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HYERS-ULAM STABILITY OF K-FIBONACCI FUNCTIONAL EQUATION

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ABSTRACT. Let denote by $F_{k,n}$ the n^{th} k-Fibonacci number where $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$ with initial conditions $F_{k,0} = 0, F_{k,1} = 1$, we may derive a functional equation $f(k, x) = kf(k, x-1) + f(k, x-2)$. In this paper, we solve this equation and prove its Hyere-Ulam stability in the class of functions $f : \mathbb{N}\times\mathbb{R} \to X$, where X is a real Banach space.

1. INTRODUCTION

The stability of functional equation originated from an equaton of Ulam [11] concerning the stability of group homomorphisms. Later, the result of Ulam was generated by Rassias [10]. Since then, the stability problems of functional equations have been extensively investigated by several mathematiciens(*see*[1-9]).
For any positive real number k, the k-Fibonacci sequence, say$\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for all $n \geq 2$ with initial conditions $F_{k,0} = 0, F_{k,1} = 1$.From this famous formula, we may derive a functional equation

$$f(k, x) = kf(k, x-1) + f(k, x-2). \quad (1.1)$$

A function $f : \mathbb{N} \times \mathbb{R} \to X$, will be called a k-Fibonacci function if it satisfies in (1.1), for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$, where X is a real vector space.
Characteristic equation of k-Fibonacci sequences is $x^2 - kx - 1 = 0$. We denote the positive and negative roots of this function by γ , θ (respectively); i.e,

$$\gamma = \frac{k + \sqrt{k^2+4}}{2} \quad , \quad \theta = \frac{k - \sqrt{k^2+4}}{2}$$

for any $x \in \mathbb{R}$, $k \in \mathbb{N}$.

2. General solution of k-Fibonacci equation

Let X bea real vector space. In the following theorem, we investigate the general solution for equation of the form (1.1) which is strongly related to the $F_{k,n}$.

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Theorem 2.1. *Let X be a real vector space. A function $f : \mathbb{N} \times \mathbb{R} \to X$ is a k-Fibonacci function if and only if there exists a function $h : \mathbb{N} \times [-1, 1) \to X$ such that*

$$f(k,x) = \begin{cases} F_{k,[x]+1}h(k,x-[x]) + F_{k,[x]}h(k,x-[x]-1) & x \geq 0 \\ (-1)^{[x]}[F_{k,-[x]-1}h(k,x-[x]) - F_{k,-[x]}h(k,x-[x]-1)] & x < 0 \end{cases} \tag{2.1}$$

where $[x]$ stands for the largest integer number that does not exceed x.

Proof. From (1.1) we have

$$f(k,x) = kf(k,x-1) + f(k,x-2).$$

Since $\gamma + \theta = k$, $\gamma\theta = -1$, hence

$$f(k,x) = (\gamma + \theta)f(k,x-1) - \gamma\theta f(k,x-2)$$

$$= \gamma f(k,x-1) + \theta f(k,x-1) - \gamma\theta f(k,x-2)$$

which implies that

$$\begin{cases} f(k,x) - \gamma f(k,x-1) = \theta[f(k,x-1) - \gamma f(k,x-2)] \\ f(k,x) - \theta f(k,x-1) = \gamma[f(k,x-1) - \theta f(k,x-2)] \end{cases}. \tag{2.2}$$

By induction on n, it follows that

$$\begin{cases} f(k,x) - \gamma f(k,x-1) = \theta^n[f(k,x-n) - \gamma f(k,x-n-1)] \\ f(k,x) - \theta f(k,x-1) = \gamma^n[f(k,x-n) - \theta f(k,x-n-1)] \end{cases}. \tag{2.3}$$

