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Gradient estimate of equations with potential under the almost Ricci soliton condition

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Abstract

Using volume comparison theorems and the Sobolev inequality with almost Ricci solitons, we study an important version of the gradient estimate for the solutions of $\Delta u = f + Hu$, for some function f, H, and we obtain an upper bound for the gradient of u on almost Ricci solitons.

Keywords: Sobolev constant, Gradient estimate, Ricci soliton, Bakry-Émery Ricci curvature
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1 Introduction

Sobolev inequality enables us to obtain many applications in differential geometry. For example, it plays the main role in the maximal principle, gradient estimates and consequently upper and lower bounds of the heat kernel. The Cheeger-Colding-Naber theory has been extended to integral Ricci curvature bound in the noncollapsed case and has important results [14, 17]. In fact, Rose in [14] showed that under locally uniformly integral bounds of the negative part of Ricci curvature, the heat kernel admits a Gaussian upper bound for small times. After that, Dai et al. [5] extended many of the basic estimates for integral curvature to the collapsed case. Zhang and Zhu [19] followed their arguments to prove Sobolev inequality on manifolds with considering a lower bound of the Bakry-Émery Ricci curvature $\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g \geq -\lambda g$, where $\mathcal{L}_X$ is the Lie derivative along the vector field X, and Ric is the Ricci tensor, and λ is a positive constant. Afterwards, they used volume comparison theorem, and Sobolev inequalities for elliptic and parabolic gradient estimates (see also [9, 10] for more information). Actually, they considered the following equations

$$Ric + \frac{1}{2}\mathcal{L}_V g \geq -\lambda g, \ \ |V|(y) \leq \frac{K}{d(y,O)^{\alpha}}, \ \ \alpha \in [0,1);$$

and moreover volume noncollapsing condition $vol(B(a,1)) \geq \rho$, for some constant $\rho > 0$, when $\alpha \neq 0$. Here $d(y,O)$ represents the distance from O to y, and $K \geq 0$ is constant. By these assumptions Zhang and Zhu obtained following upper bound for the solution of $\Delta u = f$ in $B(x,r)$:

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2 \leq C(n,\lambda,K,\alpha,\rho)\left[r^{-2}(\|u\|^*_{2,B(x,r)})^2 + (\|f\|^*_{2q,B(x,r)})^2\right],$$

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for any $q > \dfrac{n}{2}$, and moreover

$$\sup_{B(x,\frac{1}{2}r)} u^2 \leq C(n,\lambda,K,\alpha,\rho)\left[(\|u\|^*_{2,B(x,r)})^2 + r^2(\|f\|^*_{2q,B(x,r)})^2\right],$$

where

$$\|f\|^*_{q,B(x,r)} = \left(\oint_{B(x,r)} |f|^q dg\right)^{\frac{1}{q}}.$$

As well, Wang and Wei in [16] studied on the Riemannian manifolds with integral Bakry-Émery Ricci cuavature, and extended the local isoperimetric constant estimate in [5] to integral Bakry-Émery Ricci curvature, and got some applications for a complete smooth metric measure space $M^n_f := (M^n, g, e^{-f}dvol)$, the Riemannian manifold (M^n, g) coupled with a weighted volume $e^{-f}dvol$ for some $f \in C^\infty(M)$, where $dvol$ is the usual Riemannian volume element on M respect to the metric g. As a prominent result, they obtained a gradient estimate for solutions of $\Delta_f u = h$, where u and h are smooth functions on M^n_f. Lately, Richard Bamler in [2] improved an important gradient bound based on Zhang and Cao-Hamilton's works at [4, 18].

1.1 Ricci almost soliton

Let (M^n, g), be a complete smooth Riemannian manifold equipped with a smooth vector field $Y \in \chi(M)$, and a smooth function $\lambda : M^n \to \mathbb{R}$. Let (M^n, g, Y, λ) satisfies in the following equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_Y g = \lambda g,$$

then it is called an almost Ricci soliton, and if the vector field $Y = \nabla h$, for a smooth function h, it is called a gradient almost Ricci soliton. Actually, almost all Ricci solitons are the generalized Ricci solitons, considering the soliton constant λ to be a smooth function introduced in [13]. An almost Ricci soliton (M, g, Y, λ) is trivial if it is a Ricci soliton, and a Ricci soliton is trivial if the soliton vector field Y is Killing. There are some articles about the sufficient condition for an almost Ricci soliton to be a Ricci soliton, see [7, 11, 15]. Lately, in [8], necessary and sufficient conditions for a compact almost Ricci soliton endowed with a geodesic soliton vector field were examined to be a trivial Ricci soliton. Also in [6], had been shown that under some certain conditions a compact gradient almost Ricci soliton could be isometric to the unit sphere S^n, and in [3, 11] obtained some result about the condition that an almost Ricci soliton could be an Einstein manifold.

