

Laplace optimized decomposition method for solving fractional order logistic growth in a population

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Abstract

In this paper, we introduce a semi-analytical method called the Laplace optimized decomposition method, abbreviated as LODM, for solving a model of a nonlinear ordinary differential equation describing the growth of population, the so-called Logistic equation with the fractional-order type, using the Caputo fractional derivative sense. The proposed technique combines the Laplace transform (LT) with a new technique called the optimized decomposition method (ODM). The results obtained by this method have been compared with those obtained by other methods. Finally, we demonstrate our numerical results with the help of tables and figures.

Keywords: Laplace optimized decomposition method, Caputo fractional derivative, fractional differential equations, logistic equation

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1 Introduction

Fractional calculus is a specialisation of applied analysis that deals with derivatives of arbitrary (real or complex) order. The differential equations that involve fractional order have been widely used to model various phenomena in many scientific fields [23, 33, 40, 43, 44, 55]. In the literature, numerous operators of arbitrary order have been proposed; the most famous and most widely used are Riemann-Liouville (RL), Caputo derivative (CD) [38], Caputo-Fabrizio derivative (CFD) [21], and Atangana-Baleanu (AB) [15]. Applications of these fractional derivatives have been investigated by many researchers in various fields of science and engineering. (See [6, 15, 16, 27, 28, 29, 35, 36, 39, 51]). The difficulty of finding exact solutions to fractional differential equations is a major challenge for scientists and mathematicians, especially for phenomena that are modelled in the form of non-linear equations. Researchers have presented numerous numerical and analytical techniques, like the variational iteration method (VIM) [31], the differential transform method (DTM) [54], the Homotopy analysis method (HAM) [1, 42], the Homotopy perturbation method (HPM) [32], the Hussein-Jassim method (HJM) [34], the residual power series method (RPSM) [2, 9], the numerical inverse Laplace transform methods [8], the predictor corrector method [24, 48] the Daftardar-Jafari method (DJM) [19, 22] Laplace transform method [3],[37], and so on. Among these techniques in the first attempt, George Adomian introduced a semi-analytical technique called the Adomian decomposition method (ADM) in the 1980s. After that, it has been used to find approximate solutions to the nonlinear fractional differential equations [4, 5, 52]. Recently,

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Zaid Odibat was introduced and developed an effective decomposition method, called the optimized decomposition method (ODM), to produce analytically approximate solutions for nonlinear ordinary or partial differential equations. The principal concept of the ODM is the linear approximation of a nonlinear term, which is used to decompose the solution in an infinite series form. For more details, see [47, 49]. Additionally, M. Laoubi et al. [41] modified and extended the optimized decomposition method for use in the treatment of nonlinear fractional differential equations. Banan Maayah et al. [45] provide an analytical solution for a fractional order model of dengue fever disease under the Caputo-Fabrizio derivative by using the Laplace optimized decomposition method (LODM). The nonlinear ordinary differential equation describing the growth of population, the so-called logistic equation model, was first studied by Pierre Verhulst in 1938 [20]. Which have many applications in different fields of science, such as biology [56], medicine [59], economy [58], and data security in optical networks [26]. In recent years, researchers have used logistic equations to study the evolution of the COVID-19 pandemic [53], [50]. In the literature, different versions and generalizations of the logistic equation model with fractional-order type have been considered and discussed (see [12, 13, 18, 25, 30, 46, 57]). Recently, Area et al. [11] studied the Λ -fractional logistic differential equation in the Λ -space. The authors of the article [17] have presented an efficient computational technique based on the reproducing kernel theory for approximating the solutions of the logistic differential equation of fractional order. Alshammari et al. [10] established a numerical solution of a logistic equation with fractional order using the residual power series method. The fractional Euler's method is presented to obtain the approximate solution of the fractional logistic equation [60]. Ahmed [7] developed a new application of the Laplace transform method (LTM) and used the series expansion of the dependent variable for solving the fractional logistic growth model in a population and fractional prey-predator models. The fractional logistic ordinary differential equation has the form

$${}^c D_t^\alpha y(t) = \rho y(t)(1 - y(t)), \quad t > 0, \quad \rho > 0, \quad 0 < \alpha \leq 1, \quad (1.1)$$

