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Existence and uniqueness of solutions for differential equations with causal operators

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Abstract

In this paper, we consider the existence and uniqueness of a solution for interval-valued differential functions with a causal operator.

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1 Introduction

In recent years, interval algorithms have played an important role in the study of functional differential equations and their applications, for example, in biology, physics, engineering problems, and computer-aided design. The uncertain parameters are considered as intervals, where the upper and lower bounds of the parameters are estimated from the historical data. The interval valued functions are a particular case of set valued functions and are the functions involved in the interval parameters. Many scientists in many areas have studied the interval valued analysis and interval valued differential equations [2, 5, 6] and [9]. This paper is focused on the study of differential equations involving a causal operator and proving the existence and uniqueness of a solution for the problem (3.2) with respect to initial values. In [1], using the Hukuhara derivative, the authors studied the existence, uniqueness and continuity of solutions of the following problem:

$$D_H f(t) = (Qf)(t), \quad f(t_0) = f_0 \in K_C(\mathbb{R}^n), t_0 \geq 0. \tag{1.1}$$

In Section 2, we introduce some definitions and some preliminary results about interval values that will be used later. In Section 3, we prove the existence and uniqueness of a solution for the problem (3.2).

2 Preliminaries

Let K_C be the family of all non-empty compact convex subsets of $\mathbb{R}$, that is, $K_C = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \leq a^+\}$. If $A = [a^-, a^+]$, $B = [b^-, b^+]$ are in K_C, then the usual interval operations, i.e. Minkowski addition and scalar

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multiplication, are defined by

$$A + B = \left[a^-, a^+\right] + \left[b^-, b^+\right] := \left[a^- + b^-, a^+ + b^+\right] \tag{2.1}$$

and

$$\lambda A := \begin{cases} [\lambda a^-, \lambda a^+] & \lambda > 0, \\ [0,0], & \lambda = 0, \\ [\lambda a^+, \lambda a^-] & \lambda < 0. \end{cases} \tag{2.2}$$

So in special case if $\lambda = -1$, scalar multiplication gives the opposite $-A := (-1)A = (-1)\left[a^-, a^+\right] = [-a^+, -a^-]$. In general, $A + (-A) \neq \{0\}$; that is, the opposite of A is not the inverse of A with respect to the Minkowski addition (unless $A = \{a\}$ is a singleton). Minkowski difference is

$$A - B = A + (-1)B = \left[a^- - b^+, a^+ - b^-\right]. \tag{2.3}$$

The generalized Hukuhara difference (or $gH-$difference) of two intervals $[a^-, a^+]$, $[b^-, b^+]$ in K_C is defined as follows:

$$\left[a^-, a^+\right] \ominus_{gH} \left[b^-, b^+\right] = \left[\min\left\{a^- - b^-, a^+ - b^+\right\}, \max\left\{a^- - b^-, a^+ - b^+\right\}\right].$$

The width of interval A is defined and denote by $W_A = a^+ - a^-$ and $r_A = \dfrac{1}{2}(a^+ - a^-)$ which is the radius of interval of A. We denote the midpoint of A by $A_C = \dfrac{1}{2}(a^+ + a^-)$. Then interval can be represented by $A = [a^-, a^+] = [A_C - r_A, A_C + r_A]$, which simply we denote by $(A_C; r_A)$. For $A = [a^-, a^+]$ and $B = [b^-, b^+]$, we have

$$A \ominus_{gH} B = \begin{cases} [a^- - b^-, a^+ - b^+], & if\ W_A \geq W_B, \\ [a^+ - b^+, a^- - b^-], & if\ W_A < W_B. \end{cases}$$

If $A, B, C \in K_C$ then it is easy to see that

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} A = B + C, & if\ W_A \geq W_B, \\ B = A + (-C), & if\ W_A < W_B. \end{cases}$$

Proposition 2.1. [8] Let $A = [a^-, a^+] = (A_C, r_A)$ and $B = [b^-, b^+] = (B_C, r_B)$. The following hold:

(1) $A + B = [\min\{a^- + B_C, b^- + A_C\}, \max\{a^+ + B_C, b^+ + A_C\}]$.