In this paper, we consider a condition on the Ricci curvature involving vector fields, which is weaker than almost Ricci soliton, and hence can be applied to almost Ricci soliton. We stata ed new version of the gradient estimate by solving the following equation

$$L_1 u = f, \tag{1.1}$$

here, $L_1 u = \Delta u - Hu$ and $H : M \longrightarrow \mathbb{R}$ is a smooth function under. Proving this type of gradient estimate, first we had used Sobolev inequality on a manifold M^n that its Ricci cuavature tensor satisfies

$$Ric + \frac{1}{2}\mathcal{L}_V g \geq -\lambda g, \tag{1.2}$$

where Ric is the Ricci tensor, λ is a smooth function, and V is a smooth vector field which satisfies

$$|V|(y) \leq \frac{K}{d(y,O)^\alpha}, \tag{1.3}$$

for any $y \in M$. Here we denote the distance between two points $y, O \in M$ by $d(y, O)$, $K \geq 0$, and $0 \leq \alpha < 1$ are constants. Here is our main result:

Theorem 1.1. Suppose that on a Riemannian manifold M^n, (1.2), (1.3) hold. Mreover, let the volume non-collapsing condition holds

$$Vol(B(x,1)) \geq \rho.$$

For $q > \frac{n}{2}$, if u and f be smooth functions, and $|\lambda| \leq N$ for a constant N such that (1.1) holds with $|H| \leq l_1$, $0 \leq u \leq l_2$, and $|\nabla H| \leq l_3$ for constants l_1, l_2, l_3, then there exists a positive constant $r_0 = r_0(n, N, K, \alpha, \rho, l_1, l_2, l_3)$ such that for any $x \in M$ and $0 < r \leq r_0$, we have

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2 \leq C(n, N, K, \alpha, \rho, l_1, l_2, l_3)\big[(\|f\|^*_{2q,B(x,r)})^2 + r^{-2}(\|u\|^*_{2,B(x,r)})^2\big].$$

2 Main Results and Proofs

We may use following isoperimetric and sobolev inequality. The proof process is just like [5], we can prove the theorem for any $r \leq r_0 = r_0(n, K_1, K, \alpha, \rho)$.

Theorem 2.1 (Isoperimetric inequality). Let M be an almost Ricci soliton which next three conditions hold on it.

$$Ric + \frac{1}{2}\mathcal{L}_X g \geq -\lambda g, \qquad |V|(y) \leq \frac{K}{d(y,O)^\alpha}, \ \ Vol(B(x,1)) \geq \rho,$$

for all $x \in M$ and some constant $\rho > 0$ and $K \geq 0$. (we could just have the first two equations when $\alpha = 0$). In addition suppose that function λ is bounded from above by K_1, then there is a constant $r_0 = r_0(n, K_1, K, \alpha, \rho)$ such that for any $r \leq r_0$ and $f \in C_0^\infty(B(x,r))$, we have

$$\mathrm{ID}^*_n(B(x,r)) \leq C(n)r.$$

Here $\mathrm{ID}^*_n(B(x,r))$ is the isoperimetric constant defined by

$$\mathrm{ID}^*_n(B(x,r)) = Vol(B(x,r))^{\frac{1}{n}} . \sup_\Omega \left\{ \frac{Vol(\Omega)^{\frac{n-1}{n}}}{Vol(\partial\Omega)} \right\},$$

where the supremum is taken over all smooth domains $\Omega \subset B(x,r)$ with $\partial\Omega \cap \partial B(x,r) = \emptyset$.

