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Mazur-Ulam theorem in intuitionistic fuzzy normed spaces

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Abstract

In this paper, introducing the notion of intuitionistic strictly convex sets, and using functions that preserve equality of intuitionistic fuzzy distance, a new generalization of Mazur-Ulam theorem in intuitionistic fuzzy normed spaces is presented. Moreover, intuitionistic fuzzy isometries, segment-preserving functions, and norm-additive maps are used to study the prerequisites needed to prove the main theorem.

Keywords: Mazur-Ulam theorem, Intuitionistic fuzzy normed space, Intuitionistic strictly convex sets, Intuitionistic fuzzy distance
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1 Introduction and preliminaries

Mazur and Ulam introduced their classical theorem in 1932. The theorem establishes that any isometry, which is also a bijective map from a real normed space to another real normed space, must be classified as an affine mapping [15]. Researchers have made numerous efforts to establish a comprehensive assertion regarding this conclusion. Consequently, to provide a broader perspective, an overview of various methodologies leading to this theorem is presented here. In 1971, Baker demonstrated that isometric mappings from real normed linear spaces to strictly convex real normed linear spaces are affine [6]. The result of the Mazur-Ulam theorem was developed in different ways. Vogt established the Mazur-Ulam theorem [24] by considering a real vector space denoted as X, where isometries are replaced by mappings that exhibit the property of functional dependence between the distances of the image points and the distances of the domain points. The Mazur-Ulam theorem for probabilistic 2-normed spaces was demonstrated by Mahai Postolache and Wasfi Shatanawi in 2008 [21]. Additionally, Pourmoslemi and Nourouzi asserted this theorem in probabilistic normed groups in 2017 [18].

Furthermore, some efforts have been made to generalize the theorem in fuzzy normed and metric spaces (For more details, see [8, 9, 13, 17]). Zadeh introduced the concept of fuzzy sets in 1965 [25], while Kramosil et al. presented the concept of fuzzy metric spaces in 1975, which can be viewed as a generalization of statistical metric spaces [14]. The introduction of intuitionistic fuzzy sets was demonstrated by Atanassov in 1983 [5]. He extensively contributed to the intuitionistic fuzzy sets through various papers, such as those published in 1986 [4], 1989 [3], and 2012 [2]. Following the examination of intuitionistic fuzzy distances by Szmith and Casparzic [22], Park introduced the concept

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of intuitionistic fuzzy metric spaces in 2004 [16]. Park demonstrated that the topology induced by the intuitionistic fuzzy metric coincides with the topology induced by the fuzzy metric in every intuitionistic fuzzy metric space [1, 7]. In 2006, Saadati and Park focused their research on fuzzy normed spaces [19].

This paper presents the Mazur-Ulam theorem for maps that may not necessarily be bijective, preserving the equality of intuitionistic fuzzy distance for strictly convex sets between two intuitionistic fuzzy normed spaces. Hence, it becomes essential to introduce the notion of intuitionistically strictly convex sets. Initially, we explore fundamental definitions and concepts utilized throughout the subsequent sections.

We define a fuzzy set named A within a set X in the following manner.

$$A = \{(x, \psi_A(x))| \quad x \in X\}.$$

Here, $\mu_A : X \to [0,1]$ represents the membership function of the fuzzy set A. Every element in X is assigned a value between 0 and 1 through the membership function, indicating the degree of membership of that element in the fuzzy set A. Generalizing the concept of fuzzy sets, we obtain the notion of intuitionistic fuzzy sets as follows.

Consider an arbitrary set X. An intuitionistic fuzzy set A of X is represented as

$$A = \{(x, \mu_A(x), \vartheta_A(x))| \quad x \in X\},$$

where the functions $\mu_A : X \to [0,1]$ and $\vartheta_A : X \to [0,1]$ denote the degree of membership and non-membership of the element $x \in X$, such that $\mu(x,t) + \vartheta(x,t) \leq 1$ for every $x \in X$, respectively[12].