(2) $A.B = [\min\{a^-.B_C, b^-.A_C\}, \max\{a^+.B_C, b^+.A_C\}]$.

Definition 2.2. [2] Let $A = [a^-, a^+]$ be any arbitrary element ofK_C .Then, the norm of the set A is denoted by $\|A\|$ and is defined by

$$\|A\| := \max\{|a^-|, |a^+|\}.$$

The metric structure is given usually by the Hausdorff-Pompeiu distance by $D : K_C \times K_C \to [0, \infty)$ which is defined by $D(A, B) := \max\{|a^- - b^-|, |a^+ - b^+\}$, where $A = \left[a^-, a^+\right]$ and $B = \left[b^-, b^+\right]$. Obviously, the metric D induces a norm $\|.\|$ by $\|A\| = D(A, \{0\})$ and it is direct to see that $D(A, B) = \|A \ominus_{gH} B\|$.

Proposition 2.3. [2] Let $A, B, C, E \in K_C$. The Hausdorff-Pompeiu distance has the following properties

(1) $D(A + C, B + C) = D(A, B)$.

(2) $D(A + B, C + E) \leq D(A, C) + D(B, E)$.

(3) $D(\alpha A, \alpha B) = |\alpha| D(A, B);\ \alpha \in \mathbb{R}$.

It is well known that (K_C, D) is a complete metric space.

Definition 2.4. [5] $f : [a, b] \to K_C$. The function f is said to be continuous at $x_0 \in [a, b]$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$ for $x \in [a, b]$ with $|x - x_0| < \delta$. Moreover, f is said to be continuous on $[a, b]$ if f is continuous at each point in $[a, b]$.

Proposition 2.5. [8] Let $f : [a, b] \to K_C$ be such that $f(x) = [f^-(x), f^+(x)]$ and let $x_0 \in (a, b)$, then

$$\lim_{x\to x_0} f(x) = \left[\lim_{x\to x_0} f^-(x), \lim_{x\to x_0} f^+(x)\right]$$

and

$$\lim_{x\to x_0} f(x) = f(x_0) \Leftrightarrow \lim_{x\to x_0} (f(x) \ominus_{gH} f(x_0)) = \{0\},$$

where the limits are in the metric D for intervals.

Definition 2.6. [7] Let $x_0 \in (a, b)$. The $gH-$derivative of a function $f : [a, b] \to K_C$ at x_0 is defined as

$$D_H f(x_0)) = \lim_{h\to 0^+} \frac{1}{h} \left[f(x_0 + h) \ominus_{gH} f(x_0)\right],$$

or

$$D_H f(x_0)) = \lim_{h\to 0^+} \frac{1}{h} \left[f(x_0) \ominus_{gH} f(x_0 - h)\right].$$

Lemma 2.7. [7] Let $f : [a, b] \to K_C$ be an interval-valued function such that $f(x) = [f^-(x), f^+(x)]$. If f is $gH-$differentiable at $x_0 \in (a, b)$, then f is continuous at x_0 and f^- and f^+ are differentiable at x_0 and

$$D_H f(x_0)) = \left[\min\left\{D_H f^-(x_0), D_H f^+(x_0)\right\}, \max\left\{D_H f^-(x_0), D_H f^+(x_0)\right\}\right].$$

Definition 2.8. [6] Given $f : \mathbb{R}^n \to K_C$ as an interval-valued function defined by$f(x) = [f^-(x), f^+(x)]$, $\forall\ x \in \mathbb{R}^n$, it is said to be an interval-valued linear function if it satisfied the following properties:

(1) $f(x + y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}^n$.