Theorem 2.2 (Sobolev inequality). Under the same conditions as in the above theorem, we have the following Sobolev inequalities for any $f \in C_0^\infty(B(x,r))$, and $r \leq r_0$:

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}} dg\right)^{\frac{n-1}{n}} \leq C(n)r \oint_{B(x,r)} |\nabla f| dg, \tag{2.1}$$

and

$$\left(\oint_{B(x,r)} |f|^{\frac{2n}{n-2}} dg\right)^{\frac{n-2}{n}} \leq C(n)r^2 \oint_{B(x,r)} |\nabla f|^2 dg. \tag{2.2}$$

Moreover, for the case that $X = \nabla f$ for some smooth function f, we get

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}} dg\right)^{\frac{n-1}{n}} \leq C(n)r \oint_{B(x,r)} |\nabla f| dg. \tag{2.3}$$

In addition we need the volume comparison theorem which was state in [1] as follows:

Theorem 2.3. Assume that for an n-dimension almost Ricci soliton (1.2) and (1.3) hold. Moreover consider a positive constant K_1 as an upper bound for λ. Suppose in addition that the volume non-collapsing condition holds

$$Vol(B(x,1)) \geq \rho, \tag{2.4}$$

for positive constants $\rho > 0$, $K \geq 0$ and $0 \leq \alpha < 1$, then for any $0 < r_1 < r_2 \leq 1$, we have the volume ratio bound as follows

$$\frac{Vol(B(x,r_2))}{r_2^n} \leq e^{C(n,K_1,K,\alpha,\rho)[K_1(r_2^2 - r_1^2) + K(r_2 - r_1)^{1-\alpha}]} . \frac{Vol(B(x,r_1))}{r_1^n}. \tag{2.5}$$

In particular, this result are true by considering the gradient soliton vector field $V = \nabla f$.

Proof . Take $v=|\nabla u|^2+\|f^2\|^*_{q,B(x,r)}$. Due to the Bochner formula, we have

$$\frac{1}{2}\Delta|\nabla u|^2=|\nabla^2u|^2+<\nabla u,\Delta\nabla u>+\mathrm{Ric}(\nabla u,\nabla u), \tag{2.6}$$

then

$$\Delta v=2|\nabla^2u|^2+2<\nabla u,\nabla\Delta u>+2\mathrm{Ric}(\nabla u,\nabla u). \tag{2.7}$$

Since $\Delta u=f+Hu$, we infer

$$\Delta v\geq -2<\nabla(Hu),\nabla u>+2<\nabla u,\nabla f>+2Ric(\nabla u,\nabla u). \tag{2.8}$$

Applying conditions that stated in theorem, we obtain

$$\Delta v\geq 2u_if_i-2\lambda v-(\mathcal{L}_Vg)_{ij}u_iu_j-2l_1v-2uH_iu_i. \tag{2.9}$$

For any positive p, we get

$$\begin{aligned}\Delta v^p &= pv^{p-1}\Delta v+p(p-1)v^{p-2}|\nabla v|^2\\ &\geq 2pv^{p-1}u_if_i-2\lambda pv^p-pv^{p-1}(\mathcal{L}_Vg)_{ij}u_iu_j-2pv^{p-1}uH_iu_i-2l_1pv^p+\frac{p-1}{p}v^{-p}|\nabla v^p|^2.\end{aligned} \tag{2.10}$$

Let $B=B(x,r)$, then by (2.10) for any $\eta\in C_0^\infty(B_x(1))$, and $p\geq 1$, we have

$$\begin{aligned}\int_B|\nabla(\eta v^p)|^2 &=\int_B|\eta\nabla v^p+v^p\nabla\eta|^2\\ &=\int_B v^{2p}|\nabla\eta|^2-\eta^2v^p\Delta v^p\\ &\leq\int_B v^{2p}|\nabla\eta|^2-2p\eta^2v^{2p-1}u_if_i+2\lambda p\eta^2v^{2p}+p\eta^2v^{2p-1}(\mathcal{L}_Vg)_{ij}u_iu_j+2p\eta^2v^{2p-1}uH_iu_i+2l_1p\eta^2v^{2p}.\end{aligned} \tag{2.11}$$

Since $(\mathcal{L}_Vg)_{ij}=\nabla_iV_j+\nabla_jV_i$, we get

$$\frac{1}{2}\int_B\eta^2v^{2p-1}(\mathcal{L}_Vg)_{ij}u_iu_j=-\int_B 2\eta v^{2p-1}\eta_jV_iu_iu_j+(2p-1)\eta^2v^{2p-2}v_jV_iu_iu_j+\eta^2v^{2p-1}V_iu_{ij}u_j+\eta^2v^{2p-1}V_iu_iu_{jj}. \tag{2.12}$$