Example 1.1. Consider the real number set $X = \mathbb{R}$, and let $\psi_A(x)$ and $\vartheta_A(x)$ be functions that can be expressed as follows.

$$\psi_A(x) = \begin{cases} 0.6 & \text{if } 1 \leq x \leq 2 \text{ or } 4 \leq x \leq 7, \\ 0.5 & \text{if } 3 < x < 4, \\ 0 & \text{Otherwise.} \end{cases}$$

$$\vartheta_A(x) = \begin{cases} 0.3 & \text{if } 1 \leq x \leq 2 \text{ or } 4 \leq x \leq 7, \\ 0.4 & \text{if } 3 < x < 4, \\ 0.9 & \text{Otherwise.} \end{cases}$$

In this case, $A = \{(x, \psi_A(x), \vartheta_A(x)); x \in \mathbb{R}\}$ will be an intuitionistic fuzzy set on $\mathbb{R}$.

Triangular norms, commonly referred to as t-norms, and triangular conorms, referred to as t-conorms, hold vital importance in the formulation of probabilistic and fuzzy normed spaces. Indeed, continuous t-norms and t-conorms become crucial when defining intuitionistic fuzzy normed spaces.

A t-norm can be defined as a binary function T mapping from the Cartesian product $[0,1] \times [0,1]$ to the interval $[0,1]$. It possesses the properties of associativity, commutativity, non-decreasing behavior, and satisfies the condition $T(a,1) = a$ for all a belonging to the interval $[0,1]$. Furthermore, a continuous t-conorm, denoted as T^*, can be characterized as a continuous binary operation on the interval $[0,1]$ that is closely connected to the continuous t-norm T through the expression

$$T^*(a,b) = 1 - T(1-a, 1-b).$$

Intuitionistic fuzzy normed spaces (IFNS) extend the traditional notion of normed spaces by incorporating intuitionistic fuzzy sets, which allow for a more precise representation of uncertainty. Building upon these foundational concepts, we can now proceed with the definition of intuitionistic fuzzy normed spaces as follows.

Definition 1.2. (Intuitionistic Fuzzy Normed Spaces)[16] Let X be a vector space, $\circ$ be a continuous t-norm, and $\otimes$ be a continuous t-conorm. We say that the 5-tuple $(X, G, H, \circ, \otimes)$ forms an Intuitionistic Fuzzy Normed Space (IFNS) if the following conditions hold for all $a, b \in X$ and $s, t > 0$.

1. $G(a,t) + H(a,t) \leq 1$,
2. $G(a,t) > 0$,
3. $G(a,t) = 1$ if and only if $a = \bar{0}$,

4. $G(\alpha x, t) = G(a, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$, $\alpha \in \mathbb{R}$,
5. $G(a, t) \circ G(b, s) \leq G(a + b, t + s)$,
6. $G(a, .) : (0, \infty) \to [0, 1]$ is continuous,
7. $\lim_{t\to\infty} G(a, t) = 1$ and $\lim_{t\to 0} G(a, t) = \bar{0}$,
8. $H(a, t) < 1$,
9. $H(a, t) = 0$ if and only if $a = \bar{0}$,
10. $H(\alpha a, t) = H(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0, \alpha \in \mathbb{R}$,
11. $H(a, t) \otimes H(b, s) \geq H(a + b, t + s)$,
12. $H(a, .) : (0, \infty) \to [0, 1]$ is continuous,
13. $\lim_{t\to\infty} H(a, t) = 0$ and $\lim_{t\to 0} H(a, t) = 1$.

In this context, the pair (G, H) is referred to as an Intuitionistic Fuzzy Norm (IFN).

In an IFNS, we define a vector space X equipped with a pair of functions, G and H, that serve as the intuitionistic fuzzy norm. These functions satisfy specific properties that generalize the properties of classical norms. For instance, the function $G(a, t)$ reflects the degree of membership of an element a in the space at a given "time" t, while $H(a, t)$ captures the degree of non-membership. The conditions imposed on these functions ensure that they behave consistently with the underlying structure of the space, allowing for the exploration of geometric properties and theorems, such as the Mazur-Ulam theorem, within this new context. Now, Consider the following examples.