subject to the initial condition

$$y(0) = \mu, \quad \mu > 0. \quad (1.2)$$

In particular, if we put $\alpha = 1$, in equation (1.1), we have the following classical logistic differential equation

$$\frac{dy}{dt} = \rho y(t)(1 - y(t)), \quad t > 0, \quad (1.3)$$

has an exact solution in the form

$$y(t) = \frac{\mu}{\mu + (1 - \mu)e^{-\rho t}}, \quad (1.4)$$

The main objective of this work is to find an approximate solution for the model of nonlinear differential equations describing the growth of population, called the logistic equation, for the fractional-order model using the Caputo fractional derivative sense. For that purpose, the Laplace Optimized Decomposition method (LODM) is utilized to obtain numerical results. Further, we compare our results with those obtained by other methods.

The rest of the paper is organized as follows: in Section 2, we recall some fundamental definitions given. In Section 3, the formulation of the proposed technique is described. In section 4, some numerical applications and discussion are given.

2 Preliminaries

This section contains some fundamental concepts and definitions that, in order to be needed throughout this manuscript, are recollected from [34, 14]

Definition 2.1. If $f(t) \in C([a, b])$, and $a < t < b$, then the Riemann-Liouville fractional integral operator of order $\alpha > 0$, is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t - \tau)^{1-\alpha}} d\tau, \quad \alpha > 0, \quad (2.1)$$

where Γ is the well-known gamma function. In addition, some properties of the Riemann-Liouville fractional integral can be found in [34].

Definition 2.2. [34] The fractional Caputo derivative of $f(t)$, is defined as

$${}^c D_t^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_a^t (t - \tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad \text{for } m - 1 < \alpha \leq m, m \in N. \quad (2.2)$$

In addition, some properties of the Caputo fractional derivative can be found in [34].

Definition 2.3. Let $y(t)$ be piecewise continuous function is defined for $t > 0$. The Laplace transform of $y(t)$ is defined in [14] as

$$L[y(t)] = \int_0^{\infty} \exp(-st)y(t)dt = Y(s). \quad (2.3)$$

The Laplace transform of the Caputo fractional derivative of order $m - 1 < \alpha \leq m$, is given as

$$L[{}^c D_t^\alpha y(t)] = s^\alpha L[y(t)] - \sum_{i=1}^m y^{(i-1)}(0)s^{\alpha-i}. \quad (2.4)$$

3 Laplace Optimized Decomposition Method (LODM)

In this section, we will describe the basic steps of the (LODM) to solve fractional differential equations. To achieve this goal, we consider the following nonlinear fractional order differential equation of the form

$${}^c D_t^\alpha y(t) = \chi[y(t)] + f(t), \quad t > 0, \quad 0 < \alpha \leq 1, \quad (3.1)$$

with the initial condition $y(0) = \mu$, the function $y(t)$ is an analytical function, ${}^c D_t^\alpha(\cdot)$ is the Caputo fractional derivative, χ indicates the nonlinear operator, and $f(t)$ is a known function. Now, applying the Laplace transform to both sides of (3.1) and using the initial condition, we get

$$L[y(t)] = \frac{\mu}{s} + \frac{1}{s^\alpha} (L[f(t)] + L[\chi[y(t)]]). \quad (3.2)$$

Applying the inverse Laplace transform to (3.2), we get

$$y(t) = \mu + L^{-1} \left[\frac{1}{s^\alpha} (L[f(t)] + L[\chi[y(t)]]) \right]. \quad (3.3)$$

The Laplace optimized decomposition method suggests that the solution $y(t)$ be expressed by the decomposition series

$$y(t) = \sum_{k=0}^{\infty} y_k(t), \quad (3.4)$$

and the nonlinear terms $\chi[y(t)]$ is represented by

$$\chi[y(t)] = \sum_{k=0}^{\infty} P_k(t), \quad (3.5)$$

where $k \geq 0$ such that $y_k(t)$ are the components of $y(t)$ that will be determined recursively, and $P_k(t)$ are called the Adomian polynomials that represent the nonlinear $\chi[y(t)]$ and can be determined from the relation

$$P_k(t) = \frac{1}{k!} \left[\frac{d^k}{d\lambda^k} \chi \left[\sum_{k=0}^{\infty} \lambda^k y_k(t) \right] \right]_{\lambda=0}, \quad k \geq 0. \quad (3.6)$$

