Int. J. Nonlinear Anal. Appl. 16 (2025) 9, 87–93
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2024.35416.5275

Nil Armendariz rings of Hurwitz series type 1

Kianoosh Sabzipour[a], Hamid Haj Seyyed Javadi[b,*]

[a] *Department of Pure Mathematics, Tarbiat Modares University, Tehran, Iran*

[b] *Department of Mathematics and Computer Science, Shahed University, Tehran, Iran*

(Communicated by Abasalt Bodaghi)

Abstract

In this paper, we study the structure of the set of nilpotent elements in Armendariz rings of Hurwitz series type and introduce nil Armendariz as a generalization. It is proved that a ring R is nil Armendariz of Hurwitz series type if and only if R has characteristic zero and $Nil(R)$ is an ideal. We provide many examples of nil Armendariz rings of Hurwitz series type and extend the class of nil Armendariz rings of Hurwitz series type through various ring extensions.

Keywords: nil Armendariz ring, nil Armendariz ring of Hurwitz series type, nil Armendariz ring of skew Hurwitz series type
2020 MSC: 13F20

1 Introduction

In an earlier paper by Keigher [14], a variant of the ring of formal power series was introduced, and some of its properties, especially categorical properties, were studied. In the papers [15], [16], Keigher demonstrated that the ring of Hurwitz series has many interesting applications in differential algebra. Hurwitz series rings are similar to formal power series rings, except that binomial coefficients are introduced in each term of the product.

While there are many studies of these rings over a commutative ring, very little is known about them over a noncommutative ring. In the present paper we study Hurwitz series over a noncommutative ring with identity, examine its structure and properties. The definition of Hurwitz series originally allowed the ring to be noncommutative, but most authors restrict them to be commutative, therefore all of the basic definitions are still true under the restriction that the ring is noncommutative.

Throughout this paper, all rings are associative with identity, and assume that R is a ring with identity. The Hurwitz polynomial ring and the Hurwitz series ring with an indeterminate X over a ring R are denoted by hR and HR respectively. We use $Nil(HR)$ to represent the set of nilpotent elements (the nilradical) of R. A subring will refer to a subring without unit.

We denote with $H(R)$, or simply HR, the ring of Hurwitz series over R. The elements of HR are sequences of the form $a = (a_n) = (a_0, a_1, a_2, \cdots)$, where $a_n \in R$ for each $n \in \mathbb{N}$. An element in HR can be thought of as a function from $\mathbb{N}$ to R. Two elements (a_n) and (b_n) in HR are equal if they are equal as functions from $\mathbb{N}$ to R, i.e., if $a_n = b_n$, for all $n \in \mathbb{N}$. The element $a_m \in R$ is called the mth term of (a_n). Addition in HR is defined termwise, so that

*Corresponding author
Email addresses: sabzipour1977@gmail.com (Kianoosh Sabzipour), h.s.javadi@shahed.ac.ir (Hamid Haj Seyyed Javadi)

Received: July 2024 *Accepted:* September 2024

$(a_n)+(b_n)=(c_n)$, where $c_n=a_n+b_n$, for all $n\in\mathbb{N}$. If one identifies a formal power series $\Sigma_{n=0}^{\infty}a_nt^n\in R[[t]]$ with the sequence of its coefficients (a_n), then multiplication in HR is similar to the usual product of formal power series, except that binomial coefficients are introduced at each term in the product as follows. The (Hurwitz) product of (a_n) and (b_n) is given by $(a_n).(b_n)=(c_n)$, where

$$c_n=\Sigma_{k=0}^{n}C_n^k a_k b_{n-k}.$$

Hence

$$(a_0,a_1,a_2,a_3,\cdots).(b_0,b_1,b_2,b_3,\cdots)=(a_0b_0,a_0b_1+a_1b_0,a_0b_2+2a_1b_1+a_2b_0,a_0b_3+3a_1b_2+3a_2b_1+a_3b_0,\cdots).$$

The zero element in HR is $0=(0,0,0,\cdots)$, the sequence with all terms 0, and the identity is $1=(1,0,0,\cdots)$, the sequence with 0th term 1 and nth term 0 for all $n\geq 1$.