Example 1.3. [16] Let $(X, \|\cdot\|)$ denote a normed space. Suppose a and b are elements in the interval $[0, 1]$, and let x be an element of X. For any $t > 0$, we define the operations $a \circ b = ab$ and $a \otimes b = \min\{a + b, 1\}$.

With these definitions, we can now define the functions

$$G(x, t) = \frac{t}{t + \|x\|},$$

and

$$H(x, t) = \frac{\|x\|}{t + \|x\|}.$$

These functions satisfy the given properties by the definition. Therefore, we can conclude that the 5-tuple $(X, G, H, \circ, \otimes)$ represents an IFNS.

Example 1.4. Let X be a vector space. For any $a \in X$ and $t > 0$, we define the intuitionistic fuzzy norm (G, H) as follows.

$$(G, H)(a, t) = (\frac{1}{1 + ||a|| \cdot t}, \frac{||a|| \cdot t}{1 + ||a|| \cdot t}).$$

Now, let $x \circ y = \min(x, y)$ and $x \otimes y = \max(x, y)$ for any $x, y \in [0, 1]$. Then $(X, G, H, \min, \max)$ will be an IFNS.

Within the forthcoming section, we shall elaborate on the intuitionistic fuzzy variation of the Mazur-Ulam theorem within an IFNS.

2 Main Results

Given the significant importance of establishing an exact definition for an intuitionistically strictly convex set, we begin by providing a definition for this class of sets. Subsequently, we will prove the essential lemmas and corollaries required to present the Mazur-Ulam theorem. In the remainder of this section, we consider the t-norm $\circ$ and t-conorm $\otimes$ as minimum and maximum, respectively, unless otherwise stated. Notably, despite the similarities between our proofs and conventional Mazur-Ulam proofs, it is crucial to acknowledge that the introduction of the degree of non-membership has led to fundamental alterations in both the definitions and outcomes.

Intuitionistic strict convexity is a geometric property of sets in intuitionistic fuzzy normed spaces that generalizes the concept of strict convexity from classical convex analysis. A set X is said to be intuitionistically strictly convex if, for any two distinct points a and b in X, the intuitionistic fuzzy norms G and H satisfy certain conditions that imply the uniqueness of the midpoint between a and b.

Specifically, if the equality of the norms at the midpoint is achieved through the operations defined in the IFNS, then a must equal b. This property is crucial for establishing the behavior of mappings and functions defined on these spaces, particularly in the context of isometries and affine mappings. Understanding intuitionistic strict convexity is essential for applying the Mazur-Ulam theorem, as it ensures that the geometric structure of the space allows for the preservation of certain properties under mappings.

Definition 2.1. Let $(X, G, H, \circ, \otimes)$ be an IFNS. We define X to be intuitionistic strictly convex if, for all $t > 0$ and $a, b \in X$, the following conditions hold true:

1. If $G(a+b,t) = G(a,t) \circ G(b,t)$ and $G(a,t) = G(b,t)$, then a is equal to b.
2. If $H(a+b,t) = H(a,t) \otimes H(b,t)$ and $H(a,t) = H(b,t)$, then a is equal to b.

Example 2.2. Consider the vector space $X = \mathbb{R}^2$ with the intuitionistic fuzzy norm defined by the functions G and H as follows:

For any $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $t > 0$, define:

$$G(\mathbf{x}, t) = \frac{t}{t + \sqrt{x_1^2 + x_2^2}}, \quad H(\mathbf{x}, t) = \frac{\sqrt{x_1^2 + x_2^2}}{t + \sqrt{x_1^2 + x_2^2}}.$$

Now, let us consider two distinct points $\mathbf{a} = (1,0)$ and $\mathbf{b} = (0,1)$ in $\mathbb{R}^2$. The midpoint $\mathbf{c}$ of the segment connecting $\mathbf{a}$ and $\mathbf{b}$ is given by:

$$\mathbf{c} = \left(\frac{1}{2}, \frac{1}{2}\right).$$

We will show that the space is intuitionistically strictly convex. So, we need to check if $G(\mathbf{a}+\mathbf{b}, t) = G(\mathbf{a}, t) \circ G(\mathbf{b}, t)$ holds. Therefore,

$$\mathbf{a} + \mathbf{b} = (1,0) + (0,1) = (1,1).$$

Thus,

$$G(\mathbf{a} + \mathbf{b}, t) = G((1,1), t) = \frac{t}{t + \sqrt{1^2 + 1^2}} = \frac{t}{t + \sqrt{2}}.$$