If we replace x by $x+n$ ($n \geq 0$) in (2.3), divide the resulting equation by θ^n (resp.γ^n) and replace n by $-m$ in the resulting equation, then we obtain a equation with m in place of n, where $m \in \{0, -1, -2, ...\}$. Therefore, (2.3) is true for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.
Now by multiplying the first and second equations of (2.3) by θ and $-\gamma$ (respectively) and then adding with together, we get

$$f(k,x) = \frac{\theta^{n+1} - \gamma^{n+1}}{\theta - \gamma} f(k,x-n) + \frac{\theta^n - \gamma^n}{\theta - \gamma} f(k,x-n-1) \tag{2.4}$$

for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$ *and* $k \in \mathbb{N}$. For $n = [x]$, $x \geq 0$ in (2.4) and using Binet's formula

$$F_{k,n} = \frac{\theta^n - \gamma^n}{\theta - \gamma},$$

we have

$$f(k,x) = F_{k,[x]+1}f(k,x-[x]) + F_{k,[x]}f(k,x-[x]-1)$$

and if $x < 0$, then for $n = [x] = -|[x]|$, we have

$$
\begin{aligned}
f(k,x) &= \frac{\theta^{-|[x]|+1}-\gamma^{-|[x]|+1}}{\theta-\gamma} f(k,x-[x]) \\
&+ \frac{\theta^{-|[x]|}-\gamma^{-|[x]|}}{\theta-\gamma} f(k,x-[x]-1) \\
&= \frac{-1}{(\gamma\theta)^{|[x]|-1}} \frac{\theta^{|[x]|-1}-\gamma^{|[x]|-1}}{\theta-\gamma} f(k,x-[x]) \\
&+ \frac{-1}{(\gamma\theta)^{|[x]|}} \frac{\theta^{|[x]|}-\gamma^{|[x]|}}{\theta-\gamma} f(k,x-[x]-1) \\
&= (-1)^{[x]} F_{k,|[x]|-1} f(k,x-[x]) + (-1)^{1+[x]} F_{k,|[x]|} f(k,x-[x]-1) \\
&= (-1)^{[x]} [F_{k,-[x]-1} f(k,x-[x]) - F_{k,-[x]} f(k,x-[x]-1)].
\end{aligned}
$$

Since $0 \leq x-[x] < 1$, and $-1 \leq x-[x]-1 < 0$, if we define a function $h : \mathbb{N} \times [-1,1) \to X$, by $h := f|_{\mathbb{N}\times[-1,1)}$, then f is a function of the form (2.1).

Now,Let f be a function of the form (2.1), where $h : \mathbb{N} \times [-1,1) \to X$ is an arbitrary function, we want to show that

$$f(k,x) = kf(k,x-1) + f(k,x-2)$$

and so f is a k-Fibonacci function.

If $x \geq 2$, then $x-1 \geq 1, \quad x-2 \geq 0$.
and by (2.1) we have

$$f(k,x) = F_{k,[x]+1} h(k,x-[x]) + F_{k,[x]} h(k,x-[x]-1)$$

$$f(k,x-1) = F_{k,[x-1]+1} h(k,x-1-[x-1]) + F_{k,[x-1]} h(k,x-1-[x-1]-1)$$

Since $(x-1)-[x-1] = x-[x]$, hence

$$f(k,x-1) = F_{k,[x]} h(k,x-[x]) + F_{k,[x]-1} h(k,x-[x]-1),$$

$$f(k,x-2) = F_{k,[x]-1} h(k,x-[x]) + F_{k,[x]-2} h(k,x-[x]-1).$$

Therefore

$$
\begin{aligned}
kf(k,x-1) + f(k,x-2) &= kF_{k,[x]} h(k,x-[x]) + kF_{k,[x]-1} h(k,x-[x]-1) \\
&+ F_{k,[x]-1} h(k,x-[x]) + F_{k,[x]-2} h(k,x-[x]-1) \\
&= (kF_{k,[x]} + F_{k,[x]-1}) h(k,x-[x]) + (kF_{k,[x]-1} + F_{k,[x]-2}) h(k,x-[x]-1) \\
&= F_{k,[x]+1} h(k,x-[x]) + F_{k,[x]} h(k,x-[x]-1) = f(k,x).
\end{aligned}
$$