In [20], Rege and Chhawchharia introduce the notion of an Armendariz ring. A ring R is Armendariz if whenever $f(x)g(x)=0$ where $f(x)=\Sigma_{i=0}^{m}a_iX^i$ and $g(x)=\Sigma_{j=0}^{n}b_jX^j\in R[x]$, then $a_ib_j=0$ for all i and j. The ring is named after E. Armendariz, who proved in [4] that reduced rings (i.e., rings without nonzero nilpotent elements) satisfy this condition.Armendariz rings are a generalization of reduced rings; therefore, nilpotent elements play an important role in this class of rings. There are many examples of rings with nilpotent elements which are Armendariz. In [2], Anderson and Camillo proved that if $n\geq 2$, then $R[x]/(x^n)$ is an Armendariz ring if and only if R is reduced.

In [3], Antoine introduce and study nil Armendariz rings. A ring is nil Armendariz if the product of two polynomials has coefficients in the set of nilpotent elements, then the product of the coefficients of the polynomials is also nilpotent.

We denote the set of nilpotent elements of HR by $Nil(HR)$. We study the structure of the set of nilpotent elements in Armendariz rings of Hurwitz series type and introduce nil Armendariz as a generalization. A ring R is called nil Armendariz of Hurwitz series type if for each $f=\Sigma_{i=0}^{\infty}a_iX^i, g=\Sigma_{i=0}^{\infty}b_iX^i\in HR, fg\in H(Nil(R))$ implies $a_ib_j\in Nil(R)$ for each i,j, where $Nil(R)$ denotes the set of nilpotent elements of R. It is proved that a ring R is nil Armendariz of Hurwitz series type if and only if R has charateristic zero and $Nil(R)$ is an ideal. We provide many examples of nil Armendariz rings of Hurwitz series type and extend the class of nil Armendariz rings of Hurwitz series type through various ring extensions.

2 Nil Armendariz rings of Hurwitz series type

In this section we initiate the notion of nil Armendariz rings of Hurwitz seires type and we consider the Armendariz property of Hurwitz series rings.

Definition 2.1. A ring R is called Armendariz of skew Hurwitz series type if for any $f=(a_0,a_1,\cdots), g=(b_0,b_1,\cdots)\in(HR,\alpha), fg=0$ implies $a_i\alpha^ib_j=0$, for all i,j.

Definition 2.2. We say a ring R is nil Armendariz of Hurwitz series type if for each $f=(a_0,a_1,\cdots)$, $g=(b_0,b_1,\cdots)\in(HR,\alpha), fg\in(H(Nil(R)),\alpha))$, implies $a_i\alpha^ib_j\in Nil(R)$, for all i,j.

Armendariz ring of skew Hurwitz series type $\Rightarrow$ nil Armendariz ring of skew Hurwitz series type.

Lemma 2.3. Let R be a reduced $\alpha-$compatible (rigid) ring. If R is torsion free as a $\mathbb{Z}-$module, then R is Armendariz of skew Hurwitz series type.

Proof . Let R be reduced and $f=(a_0,a_1,\ldots,a_n,\cdots)$, $g=(b_0,b_1,\cdots,b_m,\cdots)\in(HR,\alpha)$ such that $fg=0$. Then we have
(1) $a_0b_0=0$,
(2) $a_0b_1+a_1\alpha(b_0)=0$,
(3) $a_0b_2+2a_1\alpha(b_1)+a_2\alpha^2(b_0)=0$,
(4) $a_0b_3+3a_1\alpha(b_2)+3a_2\alpha^2(b_1)+a_3\alpha^3(b_0)=0$,
$\vdots$

From equation (1) we get $(b_0a_0)^2=b_0a_0b_0a_0=0$ so $b_0a_0=0$, since R is reduced. Now multiplying equation (2) from left by b_0, we get $b_0a_1\alpha(b_0)=0$. As R is a $\alpha-$rigid ring, we get $a_1\alpha(b_0)\alpha(a_1\alpha(b_0))=a_1\alpha(b_0a_1\alpha(b_0))=0$ and so $a_1\alpha(b_0)=0$. From this and equation (2), we obtain $a_0b_1=0$ and so $b_1a_0=0$.