Inserting (3.4) and (3.5) into (3.3), we get

$$\sum_{k=0}^{\infty} y_k(t) = \mu + L^{-1} \left[\frac{1}{s^\alpha} \left(L[f(t)] + L \left[\sum_{k=0}^{\infty} P_k(t) \right] \right) \right]. \quad (3.7)$$

Consequently, the components of $y(t)$ can be elegantly determined by using the recursive iteration relation

$$\begin{cases} y_0(t) = \mu + L^{-1} \left[\frac{1}{s^\alpha} L[f(t)] \right], \\ y_1(t) = L^{-1} \left[\frac{1}{s^\alpha} L[P_0(t)] \right], \\ y_2(t) = L^{-1} \left[\frac{1}{s^\alpha} L[P_1(t) + \zeta(y_1(t))] \right], \\ \vdots \\ y_{k+1}(t) = L^{-1} \left[\frac{1}{s^\alpha} L[P_k(t) + \zeta(y_k(t)) - \zeta(y_{k-1}(t))] \right], \quad k \geq 2, \end{cases} \quad (3.8)$$

where

$$\zeta = \frac{\frac{\partial}{\partial y} \varphi({}^c D_t^\alpha y(t), y(t))}{\frac{\partial}{\partial {}^c D_t^\alpha y} \varphi({}^c D_t^\alpha y(t), y(t))} \Big|_{t=0} = - \frac{\partial}{\partial y} \varphi({}^c D_t^\alpha y(t), y(t)) \Big|_{t=0}, \quad (3.9)$$

such that we assume that the function $\varphi({}^c D_t^\alpha y(t), y(t)) = {}^c D_t^\alpha y(t) - \chi[y(t)]$ can be linearized by a first-order Taylor series expansion at $t = 0$. Solving $\varphi({}^c D_t^\alpha y(0), y(0)) = 0$ thus, the Taylor series expansion of the function $\varphi({}^c D_t^\alpha y(t), y(t))$ near (Y, μ) where $Y = {}^c D_t^\alpha y(0)$ and $\mu = y(0)$ is

$$\varphi({}^c D_t^\alpha y(t), y(t)) \approx {}^c D_t^\alpha y + \frac{\partial \varphi}{\partial y(t)}(Y, \mu)y(t). \quad (3.10)$$

4 Numerical Applications and Discussion

In this section, we consider the fractional logistic differential equation, then the (LODM) is applied in order to obtain the approximate solutions.

Example 4.1. Consider the following fractional-order logistic differential equation [10]

$${}^c D_t^\alpha y(t) = \frac{1}{4}y(t)(1 - y(t)), \quad t > 0, \quad 0 < \alpha \leq 1 \quad (4.1)$$

with the initial condition

$$y(0) = \frac{1}{3}, \quad (4.2)$$

In particular, if we put $\alpha = 1$, in equation (4.1), the exact solution given by

$$y(t) = \frac{1}{1 + 2e^{-\frac{1}{4}t}}, \quad (4.3)$$

In view of (3.3), we have

$$y(t) = \frac{1}{3} + L^{-1} \left[\frac{1}{s^\alpha} \left(\frac{1}{4} L [y(t) - y^2(t)] \right) \right], \quad (4.4)$$

Linearizing the function $\varphi({}^c D_t^\alpha y(t), y(t)) = {}^c D_t^\alpha y(t) - \frac{1}{4}y(t) + \frac{1}{4}y^2(t)$, near the point (Y, μ) , we obtain the linear approximation

$$\varphi({}^c D_t^\alpha y(t), y(t)) \approx {}^c D_t^\alpha y(t) + \zeta y(t), \quad (4.5)$$

where

$$\zeta = \frac{\partial}{\partial y} (\varphi({}^c D_t^\alpha y(t), y(t))) \Big|_{t=0} = -\frac{1}{12},$$