If $1 \leq x \leq 2$,then $0 \leq x-1 \leq 1$, $-1 \leq x-2 \leq 0$ and by (2.1), we have

$$
\begin{aligned}
f(k,x) &= F_{k,[x]+1} h(k,x-[x]) + F_{k,[x]} h(k,x-[x]-1) \\
&= F_{k,2} h(k,x-[x]) + F_{k,1} h(k,x-[x]-1) \\
&= kh(k,x-[x]) + h(k,x-[x]-1)
\end{aligned}
$$

$$\begin{aligned}f(k,x-1) &= F_{k,[x-1]+1}h(k,x-1-[x-1]) + F_{k,[x-1]}h(k,x-1-[x-1]-1)\\ &= F_{k,1}h(k,x-[x]) + F_{k,0}h(k,x-[x]-1)\\ &= h(k,x-[x])\end{aligned}$$

$$\begin{aligned}f(k,x-2) &= (-1)^{[x-2]}[F_{k,(-[x]-1)}h(k,x-[x]) - F_{k,2-[x]}h(k,x-[x]-1)]\\ &= -[F_{k,0}h(k,x-[x]) - F_{k,1}h(k,x-[x]-1)]\\ &= h(k,x-[x]-1).\end{aligned}$$

Hence

$$kf(k,x-1) + f(k,x-2) = kh(k,x-[x]) + h(k,x-[x]-1) = f(k,x).$$

If $0 \le x < 1$, then $-1 \le x-1 < 0$, $-2 \le x-2 < -1$ and by (2.1), we have

$$f(k,x) = F_{k,1}h(k,x-[x]) + F_{k,0}h(k,x-[x]-1) = h(k,x-[x])$$

$$f(k,x-1) = (-1)^{-1}[F_{k,0}h(k,x-[x]) - F_{k,1}h(k,x-[x]-1)] = h(k,x-[x]-1)$$

$$f(k,x-2) = (-1)^{-2}[F_{k,1}h(k,x-[x]) + F_{k,2}h(k,x-[x]-1)] = h(k,x-[x]) - kh(k,x-[x]-1).$$

Thus, we get

$$kf(k,x-1) + f(k,x-2) = h(k,x-[x]) = f(k,x).$$

Finally, if $x < 0$, then we have

$$f(k,x) = (-1)^{[x]}[F_{k,-[x]-1}h(k,x-[x]) - F_{k,-[x]}h(k,x-[x]-1)]$$

$$\begin{aligned}f(k,x-1) &= (-1)^{[x-1]}[F_{k,-[x-1]-1}h(k,x-1-[x-1]) - F_{k,-[x-1]}h(k,x-1-[x-1]-1)]\\ &= (-1)^{[x]-1}[F_{k,-[x]}h(k,x-[x]) - F_{k,-[x]-1}h(k,x-[x]-1)]\end{aligned}$$

$$\begin{aligned}f(k,x-2) &= (-1)^{[x-2]}[F_{k,-[x-2]-1}h(k,x-2-[x-2]) - F_{k,-[x-2]}h(k,x-2-[x-2]-1)]\\ &= (-1)^{[x]-2}[F_{k,-[x]}h(k,x-[x]) - F_{k,-[x]+2}h(k,x-[x]-1)].\end{aligned}$$

Therefore

$$f(k,x) = kf(k,x-1) + f(k,x-2).$$

□

3. Hyers-Ulam stability of k-Fibonacci equation

In the following theorem, we investigate the Hyers-Ulam stability for equations of the form (1.1).

Theorem 3.1. *Let $(X, ||.||)$ be a real Banach space. If a function $f : \mathbb{N} \times \mathbb{R} \to X$ satisfies the inequality*

$$||f(k,x) - kf(k,x-1) - f(k,x-2)|| \leq \varepsilon, \quad (3.1)$$

for all $x \in \mathbb{R}$, $k \in \mathbb{N}$, and for some $\varepsilon > 0$, then there exists a k-Fibonacci function $G : \mathbb{N} \times \mathbb{R} \to X$ such that

$$\|f(k,x) - G(k,x)\| \leq \frac{\varepsilon}{2k}(k + 1 - \frac{k^2 - 3k - 2}{\sqrt{k^2+4}}), \quad (3.2)$$

for all $x \in \mathbb{R}$, $k \in \mathbb{N}$.