Now multiplying equation (3) on the left side by $a_1\alpha(b_1)$, we get $a_1\alpha(b_1)a_0b_2 + 2(a_1\alpha(b_1))^2 + a_1\alpha(b_1)a_2\alpha^2(b_0) = 0$. As R is a $\alpha-$rigid ring and $b_1a_0 = a_1b_0 = 0$, we get $\alpha(b_1)a_0 = 0, a_1\alpha^2(b_0) = 0$. As R is a reduced ring, $a_1\alpha(b_1)a_0b_2 = a_1\alpha(b_1)a_2\alpha^2(b_0) = 0$ and hence $2(a_1\alpha(b_1))^2 = 0$. So $(2a_1\alpha(b_1))^2 = 0$ and hence $2a_1\alpha(b_1) = 0$, because R is reduced. Since R, as a $\mathbb{Z}-$module, is torsion free we get $a_1\alpha(b_1) = 0$. Continuing in this way, we get $a_i\alpha^i(b_j) = 0$ for each i, j. □

Proposition 2.4. Let R be a ring such that $Nil(R)$ is an ideal of R and R is torsion free as a $\mathbb{Z}$-module. If R is a $\alpha-$compatible ring, then R is nil Armendariz ring of skew Hurwitz series type.

Proof . From Lemma 2.3 every $\alpha-$rigid ring with torsion free as a $\mathbb{Z}$-module, is a Armendariz ring of skew Hurwitz series type. As $R/Nil(R)$ is reduced ring and $\alpha(Nil(R)) = Nil(R), \bar{\alpha} : R/Nil(R) \to R/Nil(R)$ is well defined ring endomorphism. Next we show that $R/Nil(R)$ is an $\bar{\alpha}-$compatible. Consider the elements $\bar{a}, \bar{b} \in R/Nil(R)$. Then $\bar{a}\bar{b} = 0$ if and only if $ab \in Nil(R)$ if and only if $a\alpha(b) \in Nil(R)$ if and only if $\bar{a}\bar{\alpha}(\bar{b}) = 0$. Therefore $R/Nil(R)$ is an $\bar{\alpha}-$rigid and so it is Armendariz ring of skew Hurwitz series type. Let $f = (a_0, a_1, \cdots, a_n, \cdots)$ and $g = (b_0, b_1, \cdots, b_m, \cdots) in (HR, \alpha)$ such that $fg \in (H(NilR), \alpha)$, $\bar{f}, \bar{g}$ the corresponding in $(H(R/NilR), \bar{\alpha}))$, so $\bar{f}\bar{g} = \bar{0}$. As $R/Nil(R)$ is Armendariz ring of skew Hurwitz series type, for each coefficient a_i of f and each coefficient b_j of g, $a_i\alpha^i b_j \in Nil(R)$. □

Lemma 2.5. Let R be a nil Armendariz ring of skew Hurwitz series type and $n \geq 2$. If R is an $\alpha-$compatible, $f_1, f_2, \cdots, f_n \in hR$ with $f_1f_2\cdots f_n \in (H(Nil(R), \alpha))$, then $a_{i1}a_{i2}\cdots a_{in} \in Nil(R)$, where $a_{ik} \in Coef(f_k), 1 \leq k \leq n$.

Proof . We prove the result by induction on $n \geq 2$. For $n = 2$ it follows by the definition and $\alpha-$compatiblity. Let $n > 2$ and $g = f_2f_3\cdots f_n$. Then $f_1g \in (H(Nil(R), \alpha))$. Since R is nil Armendariz of skew Hurwitz series type, $a_{i1}\alpha^{i1}(a_g) \in Nil(R)$, for every coefficient a_g of g and every a_{i1} of f_1. As R is $\alpha-$compatible, then $a_{i1}a_g \in Nil(R)$, for every coefficient a_g of g. Hence $a_{i1}f_2\cdots f_{n-1}f_n = a_{i1}g \in (H(NilR), \alpha)$. Since the coefficients of $a_{i1}f_2$ are $a_{i1}a_{i2}$, where a_{i2} is a coefficient of f_2, by induction, we obtain $a_{i1}a_{i2}\cdots a_{i(n-1)}a_{in} \in Nil(R)$, where a_{ik} is the coefficient of f_k, for $k = 1, \cdots, n$. □

Recall from [12], that a ring R is semicommutative if $ab = 0$ implies $aRb = 0$, for each $a, b \in R$.