Assume the series solution has the form

$$y(t) = \sum_{k=0}^{\infty} y_k(t). \quad (4.6)$$

The Adomian polynomials P_k for $y - y^2$ have been calculated before and given by

$$P_k(t) = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[\sum_{k=0}^{\infty} \lambda^k y_k(t) - \left(\sum_{k=0}^{\infty} \lambda^k y_k(t) \right)^2 \right]_{\lambda=0}, \quad k \geq 0. \quad (4.7)$$

According to equation (3.8) we obtain

$$\left\{ \begin{array}{l} y_0(t) = \frac{1}{3}, \\ y_1(t) = \frac{0.0555555556}{\Gamma(\alpha+1)} t^\alpha, \\ y_2(t) = 0, \\ y_3(t) = \frac{0.0046296296}{\Gamma(2\alpha+1)} t^{2\alpha} - \frac{0.0007716049\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha}, \\ \vdots \end{array} \right. \quad (4.8)$$

Using equation (4.6), we obtain the 4th order approximation of $y(t)$ is given by

$$\begin{aligned}
 y_{LODM}(t) &= y_0(t) + \sum_{k=1}^3 y_k(t) \\
 &= \frac{1}{3} + \frac{0.0555555556}{\Gamma(\alpha + 1)} t^\alpha + \frac{0.0046296296}{\Gamma(2\alpha + 1)} t^{2\alpha} - \frac{0.0007716049\Gamma(2\alpha + 1)}{\Gamma^2(\alpha + 1)\Gamma(3\alpha + 1)} t^{3\alpha}.
 \end{aligned}
 \tag{4.9}$$

Table 1 shows our numerical results for the approximate solution of example 4.1 (4-term solution), for $\alpha = 1$ then we compared our results with those obtained by exact solution, the fractional residual power series method (FRPSM) [10], the variational iteration method (VIM) [17], and the reproducing kernel Hilbert space method (RKHSM) [17]. We present the comparison of absolute errors in Table 2. Further, in Table 3, we show the approximate solutions that are obtained by (LODM) of example 4.1 at different values of α . Fig. ?? shows the (LODM) solutions for different values of α .

Table 1: Comparison of numerical results that are obtained for example 4.1, by using (LODM) when $\alpha = 1$, by other methods

t	$y_{EXACT}(t)$	$y_{VIM}(t)$	$y_{LODM}(t)$	$y_{RKHSM}(t)$	$y_{FRPSM}(t)$
0.0	0.3333333333	0.3333333333	0.3333333333	0.3333333333	0.3333333333
0.3	0.3502029635	0.3502013889	0.3502013889	0.3502029364	0.3502029634
0.5	0.3616644631	0.3616576645	0.3616576646	0.3616644354	0.3616644609
0.8	0.3791524531	0.3791275720	0.3791275720	0.3791524268	0.3791524170
1.0	0.3909913152	0.3909465021	0.3909465021	0.3909912851	0.3909911774

Table 2: Comparison of absolute errors that are obtained for example 4.1, by using (LODM) when $\alpha = 1$, by other methods.

t	$y_{VIM}(t)$	$y_{LODM}(t)$	$y_{RKHSM}(t)$	$y_{FRPSM}(t)$
0.0	0.0000000	0.0000000	0.0000000	0.0000000
0.3	1.5746×10^{-6}	1.5746×10^{-6}	2.71×10^{-8}	1.00×10^{-8}
0.5	6.7986×10^{-6}	6.7985×10^{-6}	2.77×10^{-8}	2.15×10^{-8}
0.8	2.4881×10^{-5}	2.4881×10^{-5}	2.63×10^{-8}	3.61×10^{-8}
1.0	4.4813×10^{-5}	4.4813×10^{-5}	3.01×10^{-8}	1.378×10^{-7}

