin P$. It is clear that $(1-e)Re = 0$, then $(1-e) \in O(P)$, hence $R(1-e) \subseteq O(P)$. Therefore $\sum R(1-e) \subseteq O(P)$. Set $1 - e = f$, then $O(P) = \sum Rf$, where the sum is taken for all central idempotents $f \in R$ such that $f \in P$. □

Lemma 5.5. Let P and Q be prime ideals of an abelian right $p.\pi$-Baer ring R, such that $Q \subseteq P$. Then $O(P) = O(Q)$.

Proof . By definition it is clear that $O(P) \subseteq O(Q)$. From Lemma 5.4, $O(Q) \subseteq O(P)$, thus $O(Q) = O(P)$. □

Theorem 5.6. Let R be an abelian right $p.\pi$-Baer ring such that for any minimal prime ideal B of R, $\overline{O}(B) = B$. Then the following hold:

(i) R is 2-primal ring.

(ii) Every prime ideal of R contains a unique minimal prime ideal.

(iii) For every prime ideal P of R, the ring $R/O(P)$ is 2-primal and $P(R/O(P))$ is a completely prime ideal.

Proof . (i) From Proposition 2.12 ring R is reduced. So, R is 2-primal.

(ii) Assume Q be a prime ideal of R which contains minimal prime ideals P_1 and P_2. Lemma 5.5 yields that $O(P_1) = O(Q)$ and $O(P_2) = O(Q)$. So $P_1 = \overline{O}(P_1) = \overline{O}(P_2) = P_2$.

(iii) Let $Q/O(P)$ be a prime ideal of $R/O(P)$. By Lemma 5.4, $O(P) \subsetneq O(Q)$. Let B be the unique minimal prime ideal of R contained in P. From Lemma 5.5, we have $B = \overline{O}(B) = \overline{O}(P) \subseteq \overline{O}(Q) = K \subseteq Q$, where K is the unique minimal prime ideal of R contained in Q. Therefore $B = K = \overline{O}(P)$. Then $\overline{O}(P)/O(P)$ is the unique minimal prime ideal of $R/O(P)$. Consequently, $N(R/O(P)) = \overline{O}(P)/O(P) = P(R/O(P))$. Thus $R/O(P)$ is 2-primal. From [3, Lemma 2.1] B is completely prime in R. Then $P(R/O(P))$ is completely prime ideal in $R/O(P)$, as $B = \overline{O}(P)$. □

Proposition 5.7. Let P be a prime ideal of an abelian right $p.\pi$-Baer ring R with $R = I(R)$. Then P is a left essential ideal of R or $P = eR$ for some idempotent $e \in R$.

Proof . Let P be a prime ideal of R and it is not left essential ideal. Then there is a nonzero projection invariant principal left ideal Ra, where $a \in R$, such that $P \cap Ra = 0$. So $P \subseteq r_R(Ra) = eR$. But $RaeR = 0 \subseteq P$. Hence $eR \subseteq P$. So $P = eR$. □

Proposition 5.8. Let P be a prime ideal of an abelian right $p.\pi$-Baer ring R with $R = I(R)$. Then either

(i) $O(P)$ is left essential in P; or

(ii) $O(P) = eR$.

Proof . Let $O(P)$ is not left essential in P. Then there is a nonzero projection invariant principal left ideal Ra of R such that $Ra \subseteq P$ and $O(P) \cap Ra = 0$. So $O(P) \subseteq r_R(Ra) = eR$. But $RaeR = 0$ or $aRe = 0$. If $e \notin P$, then $a \in O(P)$, a contradiction. Hence $e \in P$, then $eR \subseteq P$. It is clear that $eR(1-e) = 0 \subseteq P$. Then $e \in O(P)$. □

A ring R [3] satisfies the CZ2 condition if whenever $(xy)^n = 0$ with $x, y \in R$ and n a positive integer, then $x^m R y^m = 0$ for some positive integer m.

Corollary 5.9. If R is an abelian right $p.\pi$-Baer and P is a prime ideal of R, then $O(P)$ is the unique minimal prime ideal of R contained in P and $O(P)$ is completely prime.

Proof . It is clear that R is reduced. Hence R is 2-primal and satisfies CZ2 condition. By [[3], Theorem 2.3(ii)], for any minimal prime ideal Q of R, $\overline{O}(Q) = Q$. From Theorem 5.6, there exists a unique minimal prime ideal B of R such that $B \subseteq P$. By Lemma 5.5 $O(B) = O(P)$. So $B = \overline{O}(B) = \overline{O}(P)$. Since R is reduced, for every prime ideal P of R, $O(P) = \overline{O}(P)$, then $O(P)$ is a minimal prime ideal, and hence by [[3], Proposition 1.1 (iii)], $O(P)$ is a completely prime ideal. □

Proposition 5.10. Let R be an abelian right $p.\pi$-Baer ring. If $N(R)$ is an ideal, then $\overline{O}(P)$ is an ideal of R for every prime ideal P of R.