Proposition 2.6. Let R be a Armendariz ring of skew Hurwitz series type with $\alpha-$compatible. Then R is a semicommutative ring.

Proof . Let R be an Armendariz ring of Hurwitz series type and suppose that $ab = 0$ for $a, b \in R$. As for every $r \in R$

$$(1, -r, 0, 0, \cdots)(1, 1!r, 2!r\alpha(r), 3!r\alpha(r\alpha(r)), \cdots, n!r\alpha(r)\alpha^2(r)\cdots\alpha^{n-1}(r), \cdots) = (1, 0, 0, \cdots)$$

we get

$$(a, 0, 0, \cdots)(1, -r, 0, 0, \cdots)(1, 1!r, 2!r\alpha(r), 3!r\alpha(r\alpha(r)), \cdots, n!r\alpha(r)\cdots\alpha^{n-1}(r), \cdots)(b, 0, 0, \cdots) = (ab, 0, 0, \cdots) = (0, 0, 0, \cdots).$$

As R is an Armendariz ring of Hurwitz series type we get $aR\alpha(b) = 0$. Now applying $\alpha-$compatibility proves that R is semicommutative. □

We establish an analogous result in the case of nil Armendariz rings of Hurwitz series type. In [17] the authors introduced the concept of weakly semicommutative rings. A ring R is said to be weakly semicommutative if for each $a, b \in R$, $ab = 0$, then arb is nilpotent for all $r \in R$.

Lemma 2.7. Let R be a nil Armendariz ring of skew Hurwitz series type. If R is an $\alpha-$compatible ring, then for $a, b \in R$, $ab \in Nil(R)$ implies $aRb \subseteq Nil(R)$.

Proof . Let $r \in R$. Then we get

$$(a, 0, 0, \cdots)(1, -r, 0, \cdots)(1, 1!r, 2!r\alpha(r), \cdots, n!r\alpha(r)\alpha^2(r), \cdots\alpha^{n-1}(r)\cdots)(b, 0, \cdots) = (ab, 0, 0, \cdots) \in Nil(R)$$

Since R is nil Armendariz of skew Hurwitz series type, $ar\alpha(b) \in Nil(R)$, for every $r \in R$. As R is $\alpha-$compatible, we get $aRb \subseteq Nil(R)$ and the result follows. □

Corollary 2.8. Every $\alpha-$compatible, nil Armendariz ring of skew Hurwitz series type is a weakly semicommutative ring.

Proof . This result follows from Lemma 2.7. □

Lemma 2.9. Let R be a nil Armendariz ring of skew Hurwitz series type. If R is an $\alpha-$compatible ring, $fg \in (H(Nil(R)), \alpha)$, then $f(HR, \alpha)g \subseteq (H(Nil(R)), \alpha)$, for each $f, g \in (HR, \alpha)$.

Proof . Let $fg \in (H(Nil(R)), \alpha)$. Then $a_i\alpha^i b_j \in Nil(R)$ for all coefficient a_i of f and coefficient $b_j$of g. Applying Lemma 2.7, we get $a_i r b_j \in Nil(R)$, for each $r \in R$. As R is nil Armendariz of skew Hrwiz series type, it is nil Armendariz too, so Nil(R) is a subring of R, By Corollary 2.11. This yields $fHRg \subseteq H(Nil(R))$. □

Lemma 2.10. Let R be a nil Armendariz ring of skew Hurwitz series type. If R is an $\alpha-$compatible ring, then
(1) If $a, b \in R$ are nilpotent, then ab is nilpotent.
(2) If $a, b, c \in R$ are nilpotent, then $(a+b)c$ and $c(a+b)$ are nilpotent.
(3) If $a, b, c \in R$ are nilpotent, then $a + bc$ is nilpotent.
(4) If $a, b \in R$ are nilpotent, then $a - b$ is nilpotent.