Table 3: The numerical results that are obtained for example 4.1, by using (LODM) for different values of α

t	$\alpha = 1$		$\alpha = 0.85$	$\alpha = 0.65$	$\alpha = 0.45$	$\alpha = 0.25$
	y_{EXACT}	y_{LODM}	y_{LODM}	y_{LODM}	y_{LODM}	y_{LODM}
0	0.3333333333	0.3333333333	0.3333333333	0.3333333333	0.3333333333	0.3333333333
1	0.3909913152	0.3909465021	0.3943093549	0.3972550045	0.3971513174	0.3957052888
2	0.4518627619	0.4516460905	0.4468422902	0.4374762338	0.4219673513	0.4096353649
3	0.5142093777	0.5138888889	0.4970289020	0.4723955481	0.4396160378	0.4196395483
4	0.5761168848	0.5761316872	0.5443372321	0.5023843989	0.4508447948	0.4264196003
5	0.6357240312	0.6368312757	0.5874596345	0.5262626057	0.4546193716	0.4293356014
6	0.6914384540	0.6944444444	0.6248156990	0.5423439370	0.4493672542	0.4273726390
7	0.7420886558	0.7474279835	0.6546974071	0.5487172994	0.4332888578	0.4193656765
8	0.7869860422	0.7942386831	0.6753269548	0.5433534429	0.4044699739	0.4040768925
9	0.8259012891	0.8333333333	0.6848843257	0.5241539546	0.3609314751	0.3802285841
10	0.8589810787	0.8631687243	0.6815220860	0.4889767401	0.3006543670	0.3465192358

Example 4.2. Consider the fractional logistic differential equation [10]

$${}^c D_t^\alpha y(t) = \frac{1}{2} y(t)(1 - y(t)), \quad t > 0, \quad 0 < \alpha \leq 1,
 \tag{4.10}$$

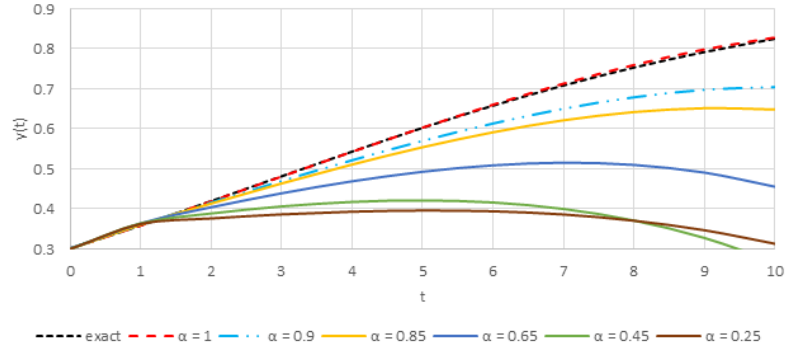


Figure 1: Plots of the approximate solutions for example 4.1 that are obtained by using (LODM) for different values of α

with the initial conditions

$$y(0) = \frac{1}{2}, \quad (4.11)$$

In particular, if we put $\alpha = 1$, in equation (4.10), the exact solution given by

$$y(t) = \frac{1}{1 + e^{-\frac{1}{2}t}}. \quad (4.12)$$

Accordinging of our method and equation (3.8), the components of the Laplace optimized decomposition series are given as follows

$$\begin{cases} y_0(t) = \frac{1}{2}, \\ y_1(t) = \frac{0.125}{\Gamma(\alpha+1)} t^\alpha, \\ y_2(t) = \frac{0.0625}{\Gamma(2\alpha+1)} t^{2\alpha}, \\ y_3(t) = -\frac{0.0625}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{0.03125}{\Gamma(3\alpha+1)} t^{3\alpha} - \frac{0.015625\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha}, \\ \vdots \end{cases} \quad (4.13)$$

and so on. Therefore, we obtain the 4th order approximation of $y(t)$ is given by

$$\begin{aligned} y_{LODM} &= y_0(t) + \sum_{k=1}^3 y_k(t) \\ &= \frac{1}{2} + \frac{0.0625}{\Gamma(2\alpha+1)} t^\alpha + \frac{0.03125}{\Gamma(3\alpha+1)} t^{3\alpha} - \frac{0.015625\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha}. \end{aligned} \quad (4.14)$$

Table 4 shows our numerical results for the approximate solution of example 4.2 (4-term solution), for $\alpha = 1$ then we compared our results with those obtained by exact solution, the (FRPSM) [10], the (RKHSM) [17], and the optimal homotopy asymptotic method (OHAM) [17]. We present the comparison of absolute errors in Table 5. Further, in Table 6, we show the approximate solutions that are obtained by (LODM) of example 4.2 at different values of α . Fig. 2 shows the (LODM) solutions for different values of α .