As we know $v_j=2u_{jj}u_j$, so (2.12) becomes

$$\begin{aligned}\frac{1}{2}\int_B\eta^2v^{2p-1}(\mathcal{L}_Vg)_{ij}u_iu_j \leq & \int_B v^{2p}|\nabla\eta|^2+\eta^2v^{2p-2}|V|^2|\nabla u|^4-\frac{2p-1}{p}\eta v^{p-1}V_iu_iu_j[(\eta v^p)_j-v^p\eta_j]\\ &-\frac{1}{2}\eta^2v^{2p-1}V_iv_i+\frac{1}{2}\eta^2v^{2p-2}f^2|\nabla u|^2+\frac{1}{2}\eta^2v^{2p}|V|^2-\eta^2v^{2p-1}V_iu_iHu.\end{aligned} \tag{2.13}$$

By the definition of v, we know that $|\nabla u|^4\leq v^2$, so

$$\begin{aligned}\frac{1}{2}\int_B\eta^2v^{2p-1}(\mathcal{L}_Vg)_{ij}u_iu_j \leq & \int_B v^{2p}|\nabla\eta|^2+\frac{3}{2}\eta^2v^{2p}|V|^2-\frac{2p-1}{p}\eta v^{p-1}V_iu_iu_j[(\eta v^p)_j-v^p\eta_j]\\ &-\frac{1}{2p}\eta v^pV_i[(\eta v^p)_i-v^p\eta_i]+\frac{1}{2}\eta^2v^{2p-2}f^2|\nabla u|^2-\eta^2v^{2p-1}V_iu_iHu.\end{aligned} \tag{2.14}$$

Since $|\lambda|\leq N$, $|H|\leq l_1$, and $|u|\leq l_2$, (2.14) changes as follows

$$\begin{aligned}\int_B\eta^2v^{2p-1}(\mathcal{L}_Vg)_{ij}u_iu_j &\leq \int_B\frac{8p-1}{4p}v^{2p}|\nabla\eta|^2+\frac{2(2p-1)^2+5p}{2p}\eta^2v^{2p}|V|^2+\frac{1}{2p}|\nabla(\eta v^p)|^2\\ &\quad+\frac{1}{2}\eta^2v^{2p-1}f^2+\frac{1}{2}l_1l_2\eta^2v^{2p}|V|^2+\frac{1}{2}l_1l_2\eta^2v^{2p-2}|\nabla u|^2\\ &\leq \int_B\frac{8p-1}{4p}v^{2p}|\nabla\eta|^2+\frac{2(2p-1)^2+5p+pl_1l_2}{2p}\eta^2v^{2p}|V|^2\\ &\quad+\frac{1}{2p}|\nabla(\eta v^p)|^2+\frac{1}{2}\eta^2v^{2p-1}f^2+\frac{1}{2}l_1l_2\eta^2v^{2p-1}.\end{aligned} \tag{2.15}$$

By the same argument of [19], we have

$$-\int_B \eta^2 v^{2p-1} u_i f_i \le \int_B \frac{4(2p-1)^2+1}{2p}\eta^2 v^{2p-1} f^2 + \frac{1}{2p} v^{2p}|\nabla\eta|^2 + \frac{1}{8p}|\nabla(\eta v^p)|^2. \tag{2.16}$$

Substituting (2.15), and (2.16) in (2.11), it follows that

$$\begin{aligned}
2\int_B |\nabla(\eta v^p)|^2 \le & \int_B 2pv^{2p}|\nabla\eta|^2 + (8(2p-1)^2+2)\eta^2 v^{2p-1} f^2 + 2v^{2p}|\nabla\eta|^2 + \frac{1}{2}|\nabla(\eta v^p)|^2 \\
& + \frac{8p-1}{2} v^{2p}|\nabla\eta|^2 + (2(2p-1)^2 + 5p + pl_1 l_2)\eta^2 v^{2p}|V|^2 + |\nabla(\eta v^p)|^2 \\
& + p\eta^2 v^{2p-1} f^2 + pl_1 l_2 \eta^2 v^{2p-1} + 4Np\eta^2 v^{2p} + 4p\eta^2 v^{2p-1} uH_i u_i + 4pl_1\eta^2 v^{2p},
\end{aligned}$$

so

$$\begin{aligned}
\int_B |\nabla(\eta v^p)|^2 \le & \int_B 16pv^{2p}|\nabla\eta|^2 + 70p^2\eta^2 v^{2p-1} f^2 + 30p^2\eta^2 v^{2p}|V|^2 \\
& + 8p(N+l_1)\eta^2 v^{2p} + 2pl_1 l_2\eta^2 v^{2p}|V|^2 + 2pl_1 l_2 \eta^2 v^{2p-1} + 8p\eta^2 v^{2p-1} uH_i u_i.
\end{aligned} \tag{2.17}$$