Proof . (1) Suppose a, b are nilpotent and $b^m = 0$. Then by $\alpha-$compatibility we get, $b\alpha(b)\cdots\alpha^{m-1}(b) = 0$ and so $(a, -ab, 0, \cdots)(1, 1!b, 2!b\alpha(b), \cdots, (m-1)!b\alpha(b)\cdots\alpha^{m-2}(b), 0, \cdots) = a \in (H(Nil(R)), \alpha)$. Since R is nil Armendariz ring of skew Hurwitz series type, we get $ab \in Nil(R)$.

(2) Suppose a, b, c are nilpotent and $a^n = b^m = 0$. Then $\alpha-$compatibility implies that

$$a\alpha(a)\cdots\alpha^{n-1}(a) = 0, \ b\alpha(b)\cdots\alpha^{m-1}(b) = 0.$$

So

$$\begin{gathered}(1, 1!a, 2!a\alpha(a), \cdots, (n-1)!a\alpha(a)\cdots\alpha^{n-2}(a), 0, \cdots)(1, -a, 0, \cdots)(1, -b, 0, \cdots)\\ (1, 1!b, 2!b\alpha(b), \cdots, (m-1)!b\alpha(b)\cdots\alpha^{m-2}(b), 0, \cdots)(c, 0, \cdots) = (c, 0, \cdots)\end{gathered}$$

Thus

$$\begin{gathered}(1, 1!a, 2!a\alpha(a), \cdots, (n-1)!a\alpha(a)\cdots\alpha^{n-2}(a), 0, \cdots)(1, -(a+b), 2ab, 0, \cdots)\\ (1, 1!b, 2!b\alpha(b), \cdots, (m-1)!b\alpha(b)\cdots\alpha^{m-2}(b), 0, \cdots)(c, 0, \cdots) = (c, 0, \cdots).\end{gathered}$$

As R is nil Armendariz ring of skew Hurwitz series type and $(c, 0, \cdots) \in Nil(R)$, we get $(a+b)\alpha(c) \in Nil(R))$. Now $\alpha-$compatibility follows that $(a+b)c \in Nil(R)$. Similarly we see that $c(a+b) \in Nil(R)$.

(3) Suppose a, b, c are nilpotent. By (1), bc is nilpotent, and by (2), $\alpha(b(a+bc))$ is also nilpotent. Hence

$$(1, -\alpha(b), 0, \cdots)(c, \alpha(a+bc), 0, \cdots) = (c, \alpha(a), -2\alpha(b(a+bc)), 0, \cdots) \in (H(Nil(R)), \alpha).$$

As R is is nil Armendariz ring of skew Hurwitz series type, we get $\alpha(a+bc) \in Nil(R)$. Now by $\alpha-$compatibility we get $(a+bc) \in Nil(R)$.

(4) Applying similar method used in [3, Lemma 3.1]. □

Corollary 2.11. If R is a nil Armendariz ring of Hurwitz series type, then $Nil(R)$ is a subrng of R.

Proof . It follows from lemma 2.10. □

Example 2.12. For each ring R, the matrix ring $M_n(R)$ is never nil Armendariz of Hurwiz series type. In fact consider $X = E_{12}$ and $Y = -E_{21}$. Then X and Y are nilpotent but $X - Y$ is not.

Each commutative ring that is torsion free, as a $\mathbb{Z}-$module, is nil Armendariz of Hurwitz series type. But there exists examples of commutative rings which are not Armendariz of Hurwitz series type. For example, let F be a field of characteristic $\neq 2$. Let $S = F[A_i, i \in \mathbb{N}]$, $I =< A_iA_jA_k >$ and $T = S/I$. Let $d_n = \Sigma_{i+j=n}A_iA_j$, $J =< d_n, n \in \mathbb{N} >_T$ and $R = T/J$. Then R is commutative but it is not Armendariz of Hurwitz series type.

Proposition 2.13. Let R be a ring and I a nil ideal of R. Then R is nil Armendariz of Hurwitz series type if and only if R/I is a nil Armendariz ring of Hurwitz series type.