(2) $f(\alpha x) = \alpha.f(x)$, for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Proposition 2.9. [7] Under the assumptions of Lemma 2.7, the $gH-$derivative is a homogeneous and sub-additive operator, i.e., for $gH-$differentiable functions $f, g : [a, b] \to K_C$ with differentiable f^-, g^-, f^+ and g^+

(1) $D_H(f + g) \subseteq D_H f + D_H g$.

(2) $D_H(\alpha f) = \alpha D_H f$, for $\alpha \in \mathbb{R}$.

Definition 2.10. [2] The integral of $f : [a, b] \to K_C$, where $f(x) = [f^-(x), f^+(x)]$, is defined by

$$\int_a^b f(t)dt := \left[\int_a^b f^-(t)dt, \int_a^b f^+(t)dt\right].$$

Proposition 2.11. [1] If $f, g : [a_0, b] \to K_C(\mathbb{R}^n)$, $a_0 \le a_1 \le a_2 \le b$, are integrable, then we have:

(1) $\int_{a_0}^{a_2} f(t)dt := \int_{a_0}^{a_1} f(t)dt + \int_{a_1}^{a_2} f(t)dt$.

(2) $\int_{a_0}^{b} \lambda f(t)dt := \lambda \int_{a_0}^{b} f(t)dt\ \lambda \in \mathbb{R}_+$.

(3) $D\left[f(.), g(.)\right] : [a_0, b] \to \mathbb{R}$ is integrable and

$$D\left[\int_{a_0}^t f(s)ds, \int_{a_0}^t g(s)ds\right] \le \int_{a_0}^t D\left[f(s), g(s)\right] ds. \tag{2.4}$$

Lemma 2.12. Let $f, g : [a,b] \to K_C$ be interval-valued functions, where $f(x) = [f^-(x), f^+(x)]$ and $g(x) = [g^-(x), g^+(x)]$. Then

$$\int_a^b (f(t)+g(t))dt = \int_a^b f(t)dt + \int_a^b g(t)dt.$$

Proof . By Definition 2.9 and (2.1) it is direct, infact

$$f(x)+g(x) = \left[f^-(x)+g^-(x), f^+(x)+g^+(x)\right].$$

So

$$\begin{aligned}\int_a^b (f(x)+g(x))dt &= \left[\int_a^b (f^-(x)+g^-(x))dt, \int_a^b (f^+(x)+g^+(x))dt\right]\\ &= \left[\int_a^b f^-(x)dt, \int_a^b f^+(x)dt\right] + \left[\int_a^b g^-(x)dt, \int_a^b g^+(x)dt\right]\\ &= \int_a^b f(x)dt + \int_a^b g(x)dt.\end{aligned}$$

□

3 Main results

Definition 3.1. [1] Suppose that $Q \in C[E,E]$, then Q is said to be a causal map or a nonanticipative map if $f(s) = g(s)$, $t_0 \le s \le t \le T$, where $U, W \in E$, then $(Qf)(s) = (Qg)(s)$, $t_0 \le s \le t$.

We study the following problem:

$$D_H f(t) = l(t)(Qf)(t) + h(t), \quad f(t_0) = f_0 \in K_C(\mathbb{R}^n), t_0 \ge 0. \tag{3.1}$$

where $h \in C[\mathbb{R}_+, K_C(\mathbb{R}^n)]$ and there exists $\alpha \in \mathbb{R}$ such that $0 < l(t) < \alpha$. The mapping $f(t) \in C^1[J, K_C(\mathbb{R}^n)]$, where $J = [t_0, t_0+a]$ is called a solution for (3.1) on J if it satisfies in (3.1) on J.