Now by the fact that $|H_i| \le l_3$, we get

$$\begin{aligned}
\int_B 8p\eta^2 v^{2p-1} uH_i u_i \le & \ 8p\int_B \eta^2 v^{2p-1} l_2 l_3 |u_i| \\
\le & \ 4p\int_B \eta^2 v^{2p}(l_2 l_3)^2 + 4p\int_B \eta^2 v^{2p-2}|u_i|^2 \\
\le & \ 4p(l_2 l_3)^2 \int_B \eta^2 v^{2p} + 4p\int_B \eta^2 v^{2p-1}.
\end{aligned} \tag{2.18}$$

Hence, we can rewrite (2.17) as

$$\begin{aligned}
\int_B |\nabla(\eta v^p)|^2 \le & \int_B 16pv^{2p}|\nabla\eta|^2 + 70p^2\eta^2 v^{2p-1} f^2 + 30p^2\eta^2 v^{2p}|V|^2 \\
& + 8p(N+l_1)\eta^2 v^{2p} + 2pl_1 l_2\eta^2 v^{2p}|V|^2 + 2pl_1 l_2\eta^2 v^{2p-1} + 4p(l_2 l_3)^2\eta^2 v^{2p} + 4p\eta^2 v^{2p-1}.
\end{aligned} \tag{2.19}$$

Constructing a cut-off function $\psi_i(s)$ such that for $r_i = (\frac{1}{2}, \frac{1}{2^{i+2}})$, $i = 0, 1, 2, ...$, $\psi_i(t) \equiv 1$ for $t \in [0, r_{i+1}]$, $supp\psi_i \subseteq [0, r_i]$, and $-\frac{52^i}{r} \le \psi_i^{'} \le 0$. Then define $\eta_i(y) = \psi_i(s)$. Thus, (2.19) becomes

$$\begin{aligned}
\int_{B(x,r_i)} |\nabla(\eta_i v^p)|^2 \le & \int_{B(x,r_i)} 16pv^{2p}|\nabla\eta_i|^2 + 70p^2\eta_i^2 v^{2p-1} f^2 + 30p^2\eta_i^2 v^{2p}|V|^2 + 8p(N+l_1)\eta_i^2 v^{2p} \\
& + 2pl_1 l_2\eta_i^2 v^{2p}|V|^2 + 2pl_1 l_2\eta_i^2 v^{2p-1} + 4p(l_2 l_3)^2\eta_i^2 v^{2p} + 4p\eta_i^2 v^{2p-1}.
\end{aligned} \tag{2.20}$$

Using volume comparison Theorem 2.3 for $\frac{r}{2} \le r_i \le \frac{3r}{4}$, we can conclude next inequalities by the use of Young's inequality.

$$\begin{aligned}
70p^2 \oint_{B(x,r_i)} \eta_i^2 v^{2p-1} f^2 \le & \frac{70p^2}{\|f^2\|^*_{q,B(x,r)}} \oint_{B(x,r_i)} \eta_i^2 v^{2p} f^2 \\
\le & C(n,N,K,\alpha,\rho)p^2 \Big(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2q}{q-1}} \Big)^{\frac{q-1}{q}}
\end{aligned}$$

$$\leq C(n,N,K,\alpha,\rho)p^2\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{a.\frac{2q}{q-1}.b}\Big)^{\frac{q-1}{q^b}}\times\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{(1-a).\frac{2q}{q-1}.\frac{b}{b-1}}\Big)^{\frac{(q-1)(b-1)}{q^b}}$$

$$\leq\epsilon\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{a.\frac{2q}{q-1}.b}\Big)^{\frac{q-1}{q^{ba}}}+\epsilon^{-\frac{a}{1-a}}C^{\frac{1}{1-a}}p^{\frac{2}{1-a}}\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{(1-a).\frac{2q}{q-1}.\frac{b}{b-1}}\Big)^{\frac{(q-1)(b-1)}{q^{b(1-a)}}}. \tag{2.21}$$