Now we have

$$G(\mathbf{a}, t) = G((1,0), t) = \frac{t}{t+1}, \quad G(\mathbf{b}, t) = G((0,1), t) = \frac{t}{t+1}.$$

Then,

$$G(\mathbf{a}, t) \circ G(\mathbf{b}, t) = \frac{t}{t+1} \cdot \frac{t}{t+1} = \frac{t^2}{(t+1)^2}.$$

Now, check if $H(\mathbf{a} + \mathbf{b}, t) = H(\mathbf{a}, t) \otimes H(\mathbf{b}, t)$ holds.

$$H(\mathbf{a} + \mathbf{b}, t) = H((1,1), t) = \frac{\sqrt{2}}{t + \sqrt{2}}.$$

So, $H(\mathbf{a}, t)$ and $H(\mathbf{b}, t)$:

$$H(\mathbf{a}, t) = H((1,0), t) = \frac{1}{t+1}, \quad H(\mathbf{b}, t) = H((0,1), t) = \frac{1}{t+1}.$$

Then,

$$H(\mathbf{a}, t) \otimes H(\mathbf{b}, t) = \max\left(\frac{1}{t+1}, \frac{1}{t+1}\right) = \frac{1}{t+1}.$$

Since $G(\mathbf{a}+\mathbf{b}, t) \neq G(\mathbf{a}, t) \circ G(\mathbf{b}, t)$ and $H(\mathbf{a}+\mathbf{b}, t) \neq H(\mathbf{a}, t) \otimes H(\mathbf{b}, t)$ for all $t > 0$, we conclude that the space is intuitionistically strictly convex. Thus, the vector space $X = \mathbb{R}^2$ equipped with the intuitionistic fuzzy norm (G, H) is an example of an intuitionistically strictly convex space.

Definition 2.3. [23] Let $(X, G, H, \circ, \otimes)$ and $(Y, G', H', \circ, \otimes)$ be sets equipped with intuitionistic fuzzy normed structures. An intuitionistic fuzzy isometry refers to a mapping $g : X \to Y$ that satisfies the following conditions for all $a, b \in X$ and $t > 0$.

$$G'(g(a) - g(b), t) = G(a - b, t),$$

$$H'(g(a) - g(b), t) = H(a - b, t).$$

These relationships can be expressed as

$$(G', H')(g(a) - g(b), t) = (G(a - b, t), H(a - b, t)) = \qquad (G'(g(a) - g(b), t), H'(g(a) - g(b), t)).$$

Furthermore, consider a function denoted as $g : X \to Y$. It can be classified as affine if for all elements a and b in X, and for any t within the range $[0, 1]$, we have

$$g((1 - t)a + tb) = (1 - t)g(a) + tg(b). \tag{2.1}$$

In other words, the function g is defined as affine when the mapping $T : X \to Y$ given by $T(x) = g(x) - g(0)$ is linear. Similarly, within the context of intuitionistic fuzzy isometries, a function denoted as $g : X \to Y$ is considered affine if the following condition holds true for any elements a and b in set X.

$$g\left(\frac{a + b}{2}\right) = \frac{g(a) + g(b)}{2}. \tag{2.2}$$

This implies that the function g preserves the midpoints of line segments.

Assuming Y is an intuitionistic strictly convex space, meaning that no sphere can encompass a line segment, we conclude that spheres centered at $g(a)$ and $g(b)$, with radii $G(\frac{a-b}{2}, t)$ and $H(\frac{a-b}{2}, t)$ respectively, intersect solely at the midpoint $\frac{g(a)+g(b)}{2}$.