Proof. As $\gamma + \theta = k,\ \gamma\theta = -1$, we get from (3.1)

$$||f(k,x) - (\gamma+\theta)f(k,x-1) + \gamma\theta f(k,x-2)|| \leq \varepsilon$$

or

$$||f(k,x) - \gamma f(k,x-1) - \theta[f(k,x-1) - \gamma f(k,x-2)]|| \leq \varepsilon,$$

for all $x \in \mathbb{R},\ k \in \mathbb{N}$.
If we replace x by $x - t$ and then multiplying the both sides of this inequality by $|\theta|^t$, we get

$$||\theta^t[f(k,x-t) - \gamma f(k,x-t-1)] - \theta^{t+1}[f(k,x-t-1) - \gamma f(k,x-t-2)]|| \leq |\theta|^t\varepsilon \quad (3.3)$$

for all $x \in \mathbb{R},\ k \in \mathbb{N}$, and $t \in \mathbb{Z}$. Since

$$||\sum_{t=0}^{n-1} \theta^t[f(k,x-t) - \gamma f(k,x-t-1)] - \theta^{t+1}[f(k,x-t-1) - \gamma f(k,x-t-2)]||$$
$$= ||f(k,x) - \gamma f(k,x-1) - \theta^n[f(k,x-n) - \gamma f(k,x-n-1)]||, \quad (3.3)$$

hence

$$||f(k,x) - \gamma f(k,x-1) - \theta^n[f(k,x-n) - \gamma f(k,x-n-1)]|| \leq \sum_{t=0}^{n-1} |\theta|^t\varepsilon, \quad (3.4)$$

for all $x \in \mathbb{R},\ k \in \mathbb{N}$, and $t \in \mathbb{Z}$.
From (3.3)for all $x \in \mathbb{R},\ k \in \mathbb{N}$, we have $\{\theta^n[f(k,x-n) - \gamma f(k,x-n-1)]\}$ is a Cauchy sequence ($|\theta| < 1$). Therefore we can define $G_1 : \mathbb{N} \times \mathbb{R} \to X$ by

$$G_1(k,x) = \lim_{n\to\infty} \theta^n[f(k,x-n) - \gamma f(k,x-n-1)].$$

Since X is a Banach space, so it is complete and G_1 is well defined function.We have

$$kG_1(k,x-1) + G_1(k,x-2)$$
$$= k\theta^{-1}G_1(k,x) + \theta^{-2}G_1(k,x) = G_1(k,x)$$

if $n \to \infty$, then from (3.4) we have

$$||f(k,x) - \gamma f(k,x-1) - G_1(k,x)|| \leq \frac{2 + k + \sqrt{k^2+4}}{2}\varepsilon, \quad (3.5)$$

for all $x \in \mathbb{R},\ k \in \mathbb{N}$.
On other hand, from (3.1), we have

$$||f(k,x) - \theta f(k,x-1) - \gamma[f(k,x-1) - \theta f(k,x-2)]|| \leq \varepsilon,$$

for all $x \in \mathbb{R},\ k \in \mathbb{N}$.
Now if we replace x by $x+t$ and then multiplying the both sides of this inequality by γ^{-t}, we get

$$||\gamma^{-t}[f(k,x+t)-\theta f(k,x+t-1)]-\gamma^{-t+1}[f(k,x+t-1)-\theta f(k,x+t-2)]|| \leq \gamma^{-t}\varepsilon, \quad (3.6)$$