Proof . From ([5], Proposition 1.2), we have $HR/HI \cong H(R/I)$. We denote $\bar{R} = R/I$. Since I is nil, $Nil(\bar{R}) = \overline{Nil(R)}$. Hence $fg \in H(Nil(R))$ if and only if $\bar{f}\bar{g} \in H(Nil(\bar{R})$. If a is a coefficient of f and b a coefficient of g, then $ab \in Nil(R)$ if and only if $\bar{a}\bar{b} \in Nil(\bar{R})$. Therefore R is nil Armendariz of Hurwitz series type if and only if $\bar{R}$ is a nil Armendariz ring of Hurwitz series type. □

Lemma 2.14. If R is a nil Armendariz ring of Hurwitz series type with no nonzero nil ideals, then R is an Armendariz ring of Hurwitz series type.

Proof . Since R is nil Armendariz of Hurwitz series type, R does not contain any nonzero nil ideal. Suppose $f, g \in HR$ such that $fg = 0$. Let a be a coefficient of f and b a coefficient of g. For all $r \in R$, since $(r, 0, \cdots)fg = 0$, R is nil Armendariz of Hurwitz series type, and ra is a coefficient of rf, we have that rab is nilpotent. Hence Rab is a nil ideal. Then $Rab = 0$ and thus $ab = 0$. Therefore R is an Armendariz ring of Hurwitz series type. □

By Proposition 2.13 and the previous lemma, we obtain a new characterization of nil Armendariz rings of Hurwitz series type.

Theorem 2.15. A ring R is nil Armendariz of Hurwitz series type if and only if $R/Nil^*(R)$ is an Armendariz ring of Hurwitz series type.

Theorem 2.16. Let R be a nil Armendariz of Hurwitz series type. Then $Nil(R)$ is an ideal of R.

Proof . By Theorem 2.15 $R/Nil^*(R)$ is Armendariz of Hurwitz series type. Now applying Proposition 2.6, we get $R/Nil^*(R)$ is semicommutaive ring. As $R/Nil^*(R)$ is semiprime, $R/Nil^*(R)$ must be reduced ring, i.e., $Nil(R) = Nil^*(R)$, and so $Nil(R)$ is an ideal of R. □

Corollary 2.17. A ring R is nil Armendariz of Hurwitz series type if and only if $R/Nil^*(R)$ is an Armendariz ring of Hurwitz series type.

Corollary 2.18. Let R be a ring that is a torsion free as a $\mathbb{Z}-$module. Then R is a nil Armendariz of Hurwitz series type if and only if $Nil(R)$ is an ideal of R.

Proof . It follows from Theorem 2.16 and Proposition 2.4. □

In [12], it is proved that if e is a central idempotent then R is Armendariz if and only if eR and $(1-e)R$ are Armendariz rings. A same result is proved for weak Armendariz rings in [18] and it is also true for nil Armendariz rings. Furthermore, Anderson and Camillo prove that Armendariz rings are abelian (i.e. all idempotents are central).

Proposition 2.19. Let R be a nil Armendariz ring of Hurwitz series type and e be an idempotent. Then $ef - fe$ is nilpotent, for all $f \in HR$.

Proof . Let $e = e^2 \in HR$ and $f \in HR$. Then $ef(1-e)$ and $(e-1)fe$ are nilpotent elements. Hence, by Corollary 2.11, $ef(1-e) + (e-1)fe = ef - fe$ is also nil. Now, since $R/Nil(R)$ is Armendariz of Hurwitz series type and it is abelian, $ef - efe \in Nil(HR)$ □

Corollary 2.20. Let R be a nil Armendariz ring. The following are equivalent:
(1) R is nil Armendariz of Hurwitz series type.
(2) for each $a \in Nil(R)$ and $b \in R$, $ab \in Nil(R)$.
(3) if $ab \in Nil(R)$ then $aRb \subset Nil(R)$, for $a, b \in R$.

Proof . ($1 \Longleftrightarrow 2$) follows from the fact that if R is nil Armendariz, then $Nil(R)$ is a subrng of R.

For ($3 \Rightarrow 2$), let $a \in Nil(R)$. Then $a = a.1$, so $aR = aR1 \subseteq Nil(R)$.

($1 \Rightarrow 3$), It follows from Lemma 2.7. □

Corollary 2.21. Each semicommutative ring is nil Armendariz of Hurwitz series type.