In order to establish the Mazur-Ulam theorem in the realm of intuitionistic fuzzy normed spaces, it is imperative to rely on the subsequent crucial lemma.

Lemma 2.4. Consider $(X, G, H, \circ, \otimes)$ as an intuitionistic fuzzy normed space with the additional assumption that X possesses intuitionistic strict convexity. Under these conditions, there exists a unique element $z = \frac{a+b}{2}$ in X, where $a, b \in X$ and $t > 0$, such that the following equalities hold for all $a, b \in X$ and $t > 0$:

$$(G, H)(a - z, t) = (G, H)(z - b, t) = (G, H)(a - b, 2t).$$

Proof . Assume that X is an intuitionistic strictly convex space. Then, the following equalities hold.

$$(G, H)(a - z, t) = (G, H)\left(a - \frac{a + b}{2}, t\right) = (G, H)\left(\frac{a - b}{2}, t\right) = (G, H)(a - b, 2t),$$
$$(G, H)(z - b, t) = (G, H)\left(\frac{a + b}{2} - b, t\right) = (G, H)\left(\frac{a - b}{2}, t\right) = (G, H)(a - b, 2t).$$

Now, suppose that there exists another $z_0 \in X$ such that

$$(G, H)(a - z_0, t) = (G, H)(z_0 - y, t) = (G, H)(a - b, 2t).$$

From this assumption, we can derive the following inequalities.

$$G(a - \frac{z + z_0}{2}, t) \geq G(a - z, t) \circ G(a - z_0, t) = \min\{G(a - z, t), G(a - z_0, t)\} = G(a - b, 2t),$$
$$G\left(\frac{z + z_0}{2} - b, t\right) \geq G(z - b, t) \otimes G(z_0 - b, t) = \min\{G(z - b, t), G(z_0 - b, t)\} = G(a - b, 2t).$$

If the above inequalities are strict, that is, if equality is not established, then

$$G(a - b, 2t) > \min\{G(a - \frac{z + z_0}{2}, t), G(\frac{z + z_0}{2} - b, t)\} = G(a - b, 2t),$$

which leads to a contradiction. Therefore, while maintaining generality, we can suppose that

$$G((a-z)+(a-z_0),2t)=\min\{G(a-z,t),G(a-z_0,t)\}.$$

Additionally, since $G(a-z,t)=G(a-b,2t)=G(a-z_0,t)$, by intuitionistic strict convexity, we have $a-z=a-z_0$, implying $z=z_0$. Similarly, we can derive the following inequalities.

$$\begin{aligned} H(a-\frac{z+z_0}{2},t) &\leq \max\{H(x-z,t),H(a-z_0,t)\}=H(a-b,2t),\\ H\left(\frac{z+z_0}{2}-b,t\right) &\leq \max\{H(z-b,t),H(z_0-b,t)\}=H(a-b,2t).\end{aligned}$$

If the above inequalities are strict, then

$$H(a-b,2t)<\max\{H(a-\frac{z+z_0}{2},t),H(\frac{z+z_0}{2}-b,t)\}=H(a-b,2t),$$

which is contradictory. Hence, we can suppose that

$$H((a-z)+(a-z_0),2t)=\max\{H(a-z,t),H(a-z_0,t)\}.$$

Moreover, since $H(a-z,t)=H(a-b,2t)=H(a-z_0,t)$, by intuitionistic strict convexity, we have $a-z=a-z_0$, implying $z=z_0$. Now, consider $z_0\in[x,y]$ such that $z_0=\theta a+(1-\theta)b$ for some $\theta\in[0,1]$. We can express the following equalities.

$$\begin{aligned}(G,H)(a-(\theta a+(1-\theta)b),t)&=(G,H)(\theta a+(1-\theta)b-b,t)\\ &=(G,H)(a-b,2t).\end{aligned}$$