Proof . Let $x, y \in \overline{O}(P)$. So there exist positive integers n, m such that $x^n, y^m \in O(P)$. From Lemma 5.3 there exist central idempotents $e, f \in R \setminus P$, such that $x^n Re = 0$ and $y^m Rf = 0$. Let $r \in R$, then $x^n re = 0 \in N(R)$ and $y^m rf = 0 \in N(R)$. Since $N(R)$ is completely semiprime, $xre \in N(R)$. So there exists a positive integer k such that $(xre)^k = 0 = (xr)^k e$. Hence $(xr)^k Re = (xr)^k eR = 0$. So $xr \in \overline{O}(P)$. Similarly $rx \in \overline{O}(P)$. Now we have $(x-y)ef = xef - yef \in N(R)$, as $N(R)$ is ideal. So there exists a positive integer t such that $[(x-y)ef]^t = 0 = (x-y)^t ef$. Therefore $(x-y)^t Ref = 0$, as ef is a central idempotent. We claim that $ef \notin P$. Assume $ef \in P$. Then $Ref \subseteq P$, hence $eRf \subseteq P$. Since P is prime, either $e \in P$ or $f \in P$. It is a contradiction. So $ef \notin P$. Then $(x-y) \in \overline{O}(P)$. Hence $\overline{O}(P)$ is an ideal of R. □

Theorem 5.11. Let R be an abelian right $p.\pi$-Baer ring. Then for every prime ideal P of R, $R[B_P^{-1}] \simeq R/O(P)$.

Proof . It is clear $\{a \in R | aRe = 0$ for some $e \in B_P\} \subseteq \{a \in R \mid ae = 0$, for some $e \in B_P\}$. Now let $a \in R$ such that $ae = 0$, for some $e \in B_P$. Then $a = a(1-e)$. So $aRe = a(1-e)Re = a(1-e)eR = 0$. Thus $\{a \in R|aRe = 0$ for some $e \in B_P\} = \{a \in R|ae = 0$ for some $e \in B_P\}$. From Lemma 5.3 $\{a \in R \mid ae = 0$ for some $e \in B_P\}$=$\{a \in R|aRe = 0$, for some $e \in B_P\} = O(P)$. Since $R[B_P^{-1}]$ is the right ring of fractions with the denominator set B_P, there is a ring homomorphism $\phi : R \to R[B_P^{-1}]$ and $R[B_P^{-1}] = \{\varphi(a)\varphi(e)^{-1} | a \in R$ and $e \in B_P\}$. Also, since $O(P) = \{a \in R|ae = 0$, for some $e \in B_P\}$, it is clear that the kernel of φ is $O(P)$. For any $e \in B_P$, $\varphi(e)$ is an idempotent in $R[B_P^{-1}]$ which is invertible. Thus $\varphi(e) = 1$ for any $e \in B_P$. So $R[B_P^{-1}] = \varphi(R)$ that is a homomorphic image of R with the kernel $O(P)$. So $R[B_P^{-1}] \simeq R/O(P)$. □

Lemma 5.12. [4, Lemma 3.1] A ring R has a representation as a subdirect product of the rings $R/O(P)$, where P ranges through all prime ideals of R.

Lemma 5.13. Let R be an abelian right $p.\pi$-Baer ring. Then R is a subdirect product of $R/O(P)$, where P ranges through all minimal prime ideals.

Proof . Assume P be a minimal prime ideal and Q be a prime ideal of R containing P. By Lemma 5.5, $O(P) = O(Q)$. So $\bigcap_{P \in Spec(R)} O(P) = \bigcap_{P \in MinSpec(R)} O(P)$. The result follows by Lemma 5.12. □

Lemma 5.14. [14, Lemma 3.9] Let R be a ring with a nontrivial central idempotent e. Then for every prime ideal P of R, $O(P) \neq 0$.

Corollary 5.15. Let R be an abelian right $p.\pi$-Baer ring which is not prime. Then for prime ideal P of R, $O(P) \neq 0$.

Proof . Since R is not prime, there exists a nonzero element $a \in R$ such that $r_R(Ra) \neq 0$. Also, since R is an abelian, Ra is projection invariant principal left ideal, so $r_R(Ra) = eR \neq 0$. Therefore by Lemma 5.14, $O(P) \neq 0$. □

Theorem 5.16. Let R be an abelian right $p.\pi$-Baer ring such that $O(P) \neq 0$ for any minimal prime ideal P of R. Then R has a nontrivial representation as a subdirect product of the rings of fractions $R[B_P^{-1}]$, where P ranges through all minimal prime ideals.

Proof . Theorem 5.11 and Lemma 5.13, yield the result. □

Proposition 5.17. Let R be an abelian left $p.\pi$-Baer ring. If P and Q are prime ideals of R, then we have the following results:

(i) $O(P) = \{x \in R | x = xe$, for some central idempotent $e \in P\}$.

(ii) If $Q \subseteq P$, then $O(P) = O(Q)$.

(iii) $R[B_P^{-1}] \simeq R/O(P)$.

(iv) If for any minimal prime ideal B of R, $\overline{O}(B) = B$, then:

  (a) R is 2-prime.

  (b) Every prime ideal of R contains a unique minimal prime ideal.

  (c) $\overline{O}(P)$ is the unique minimal prime ideal of R contained in P.

Proof . (i) Let $x \in O(P)$. Then there is $s \in R \setminus P$ such that $xRs = 0$. Thus, $x \in l_R(sR) = Re$, where $e = e^2 \in R$. Therefore, $x = xe$. Clearly, $e \in P$. Then, $O(P) \subseteq \{x \in R|x = xe$, for some central idempotent $e \in P\}$. Now, assume $x = xe$ for some idempotent $e \in P$. Then $x(1-e) = 0$, so $xR(1-e) = 0$. Since $(1-e) \notin P$, we have $x \in O(P)$. The proof is complete.

The proofs of (ii), (iii) and (iv) are similar to the proofs of Lemma 5.5, Theorems 5.6 and 5.11, respectively. □

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