Proof . By [18, Lemma 3.1], if R is semicommutative then $Nil(R)$ is an ideal. □

The converse of the last corollary is false. If F is a division ring then the triangular matrix ring $R = T_n(F)$ is nil Armendariz of Hurwitz series type (see below, Proposition 3.7). But by [12, Example 5], R is not semicommutative.

3 Hurwitz Polynomial rings over nil Armendariz rings

Anderson and Camillo in [2] prove that a ring R is Armendariz if and only if the polynomial ring $R[x]$ is Armendariz. Furthermore, the Armendariz condition is linked to many annihilator conditions being preserved in the polynomial ring, where in this paper we prove for Hurwitz polynomial rings. In [18], Liu and Zhao ask whether polynomial rings over weak Armendariz rings are weak Armendariz.

We can define both Armendariz and nil Armendariz rings for rings without identity, although most of the results we have proved need not be true, since we have strongly used the existence of the identity element. But clearly, if R is a nil ring, then R is nil Armendariz. Furthermore, adjoining an identity element, the ring $R_1 = \mathbb{Z} + Nil(R)$ ($K + R$ if R is a K-algebra) satisfies $Nil(R_1) = R$ is an ideal of R_1 and is thus nil Armendariz. The question of whether $Nil(hR) = h(Nil(R))$ for nil Armendariz rings is equivalent to the question of whether polynomial rings over nil rings are nil. Amitsur, in [1], proved that this is true for K-algebras over uncountable fields. But recently, Agata Smoktunowicz, in [21], has proved that the result is not true for algebras over countable fields. By using both the results of Amitsur and Smoktunowicz, we can prove the following.

Theorem 3.1. If R is a nil Armendariz ring, then $R[x]$ is a nil Armendariz if and only if $Nil(R)[x] = Nil(R[x])$.

Theorem 3.2. If R is a nil Armendariz algebra over an uncountable field K. Then the ring of Hurwitz polynomials hR is also nil Armendariz.

Proof . Let R be a nil Armendariz ring of Hurwitz polynomial K-algebra. Then $Nil(R)$ is a nil K-algebra by Corollary 2.11. Since K is uncountable and $Nil(hR) = h(Nil(R))$, by Theorem 3.1, hR is nil Armendariz. □

Theorem 3.3. If R is an Armendariz ring, then $hR/Nil(hR)$ is an Armendariz ring.

Proof . Note that every Armendariz ring is a nil Amrendariz ring. □

Example 3.4. For any countable field K, there exists a K-algebra R which is nil Armendariz and such that hR is not nil Armendariz. Let R_0 be the nil K-algebra constructed by Agata Smoktunowicz in [21] such that hR_0 is not nil. Let $R = K + R_0$. Clearly $Nil(R) = R_0$ is an ideal of R and hence R is nil Armendariz. By Theorem 3.1, R be a nil Armendariz ring. Then, hR is nil Armendariz. Thus hR_0 is nil and this is a contradiction. Hence hR is not nil Armendariz.

Example 3.5. Let R be a nil Armendariz ring such that hR is not nil Armendariz. Then $\bar{R} = R/Nil(R)$ is not Armendariz. Suppose otherwise that $\bar{R}$ is an Armendariz ring. Since $Nil(hR) = h(Nil(R))$, we have

$$hR/Nil(hR) \cong h\bar{R}.$$

Since R is Armendariz, by Theorem 3.3, $h\bar{R}$ is also Armendariz. Since $Nil(R)$ is nil, by Proposition 2.13, hR is nil Armendariz, which is a contradiction.

Proposition 3.6. Let R be a ring and n any positive integer. Then R is nil Armendariz of Hurwitz series type if and only if $hR/(X^n)$ is a nil Armendariz ring of Hurwitz series type. In particular if R is a semicommutative ring then $hR/(X^n)$ is a nil Armendariz ring of Hurwitz series type.