By the same argument for $q\in(\frac{n}{2},\frac{n}{2\alpha})$, we conclude that

$$\begin{aligned}30p^2\oint_{B(x,r_i)}\eta_i^2v^{2p}|V|^2 &\leq 30p^2\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{\frac{2q}{q-1}}\Big)^{\frac{q-1}{q}}.\Big(\oint_{B(x,r_i)}|V|^{2q}\Big)^{\frac{1}{q}}\\ &\leq p^2C(n,N,k,\alpha,\rho)r_i^{-2\alpha}\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{\frac{2q}{q-1}}\Big)^{\frac{q-1}{q}}\\ &\leq \epsilon r_i^{-2\alpha}\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+\epsilon^{-\frac{a}{1-a}}p^{\frac{2}{1-a}}C^{\frac{1}{1-a}}r_i^{-2\alpha}\oint_{B(x,r_i)}\eta_i^2v^{2p}. \end{aligned}\tag{2.22}$$

Therefore,

$$2pl_1l_2\oint_{B(x,r_i)}\eta_i^2v^{2p}|V|^2 \leq \Big[\epsilon r_i^{-2\alpha}\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+\epsilon^{-\frac{a}{1-a}}P^{\frac{1}{1-a}}C^{\frac{1}{1-a}}r_i^{-2\alpha}\oint_{B(x,r_i)}\eta_i^2v^{2p}\Big]. \tag{2.23}$$

So substituting (2.21), (2.22) and (2.23) in (2.20), we have

$$\begin{aligned}oint_{B(x,r_i)}|\nabla(\eta_iv^p)|^2\leq&\oint_{B(x,r_i)}8p(N+l_1)\eta_i^2v^{2p}+16pv^{2p}|\nabla\eta_i|^2+2pl_1l_2\eta_i^2v^{2p-1}+4p(l_2l_3)^2\eta_i^2v^{2p}+4p\eta_i^2v^{2p-1}\\ &+\epsilon\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{a.\frac{2q}{q-1}.b}\Big)^{\frac{q-1}{q^{ba}}}\\ &+\epsilon^{-\frac{a}{1-a}}C^{\frac{1}{1-a}}p^{\frac{2}{1-a}}\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{(1-a).\frac{2q}{q-1}.\frac{b}{b-1}}\Big)^{\frac{(q-1)(b-1)}{q^{b(1-a)}}}\\ &+2\epsilon r_i^{-2\alpha}\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+(p^{\frac{1}{1-a}}+p^{\frac{2}{1-a}})\epsilon^{-\frac{a}{1-a}}C^{\frac{1}{1-a}}r_i^{-2\alpha}\oint_{B(x,r_i)}\eta_i^2v^{2p}.\end{aligned}\tag{2.24}$$

Now using Sobolev inequality (2.1) and (2.24), if we take $a=\frac{n}{2q}$ and $b=\frac{2q-2}{n-2}$, we get

$$\begin{aligned}\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\leq&C(n)r_i^2\oint_{B(x,r_i)}|\nabla(\eta_iv^p)|^2\\ \leq&C(n)r_i^2\Big[\oint_{B(x,r_i)}8p(N+l_1)\eta_i^2v^{2p}+16pv^{2p}|\nabla\eta_i|^2+2pl_1l_2\eta_i^2v^{2p-1}+4p(l_2l_3)^2\eta_i^2v^{2p}+4p\eta_i^2v^{2p-1}\Big]\\ &+C(n)\epsilon r_i^{2-2\alpha}\Big(\oint_B(\eta v^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+C(n)\epsilon^{-\frac{a}{1-a}}C^{\frac{2q}{2q-n}}p^{\frac{4q}{2q-n}}r_i^{2-2\alpha}\oint_{B(x,r_i)}\eta_i^2v^{2p}\\ &+C(n)r_i^2\epsilon\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}+C(n)r_i^2\epsilon^{-\frac{a}{1-a}}(p^{\frac{4q}{2q-n}}+p^{\frac{2q}{2q-n}})\oint_{B(x,r_i)}\eta_i^2v^{2p}.\end{aligned}$$

We choose ϵ small so that due to $r_i\leq r\leq 1$, and $\alpha<1$, we obtain

$$\Big(\oint_{B(x,r_i)}(\eta_iv^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \leq C(n,N,K,\alpha,\rho,l_1,l_2,l_3)r_i^2\oint_{B(x,r_i)}pv^{2p}|\nabla\eta_i|^2+p\eta_i^2v^{2p}+p\eta_i^2v^{2p-1}. \tag{2.25}$$