Consequently,

$$(G,H)(a-b,\frac{t}{1-\theta})=(G,H)(a-b,\frac{t}{\theta})=(G,H)(a-b,2t).$$

Therefore, we can deduce

$$\frac{1}{1-\theta}=\frac{1}{\theta}=2,$$

which implies $\theta=\frac{1}{2}$. $\square$

Remark 2.5. Let $(X,G_1,H_1,\circ,\otimes)$ and $(Y,G_2,H_2,\circ,\otimes)$ be intuitionistic fuzzy normed spaces. Suppose that the function $g:X\to Y$ preserves equality of intuitionistic fuzzy distance. Then, for each $t>0$ there are functions

$$q_{1t},q_{2t}:F_t\subseteq[0,1]\to[0,1],$$

such that

$$\begin{aligned}G_2(g(a)-g(b),t)=&q_{1t}(G_1(a-b),t),\\ H_2(g(a)-g(b),t)=&q_{2t}(H_1(a-b),t).\end{aligned}$$

In the given remark, we are looking for functions q_{1t} and q_{2t} that satisfy the equalities involving the intuitionistic fuzzy norms. We give an example to show that the functions q_{1t} and q_{2t} can be satisfied in an IFNS.

Example 2.6. Let's take $X=\mathbb{R}^n$ as the vector space over the $\mathbb{R}$. Define the intuitionistic fuzzy norm (G_1,H_1) on X, for all $a\in X$ as

$$(G_1,H_1)(a,t)=\left(\frac{1}{1+||a||\cdot t},\frac{||a||\cdot t}{1+||a||\cdot t}\right)$$

Similarly, let $Y=\mathbb{R}^m$ be another vector space, and define the intuitionistic fuzzy norm (G_2,H_2) on Y for all $b\in Y$ as

$$(G_2,H_2)(b,t)=\left(\frac{1}{1+||b||\cdot t},\frac{||b||\cdot t}{1+||b||\cdot t}\right)$$

Now, suppose we have a function $g : X \to Y$ that preserves equality of intuitionistic fuzzy distance. This means that for any $c, d \in X$, we have

$$G_2(g(c) - g(d), t) = G_1(c - d, t)$$
$$H_2(g(c) - g(d), t) = H_1(c - d, t).$$

Now, for all $t > 0$ and $x \in [0, 1]$, let

$$q_{1t}(x, t) = \frac{1}{1 + x \cdot t},$$
$$q_{2t}(x, t) = \frac{x \cdot t}{1 + x \cdot t}.$$

Therefore, in this example, the functions q_{1t} and q_{2t} satisfy the conditions of previous remark.

Theorem 2.7. Consider two intuitionistic fuzzy normed spaces: $(X, G_1, H_1, \circ, \otimes)$ and $(Y, G_2, H_2, \circ, \otimes)$. Let $g : X \to Y$ be an intuitionistic fuzzy isometry that preserves the equality of intuitionistic fuzzy distances and satisfies $g(0) = 0$. For all $a \in X$ and $t > 0$, we have

$$(G_2, H_2)(g(\frac{a}{2}), t) = (G_2, H_2)(\frac{g(a)}{2}, t).$$

If either Y is intuitionistic strictly convex or g is segment-preserving, then g is additive.

Proof . Let Y be an intuitionistic strictly convex space. For any $a \in X$ and $t > 0$, we have

$$q_{1t}(G_1(a, 2t)) = G_2\left(g\left(\frac{a}{2}\right), t\right) = G_2\left(\frac{g(a)}{2}, t\right) = G_2(g(a), 2t),$$
$$q_{2t}(H_1(a, 2t)) = H_2\left(f\left(\frac{a}{2}\right), t\right) = H_2\left(\frac{g(a)}{2}, t\right) = H_2(g(a), 2t).$$

Furthermore, for any $a, b \in X$ and $t > 0$, we have

$$\begin{aligned} G_2(g\left(\frac{a+b}{2}\right) - g(b), t) &= q_{1t}\left(G_1\left(\frac{a+b}{2}\right) - b, t\right) \\ &= q_{1t}(G_1(\frac{a-b}{2}), t) = q_{1t}(G_1(a - b, 2t)) \\ &= G_2(g(a) - g(b), 2t), \end{aligned}$$

and

$$\begin{aligned} H_2(g\left(\frac{a+b}{2}\right) - g(b), t) &= q_{2t}\left(H_1\left(\frac{a+b}{2}\right) - b, t\right) \\ &= q_{2t}(H_1(\frac{a-b}{2}), t) = q_{2t}(H_1(a - b, 2t)) \\ &= H_2(g(a) - g(b), 2t). \end{aligned}$$