Corollary 3.2. [1, 5] The interval valued differential equation (3.1) is equivalent to the following integral equations:

$$f(t) = f_0 + \int_{t_0}^t D_H f(s)ds, \quad t \in J. \tag{3.2}$$

So by Lemma 2.11 and (3.1),

$$f(t) = f_0 + \int_{t_0}^t l(s)(Qf)(s)ds + \int_{t_0}^t h(s)ds, \quad t \in J. \tag{3.3}$$

Let $E = C\left[[t_0, T], K_C(\mathbb{R}^n)\right]$ with norm

$$D_0[f,\theta] = \sup_{t_0 \le t \le T} D[f(t), \theta]. \tag{3.4}$$

Theorem 3.3. [1] Assume that $m \in C[J, \mathbb{R}_+]$, $F \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$ and for $t \in J = [t_0, T]$,

$$D_- m(t) \le F(t, |m|_0(t)), \tag{3.5}$$

where $|m|_0(t) = \sup_{t_0 \le s \le t} |m(s)|$. Suppose that $r(t) = r(t, t_0, w_0)$ is the maximal solution of the scalar differential equation

$$w' = F(t,w), \quad w(t_0) = w_0 \ge 0, \tag{3.6}$$

existing on J. Then $m(t_0) \le w_0$ implies $m(t) \le r(t), t \in J$.

Theorem 3.4. [1] Let $Q \in C[E, E]$ be a causal map such that for $t \in J$,

$$D[(Qf)(t), (Qg)(t)] \leq F(t, D_0[f, g](t)), \tag{3.7}$$

where $F \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$. Suppose further that the maximal solution $r(t, t_0, w_0)$ of the differential equation (3.6) exists on J. Then if $f(t), g(t)$ are any two solutions of (3.6) through $f(t_0) = f_0$, $g(t_0) = g_0$, $f_0, g_0 \in K_C(\mathbb{R}^n)$ on J, respectively, then

$$D[(f)(t), (g)(t)] \leq r(t, t_0, w_0), \quad t \in J, \tag{3.8}$$

Provided that $D[f_0, g_0] \leq w_o$.

Theorem 3.5. Assume that

(1) $Q \in C[B, E]$ is a causal map, where $B = B(f_0, b) = \{f \in E : \ D_0[f, f_0] \leq b\}$ and $D_0[(Qf), \theta](t) \leq M_1$, on B,

(2) $F \in C[J \times [0, 2b], \mathbb{R}_+]$, $F(t, w) \leq M_2$ on $J \times [0, 2b]$, $F(t, 0) \equiv 0$, $F(t, w)$ is nondecreasing in w for each $t \in J$ and $w(t) = 0$ is the only solution of

$$w' = F(t, w), \quad w(t_0) = 0 \ \ on\ J, \tag{3.9}$$

(3) $D[(Qf)(t), (Qg)(t)] \leq F(t, D_0[f, g](t))$ on B,

(4) $D_0[h, \theta](t) \leq \mu;\ \mu + M_1 < b$.

Then, the successive approximations defined by

$$f_{n+1}(t) = f_0 + \int_{t_0}^{t} l(s)(Qf_n)(s)ds + \int_{t_0}^{t} h(s)ds, \quad n = 0, 1, 2, ..., \tag{3.10}$$

exist on $J_0 = [t_0, t_0 + \eta]$, where $\eta = \min\left[T - t_0, \dfrac{b}{M}\right]$, $M = \max(\alpha(\mu + M_1), M_2)$ such that $0 < l(t) < \alpha$, $\alpha < 1$ and converge uniformly to the unique solution $f(t)$ of (3.1).

Proof . For $t \in J_0$, by using Proposition 2.3 and (2.4),

$$\begin{aligned}
D[f_{n+1}(t), f_0] &= D\left[f_0 + \int_{t_0}^{t} l(s)(Qf_n)(s)ds + \int_{t_0}^{t} h(s)ds, f_0\right] \\
&\leq D\left[\int_{t_0}^{t} l(s)(Qf_n)(s)ds + \int_{t_0}^{t} h(s)ds, \theta\right] \\
&\leq \int_{t_0}^{t} D[l(s)(Qf_n)(s), \theta]\, ds + \int_{t_0}^{t} D[h(s), \theta]\, ds \\
&\leq \alpha \int_{t_0}^{t} D[(Qf_n)(s), \theta]\, ds + \int_{t_0}^{t} D[h(s), \theta]\, ds \\
&\leq \alpha \int_{t_0}^{t} D_0[(Qf_n), \theta](s)ds + \int_{t_0}^{t} D_0[h, \theta](s)ds \\
&\leq \alpha(M_1 + \mu)(t - t_0) \\
&\leq M(t - t_0) \\
&\leq b,
\end{aligned}$$