for all $x \in \mathbb{R},\ k \in \mathbb{N}$, and $t \in \mathbb{Z}$. Therefore

$$||\gamma^{-n}[f(k,x+n) - \theta f(k,x+n-1)] - [f(k,x) - \theta f(k,x-1)]||$$
$$\leq \sum_{t=1}^{n} ||\gamma^{-t}[f(k,x+t) - \theta f(k,x+t-1)] - \gamma^{-t+1}[f(k,x+t-1) - \theta f(k,x+t-2)]||$$
$$\leq \sum_{t=1}^{n} |\gamma^{-t}|\varepsilon, \quad (3.7)$$

for all $x \in \mathbb{R},\ k \in \mathbb{N}$, and $t \in \mathbb{Z}$.
For all $x \in \mathbb{R},\ k \in \mathbb{N}(3.6)$, we have $\{\gamma^{-n}[f(k,x+n) - \theta f(k,x+n-1)]\}$ is a Cauchy sequence and hence we can define $G_2 : \mathbb{N} \times \mathbb{R} \to X$ by

$$G_2(k,x) = \lim_{n\to\infty} \gamma^{-n}[f(k,x+n) - \theta f(k,x+n-1)].$$

Since X is a Banach space, so it is complete and G_2 is well defined function.We have

$$\begin{aligned} &kG_2(k,x-1) + G_2(k,x-2) \\ &= k\gamma^{-1}G_2(k,x) + \gamma^{-2}G_2(k,x) = G_2(k,x). \end{aligned}$$

If $n \to \infty$, then from (3.7), we have

$$||G_2(k,x) - f(k,x) - \theta f(k,x-1)|| \leq \frac{2-k+\sqrt{k^2+4}}{2k}\varepsilon \quad (3.8)$$

for all $x \in \mathbb{R},\ k \in \mathbb{N}$.
For

$$G(k,x) = \frac{\theta}{\theta-\gamma}G_1(k,x) - \frac{\gamma}{\theta-\gamma}G_2(k,x),$$

we have

$$\begin{aligned} ||f(k,x) - G(k,x)|| &= ||f(k,x) - \frac{\theta}{\theta-\gamma}G_1(k,x) - \frac{\gamma}{\theta-\gamma}G_2(k,x)|| \\ &= \frac{1}{|\theta-\gamma|}||(\theta-\gamma)f(k,x) - [\theta G_1(k,x) - \gamma G_2(k,x)]|| \\ &\leq \frac{1}{\gamma-\theta}||\theta[f(k,x) - \gamma f(k,x-1) - G_1(k,x)|| \\ &+ \frac{1}{\gamma-\theta}||\gamma[G_2(k,x) - f(k,x) - \theta f(k,x-1)|| \\ &\leq \frac{\varepsilon}{2k}(k+1-\frac{k^2-3k-2}{\sqrt{k^2+4}}), \quad (By\ 3.5\ and\ 3.8) \end{aligned}$$

and it is easy to see G is a k-Fibonacci function.

□

In order to prove G is also unique, we need the following lemma.

Lemma 3.2. *Let $(X, ||.||)$ be a real normed space and $u, v \in X$ are given. If for all $n \in \mathbb{N}$ and for some $C \geq 0$ we have*

$$||F_{k,n+1}u + F_{k,n}v|| \leq C$$

then,

$$\gamma u + v = 0.$$

Proof. We have,

$$\begin{aligned} F_{k,n}||\gamma u + v|| &= ||\gamma F_{k,n}u + F_{k,n}v + F_{k,n+1}u - F_{k,n+1}u|| \\ &\leq ||F_{k,n+1}u + F_{k,n}v|| + |F_{k,n+1} - \gamma F_n|||u|| \\ &\leq C + |\frac{\gamma^{n+1} - \theta^{n+1}}{\gamma - \theta} - \gamma\frac{\gamma^n - \theta^n}{\gamma - \theta}|||u|| \quad (By\ Binet's\ formula) \\ &= C + |\theta|^n||u||, \end{aligned}$$

for all $n \in \mathbb{N},\ k \in \mathbb{N}$.
Since $|\theta| < 1$, if $n \to \infty$, then $F_{k,n} \to \infty$, and so $\gamma u + v = 0$.