Similarly, we have

$$\begin{aligned} G_2(g(a) - g\left(\frac{a+b}{2}\right), t) &= q_{1t}(G_1(a - \frac{a+b}{2}), t) \\ &= q_{1t}(G_1(\frac{a-b}{2}), t) = q_{1t}(G_1(a - b, 2t)) \\ &= G_2(g(a) - g(b), 2t), \end{aligned}$$

and

$$\begin{aligned} H_2(g(a) - g\left(\frac{a+b}{2}\right), t) &= q_{2t}(H_1(a - \frac{a+b}{2}), t) \\ &= q_{2t}(H_1(\frac{a-b}{2}), t) = q_{2t}(H_1(a - b, 2t)) \\ &= H_2(g(a) - g(b), 2t). \end{aligned}$$

By applying lemma 2.4, we conclude that $g\left(\frac{a+b}{2}\right)=\frac{g(a)+g(b)}{2}$ for all $a,b\in X$. This is a direct consequence of the definition of segment-preserving maps, which require that the image of the midpoint is equal to the midpoint of the images of the endpoints, confirming that the mapping g preserves the structure of the space in terms of midpoints. Additionally, since $g(0)=0$, we have

$$g(a)=g\left(\frac{2a+0}{2}\right)=\frac{g(2a)+g(0)}{2}=\frac{g(2a)}{2}.$$

Hence, we can deduce

$$g(a+b)=\frac{g(2a+2b)}{2}=\frac{g(2a)}{2}+\frac{g(2b)}{2}=g(a)+g(b).$$

Now, Let g be segment-preserving, then we have

$$(G_2,H_2)(g(\frac{a+b}{2})-g(b),t)=(G_2,H_2)(g(a)-g(\frac{a+b}{2}),t)=(G_2,H_2)(g(a)-g(b),2t).$$

Hence, by lemma 2.4, we infer that for all $a,b\in X$,

$$g(\frac{a+b}{2})=\frac{g(a)+g(b)}{2}.$$

Therefore, the proof is complete. □

Remark 2.8. The primary role of a segment-preserving map is to maintain the geometric structure of the space. In classical normed spaces, the Mazur-Ulam theorem relies on the property that isometries preserve distances. Similarly, in intuitionistic fuzzy normed spaces, the segment-preserving property ensures that the mapping g respects the linear combinations of points. This is essential for proving that such mappings are affine, as it allows for the conclusion that the midpoint of any two points is mapped to the midpoint of their images. The assumption regarding the existence of a segment-preserving maps is necessary for the validity of the results presented and plausible given the context of the study. This assumption allows for a meaningful exploration of the Mazur-Ulam theorem in intuitionistic fuzzy normed spaces, bridging classical results with contemporary mathematical frameworks.

Now, we use the previous theorem to express the relationship between intuitionistic strict convexity and additivity. The next corollary shows these relationships.

Corollary 2.9. Let $(X,G_1,H_1,\circ,\otimes)$ and $(Y,G_2,H_2,\circ,\otimes)$ be intuitionistic fuzzy normed spaces, and let $g:X\to Y$ be an intuitionistic fuzzy isometry. If Y is intuitionistic strictly convex or g is segment-preserving then the function $h(x)=g(a)-g(0)$ is additive.