which shows the successive approximations are well defined on J_0. Next, we define successive approximations for the problem (3.9) as follows:

$$w_0(t) = M(t - t_0),$$

$$w_{n+1}(t) = \int_{t_0}^{t} F(s, w_n(s))ds, \quad t \in J_0, \quad n = 0, 1, 2, \ldots.$$

Then

$$w_1(t) = \int_{t_0}^{t} F(s, w_0(s))ds \leq M_2(t - t_0) \leq M(t - t_0) = w_0(t).$$

Assume, for some $k > 1$, $t \in J_0$, that

$$w_k(t) \leq w_{k-1}(t).$$

Then, using the monotonicity of F, we get

$$w_{k+1}(t) = \int_{t_0}^{t} F(s, w_k(s))ds \leq \int_{t_0}^{t} F(s, w_{k-1}(s))ds \leq w_k(t).$$

Hence, the sequence $\{w_k(t)\}$ is monotone decreasing.
Since $w_k'(t) = F(t, w_{k-1}(t)) \leq M_2$, $t \in J_0$, by Ascoli-Arzela theorem and the monotonicity of the sequence $\{w_k(t)\}$, we have

$$\lim_{n\to\infty} w_n(t) = w(t),$$

uniformly on J_0 for a suitable function $w(t)$. Since $w(t)$ satisfies (3.9), so from condition (b), $w(t) \equiv 0$ on J_0. Observing that for each $t \in J_0$, $J_0 \leq s \leq t$,

$$\begin{aligned}
D\left[f_1(s), f_0\right] &= D\left[f_0 + \int_{t_0}^{s} l(\varrho)(Qf_0)(\varrho)d\varrho + \int_{t_0}^{s} h(\varrho)d\varrho, f_0\right] \\
&= D\left[\int_{t_0}^{s} l(\varrho)(Qf_0)(\varrho)d\varrho + \int_{t_0}^{s} h(\varrho)d\varrho, \theta\right] \\
&\leq \int_{t_0}^{s} D\left[l(\varrho)(Qf_0)(\varrho), \theta\right] d\varrho + \int_{t_0}^{s} D\left[h(\varrho), \theta\right] d\varrho \\
&\leq \alpha \int_{t_0}^{s} D\left[(Qf_0)(\varrho), \theta\right] d\varrho + \int_{t_0}^{s} D\left[h(\varrho), \theta\right] d\varrho \\
&\leq \alpha \int_{t_0}^{s} D_0\left[(Qf_0), \theta\right](\varrho)d\varrho + \int_{t_0}^{s} D_0\left[h, \theta\right](\varrho)d\varrho \\
&\leq \alpha(M_1 + \mu)(s - t_0) \\
&\leq \alpha(M_1 + \mu)(t - t_0) \\
&\leq M(t - t_0) = w_0(t),
\end{aligned}$$

which implies that $D_0\left[f_1, f_0\right](t) \leq w_0(t)$. We assume, for some $k > 1$,