□

Theorem 3.3. *The k-Fibonacci function in Theorem (3.1) is unique.*

Proof. Let there exist k-Fibonacci functions, $G_1 : \mathbb{N} \times \mathbb{R} \longrightarrow X$, and $G_2 : \mathbb{N} \times \mathbb{R} \longrightarrow X$ satisfying

$$||f(k,x) - G_i(k,x)|| \leq \frac{\varepsilon}{2k}(k + 1 - \frac{k^2 - 3k - 2}{\sqrt{k^2 + 4}}), \quad (3.9)$$

for all $x \in \mathbb{R},\ k \in \mathbb{N}, i \in \{1, 2\}$. Since G_1 and G_2 are k-Fibonacci function,by Theorem (2.1), there exist functions $g_i : \mathbb{N} \times [-1, 1) \longrightarrow X$ $(i = \{1, 2\})$ such that

$$G_i(k,x) = \begin{cases} F_{k,[x]+1}g_i(k, x - [x]) + F_{k,[x]}g_i(k, x - [x] - 1) & x \geq 0 \\ (-1)^{[x]}[F_{k,-[x]-1}g_i(k, x - [x]) - F_{k,-[x]}g_i(k, x - [x] - 1)] & x < 0 \end{cases}, \quad (3.10)$$

for $i \in 1, 2$.
Fix a t in $[0, 1)$, from (3.9), we have

$$\begin{aligned} &||G_1(k, n + t) - G_2(k, n + t)|| \\ &\leq ||G_1(k, n + t) - f(k, n + t)|| + ||f(k, n + t) - G_2(k, n + t)|| \\ &\leq 2\frac{\varepsilon}{2k}(k + 1 - \frac{k^2 - 3k - 2}{\sqrt{k^2 + 4}}), \end{aligned}$$

for all $n \in \mathbb{Z},\ k \in \mathbb{N}$.
by (3.10), we have

$$\begin{aligned} &||F_{k,n+1}[g_1(k,t) - g_2(k,t)] + F_{k,n}[g_1(k, t - 1) - g_2(k, t - 1)|| \\ &= ||G_1(k, n + t) - G_2(k, n + t)|| \leq 2\frac{\varepsilon}{2k}(k + 1 - \frac{k^2 - 3k - 2}{\sqrt{k^2 + 4}}), \end{aligned}$$

and

$$||F_{k,n-1}[g_1(k,t)-g_2(k,t)] - F_{k,n}[g_1(k,t-1)-g_2(k,t-1)||$$
$$= ||G_1(k,-n+t) - G_2(k,-n+t)|| \leq 2\frac{\varepsilon}{2k}(k+1-\frac{k^2-3k-2}{\sqrt{k^2+4}}),$$

for all $n \in \mathbb{N},\ k \in \mathbb{N}$.
According to Lemma (3.2), we have

$$\begin{cases} \gamma[g_1(k,t)-g_2(k,t)] + [g_1(k,t-1)-g_2(k,t-1)] = 0 \\ -\gamma[g_1(k,t-1)-g_2(k,t-1)] + [g_1(k,t)-g_2(k,t)] = 0 \end{cases}$$

or

$$\begin{pmatrix} \gamma & 1 \\ 1 & -\gamma \end{pmatrix} \begin{pmatrix} g_1(k,t)-g_2(k,t) \\ g_1(k,t-1)-g_2(k,t-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $-\gamma^2 - 1 \neq 0$, hence

$$g_1(k,t)-g_2(k,t) = g_1(k,t-1)-g_2(k,t-1).$$

Since $0 \leq t < 1$ is arbitrary, therefore $g_1(k,t) = g_2(k,t)$, for any $0 \leq t < 1$, and from (3.10), we have $G_1(k,x) = G_2(k,x)$, for all $x \in \mathbb{R}$.

□

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