Proof . It is clear that $h(a)$ is an intuitionistic fuzzy isometry. Furthermore, for all $a\in X$ and $t>0$,

$$\begin{aligned}(G_2,H_2)(h(\frac{a}{2}),t)&=(G_1,H_1)(a,2t),\\&=(G_2,H_2)(\frac{h(a)}{2},t).\end{aligned}$$

The results can be derived from Theorem 2.7. □

Definition 2.10. Let $(X,G_1,H_1,\circ,\otimes)$ and $(Y,G_2,H_2,\circ,\otimes)$ be intuitionistic fuzzy normed spaces. We define a map $g:X\to Y$ as intuitionistic norm-additive if, for any finite subset $\{a_1,a_2,...,a_n\}$ of X, where $a_i\in X$ and $t>0$, the following holds.

$$(G_2,H_2)(g(\sum_1^n a_i),t)=(G_2,H_2)(\sum_1^n g(a_i),t).$$

Finally, as a conclusion of the discussions that were raised and proved in this article, we prove a form of Mazur-Ulam theorem in an IFNS.

Theorem 2.11. Suppose that $(X, G_1, H_1, \circ, \otimes)$ and $(Y, G_2, H_2, \circ, \otimes)$ are intuitionistic fuzzy normed spaces. Moreover, consider Y is intuitionistic strictly convex. Let $g : X \to Y$ be a norm-additive map with $g(0) = 0$. Then g is an affine function.

Proof . We aim to show that for all $a_i \in X$ and $t > 0$,

$$(G_2, H_2)(g(\sum_{i=1}^{n} a_i), t) = (G_2, H_2)(\sum_{i=1}^{n} g(a_i), t).$$

Additionally, for all $t > 0$,

$$\begin{aligned} 1 &= G_2(0, t) = G_2(g(0), t) = G_2(g(a - a), t) = G_2(g(a) + g(-a), t), \\ 0 &= H_2(0, t) = H_2(g(0), t) = H_2(g(a - a), t) = H_2(g(a) + g(-a), t). \end{aligned}$$

Consequently, we obtain $g(a) = -g(-a)$ for all $a \in X$. Moreover, we have

$$\begin{aligned} (G_2, H_2)(g(a + b) - \frac{1}{2}g(a), t) &= (G_2, H_2)(2g(a + b) - g(a), 2t), \\ &= (G_2, H_2)(g(a + b + b), 2t), \\ &= (G_2, H_2)(g(a + b) + g(b), 2t), \\ &= (G_2, H_2)\left(\frac{g(a + b) + g(b)}{2}, t\right). \end{aligned}$$

Similarly,

$$\begin{aligned} (G_2, H_2)\left(\frac{1}{2}g(a) + g(b), t\right) &= (G_2, H_2)(g(a) + 2g(b), 2t), \\ &= (G_2, H_2)(g(a + b + b), 2t), \\ &= (G_2, H_2)(g(a + b) + g(b), 2t), \\ &= (G_2, H_2)\left(\frac{g(a + b) + g(b)}{2}, t\right). \end{aligned}$$

Letting $u = g(a + b) - \frac{1}{2}g(a)$ and $v = \frac{1}{2}g(a) + g(b)$, we observe that

$$(G_2, H_2)(u, t) = (G_2, H_2)(v, t).$$

By applying intuitionistic strict convexity, we conclude that $u = v$. This conclusion arises because the strict convexity condition ensures that the only way the norms can equal each other under the given operations is if the points themselves are identical. Thus, the application of intuitionistic strict convexity directly leads to the conclusion $u = v$, which implies $g(x + y) = g(x) + g(y)$. □

Example 2.12. Consider $X = \mathbb{R}^2$ and $Y = \mathbb{R}^2$ equipped with the intuitionistic fuzzy norms (G_1, H_1) and (G_2, H_2) defined as follows

1. $G_1(\mathbf{a}, t) = H_1(\mathbf{a}, t) = \frac{t}{t+\|\mathbf{a}\|}$ for all $\mathbf{a} \in X$ and $t > 0$.
2. $G_2(\mathbf{b}, s) = H_2(\mathbf{b}, s) = \frac{s}{s+\|\mathbf{b}\|}$ for all $\mathbf{b} \in Y$ and $s > 0$.

Now, defining the map $g : X \to Y$ as $g(\mathbf{a}) = (2a_1, 3a_2)$ for all $\mathbf{a} = (a_1, a_2) \in X$, we conclude that g satisfies the conditions of being intuitionistic norm-additive and $g(0) = 0$. Therefore, g is an affine function.

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