$$D_0\left[f_k, f_{k-1}\right](t) \leq w_{k-1}(t) \quad t \in J_0. \tag{3.11}$$

By condition (c) and (3.11), for any $t \in J_0$, $J_0 \leq s \leq t$,

$$\begin{aligned}
D\left[f_{k+1}(s), f_k(s)\right] &\leq \int_{t_0}^{s} D\left[l(\varrho)(Qf_k)(\varrho), l(\varrho)(Qf_{k-1})(\varrho)\right] d\varrho \\
&\leq \alpha \int_{t_0}^{s} D\left[(Qf_k)(\varrho), Qf_{k-1})(\varrho)\right] d\varrho \\
&\leq \int_{t_0}^{s} F(\varrho, D_0\left[f_k, f_{k-1}\right](\varrho))d\varrho \\
&\leq \int_{t_0}^{s} F(\varrho, w_{k-1}(\varrho))d\varrho \\
&\leq \int_{t_0}^{t} F(\varrho, w_{k-1}(\varrho))d\varrho \\
&= w_k(t),
\end{aligned}$$

which further gives

$$D_0\left[f_{k+1}, f_k\right](t) \leq w_k(t) \quad t \in J_0. \tag{3.12}$$

Thus, we have

$$D_0\left[f_{n+1}, f_n\right](t) \leq w_n(t), \tag{3.13}$$

for $t \in J_0$ and for all $n = 0, 1, 2, \ldots$. We claim that $\{f_n(t)\}$ is a Cauchy sequence. To show this, let $n \leq m$. Setting $u(t) = D\left[f_n(t), f_m(t)\right]$ and using (3.10), we have

$$\begin{aligned}
D^+u(t) \leq & D\left[D_H f_n(t), D_H f_m(t)\right](t) \\
= & D\left[l(t)(Qf_{n-1})(t), l(t)(Qf_{m-1})(t)\right] \\
\leq & D\left[l(t)(Qf_{n-1})(t), l(t)(Qf_n)(t)\right] + D\left[l(t)(Qf_n)(t), l(t)(Qf_m)(t)\right] + D\left[l(t)(Qf_m)(t), l(t)(Qf_{m-1})(t)\right] \\
\leq & \alpha F(t, D_0\left[f_{n-1}, f_n\right](t)) + \alpha F(t, D_0\left[f_n, f_m\right](t)) + \alpha F(t, D_0\left[f_{m-1}, f_m\right](t)) \\
\leq & F(t, D_0\left[f_{n-1}, f_n\right](t)) + F(t, D_0\left[f_n, f_m\right](t)) + F(t, D_0\left[f_{m-1}, f_m\right](t)) \\
\leq & F(t, w_n(t)) + F(t, |u|_0(t)) + F(t, w_n(t)) \\
= & F(t, |u|_0(t)) + 2F(t, w_n(t)).
\end{aligned}$$

These inequalities together with Theorem 3.3, imply the estimate

$$u(t) \leq r_n(t), \quad t \in J_0,$$

where $r_n(t)$ is the maximal solution of

$$r_n' = F(t, r_n) + 2F(t, w_{n-1}(t)), \quad r_n(t_0) = 0,$$

for each n. Since as $n \to \infty$, $2F(t, w_{n-1}(t)) \to 0$ uniformly on J_0. It follows by [8, Lemma 1.3.1] that $r_n(t) \to 0$, as $n \to \infty$ uniformly on J_0. Then from (3.13) that $f_n(t)$ converges uniformly to $f(t)$ on J_0 and clearly $f(t)$ is a solution of (3.1).

To prove uniqueness, let $g(t)$ be another solution of (3.1) on J_0. Set $m(t) = D\left[f(t), g(t)\right]$. Then, $m(t_0) = 0$ and

$$D^+m(t) \leq F(t, |m|_0(t)), \quad t \in J_0. \tag{3.14}$$

Since $m(t_0) = 0$, it follows from Theorem 3.3 that

$$m(t) \leq r(t, t_0, 0), \quad t \in J_0, \tag{3.15}$$

where $r(t, t_0, 0)$ is the maximal solution of (3.9). The assumption (b) now shows that $f(t) = g(t)$, $t \in J_0$, proving uniqueness. □

4 Conclusion

In this article, we proved the existence and uniqueness of solutions for interval valued differential problem (3.1), involving causal operators.

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