Proof . Let $f \in Nil(R) + XhR$. Then we write $f = a_0 + a_1X + \cdots + a_sX^s$ with $a_0 \in Nil(R)$, so there exists $p \in \mathbb{N}$ such that $a_0^p = 0$. So $f^p \in XhR$ and $f^{pn} \in X^nhR$. So $Nil(hR/(X^n)) = \{\bar{f} \in hR/(X^n) | k \in \mathbb{N}^*; f^k \in X^nhR\} = Nil(R) + XhR$. Conversely, if there exists k such that $f^k \in X^nhR$ and as $n \geq 1$ then $f(0) \in Nil(R)$, so $f \in Nil(R) + XhR$. So $Nil(R)$ is an ideal of R if and only if $Nil(R) + XhR$ is an ideal of hR. □

Let $T_n(R)$ denote the n by n upper triangular matrix ring over R. By observing that

$$Nil(T_n(R)) = \begin{pmatrix} Nil(R) & R & \cdots & R \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R \\ 0 & \cdots & 0 & Nil(R) \end{pmatrix}$$

A similar proof yields that, a ring R is nil Armendariz if and only if, for any positive integer n, $T_n(R)$ is nil Armendariz. It is then easy to verify the next.

Proposition 3.7. A ring R is nil Armendariz of Hurwitz series type if and only if for each n, $T_n(R)$ is a nil Armendariz ring of Hurwitz series type.

Proof . This implication is similar to [18, Proposition 2.2]. □

References

[1] A. Amitsur, *Algebras over infinite fields*, Proc. Amer. Math. Soc. **7** (1956), 35–48.

[2] D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), no. 7, 2265–2272.

[3] R. Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), 3128–3140.

[4] E.P. Armendariz, *A note on extensions of Baer and p.p.-rings*, J. Austral. Math. Soc. textbf18 (1974), 470–473.

[5] A. Benhissi, *Ideal structure of Hurwitz series rings*, Contrib. Algebra Geom. **47** (2007), no. 1, 251–256.

[6] G.F. Birkenmeier, J.Y. Kim, and J.K. Park, *On polynomial extensions of principally quasi-Baer rings*, Kyungpook Math. J. **40** (2000), 247–253.

[7] G.F. Birkenmeier, J.Y. Kim, and J.K. Park, *Quasi-Baer ring extensions and biregular rings*, Bull. Aust. Math. Soc. **61** (2000), 39–52.

[8] G.F. Birkenmeier, J.Y. Kim, and J.K. Park, *Principally quasi-Baer rings*, Comm. Algebra **29** (2001), no. 2, 639–660.

[9] G. F. Birkenmeier, J.Y. Kim, J.K. Park, *Polynomial extensions of Baer and quasi-Baer rings*, J. Pure Appl. Algebra **159** (2001), 24–42.

[10] W.E. Clark, *Twisted matrix units semigroup algebra*, Duke Math. J. **34** (1967), 417–424.

[11] C. Faith, *Rings with zero intersection property on annihilators Zip rings*, Publ. Mat. **33** (1989), 329–332.

[12] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761.

[13] I. Kaplansky, *Rings of Operators*, New York: Benjamin, 1965.

[14] W.F. Keigher, *Adjunctions and comonads in differential algebra*, Pacific J. Math **56** (1975), 99–112.

[15] W.F. Keigher, *On the ring of Hurwitz series*, Comm. Algebra **25** (1997), no. 6, 1845–1859.

[16] W.F. Keigher and F.L. Pritchard, *Hurwitz series as formal functions*, J. Pure Appl. Algebra **146** (2000), 291–304.

[17] L. Liang, L, Wang, and Z. Ziu, *On a generalization of semi commutative rings*, Taiwanese J. Math. **11** (2007), no. 5, 1359–1368.

[18] Z. Liu and R. Zhao, *On weak Armendariz rings*, Comm. Algebra **34** (2006), 2607–2616.

[19] P. Pollingher and A. Zaks, *On Baer and quasi-Baer rings*, Duke Math. J. **37** (1970), 127–138.

[20] M.B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci, **73** (1997), 14–17.

[21] A. Smoktunowicz, *Polynomial rings over nil rings need not be nil*, J. Algebra **233** (2000), 427–436.

[22] J.M. Zelmanowitz, *The finite intersection property on annihilator right ideals*, Proc. Amer. Math. Soc. **57** (1976), 213–216.