Table 4: Comparison of numerical results that are obtained for example 4.2, by using (LODM), when $\alpha = 1$, by other methods.

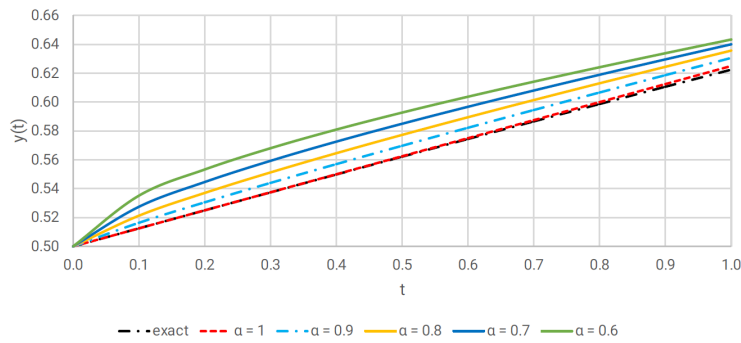
t	$y_{EXACT}(t)$	$y_{LODM}(t)$	$y_{RKHSM}(t)$	$y_{FRPSM}(t)$	$y_{OHAM}(t)$
0.0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
0.3	0.5374298453	0.5375000000	0.5374297936	0.5374298457	0.5374288935
0.5	0.5621765009	0.5625000000	0.5621764494	0.5621765137	0.5621790838
0.8	0.5986876601	0.6000000000	0.5986876097	0.5986880000	0.5986911508
1.0	0.6224593312	0.6250000000	0.6224592820	0.6224609375	0.6224603770

Table 5: Comparison of absolute errors that are obtained for example 4.2, by using (LODM) when $\alpha = 1$, by other methods.

t	$y_{LODM}(t)$	$y_{RKHSM}(t)$	$y_{FRPSM}(t)$	$y_{OHAM}(t)$
0.0	0.000000	0.000000	0.000000	0.000000
0.3	7.02E-05	5.17E-08	3.56E-10	9.52E-07
0.5	0.000323	5.15E-08	1.28E-08	2.58E-06
0.8	0.001312	5.04E-08	3.4E-07	3.49E-06
1.0	0.002541	4.92E-08	1.61E-06	1.05E-06

Table 6: The numerical results that are obtained for example 4.2, by using (LODM) for different values of α

t	$\alpha = 1$		$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.6$
	y_{EXACT}	y_{LODM}	y_{LODM}	y_{LODM}	y_{LODM}
0.0	0.5000000000	0.5000000000	0.5000000000	0.5000000000	0.5000000000
0.2	0.5249791875	0.5250000000	0.5305379320	0.5370462089	0.5532907788
0.4	0.5498339973	0.5500000000	0.5570168301	0.5645750909	0.5809535487
0.6	0.5744425168	0.5750000000	0.5822051605	0.5895059340	0.6037159692
0.8	0.5986876601	0.6000000000	0.6066454399	0.6130229215	0.6241412937
1.0	0.6224593312	0.6250000000	0.6306016932	0.6356847031	0.6433635662

Figure 2: Plots of the approximate solutions for example 4.2 that are obtained by using (LODM) for different values of α

Comment 1. Based on Tables 2 and 5 and comparing the absolute errors of the methods, there is not much difference between the numerical approximation results of the methods. Furthermore, the “RKHSM”, “FRPSM” and “OHAM” methods have achieved better results.

5 Conclusion

In this study, an analytical solution for a nonlinear fractional ordinary differential equation describing the growth of population is provided under the Caputo fractional derivative. A comparison between our method and other methods is presented. As a result, the LODM studied in this work is an efficient and powerful technique for obtaining accurate analytic approximate solutions to fractional-order logistic differential equations. In the future, we will explore applying the aforementioned technique to fuzzy differential equations and complex dynamical systems, such as infectious disease models.

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