Using volume comparison theorem for $r_2 = r_{i+1}$ and $r_1 = r_i$, we infer

$$\begin{aligned}\Big(\oint_{B(x,r_{i+1})} (v^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} &\leq C(n,N,K,\alpha,\rho)\Big(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\leq C(n,N,K,\alpha,\rho,l_1,l_2,l_3)\oint_{B(x,r_i)} 2^{2i}pv^{2p} + 2pv^{2p}.\end{aligned}$$

Then

$$\begin{aligned}\Big(\oint_{B(x,r_{i+1})} v^{\mu^{i+1}}\Big)^{\frac{n-2}{n}} &= \Big(\oint_{B(x,r_{i+1})} (v^p)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\leq C(n,N,K,\alpha,\rho,l_1,l_2,l_3)(2^{2i-1}\mu^i + \mu^i)\oint_{B(x,r_i)} v^{\mu^i}\\ &\leq C(n,N,K,\alpha,\rho,l_1,l_2,l_3)(2^{2i-1}+1)2^{2i}\oint_{B(x,r_i)} v^{\mu^i},\end{aligned}$$

as $\mu = \dfrac{n}{n-2}$ and $p = \dfrac{\mu^i}{2}$ for $i = 0,1,2,\ldots$, this means that

$$\|V\|^*_{\mu^{i+1},B(x,r_{i+1})} \leq C^{\mu^{-i}}(2^{4i-1}+2^{2i})^{\mu^{-i}}\|v\|^*_{\mu^i,B(x,r_i)}. \tag{2.26}$$

So,

$$\sup_{B(x,\frac{1}{2}r)} v \leq C^{\Sigma\mu^{-i}}(2^{4i-1}+2^{2i})^{\Sigma\mu^{-i}}\|v\|^*_{1,B(x,\frac{3}{4}r)} \leq C(n,N,K,\alpha,\rho,l_1,l_2,l_3)\|v\|^*_{1,B(x,\frac{3}{4}r)}. \tag{2.27}$$

On the other hand, we have

$$\begin{aligned}\int_{B(x,r)} \eta^2|\nabla u|^2 &= \int_{B(x,r)} -\eta^2 u(f-Hu) - 2\eta u\nabla_i u\nabla_i\eta\\ &\leq \int_{B(x,r)} \frac{1}{2}u^2\eta^2 + \frac{1}{2}f^2\eta^2 + \eta^2 l_1 l_2 + \frac{1}{2}\eta^2|\nabla u|^2 + 2u^2|\nabla\eta|^2.\end{aligned}$$

Due to the definition of η, we have

$$\begin{aligned}\oint_{B(x,r)} \eta^2|\nabla u|^2 &\leq 4\oint_{B(x,r)} u^2\eta^2 + f^2\eta^2 + \eta^2 l_1 l_2 + u^2|\nabla\eta|^2\\ &\leq 100r^{-2}(\|u|^*_{2,B(x,r)})^2 + 4\|f^2\|^*_{q,B(x,r)} + l_1 l_2.\end{aligned}$$

Accordingly, we achieve

$$\begin{aligned}\|v\|^*_{1,B(x,\frac{3}{4}r)} &\leq \frac{Vol(B(x,r))}{Vol(B(x,\frac{3}{4}r))}\oint_{B(x,r)} \eta^2(|\nabla u|^2 + \|f^2\|^*_{q,B(x,r)})\\ &\leq C(n,N,K,\alpha,\rho,l_1,l_2)\big[r^{-2}(\|u\|^*_{2,B(x,r)})^2 + (\|f\|^*_{2q,B(x,r)})^2\big].\end{aligned}$$

Thus,

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2 \leq \|v\|_{\infty,B(x,\frac{1}{2}r)} \leq C(n,N,K,\alpha,\rho,l_1,l_2,l_3)\big[r^{-2}(\|u\|^*_{2,B(x,r)})^2 + (\|f\|^*_{2q,B(x,r)})^2\big].$$

This completes the proof. □

Remark 2.4. Note that the same results hold without noncollapsing condition when $\alpha = 0$.

According to the previous theorem, it could be easy to conclude:

Corollary 2.5. Suppose that the following condition holds for a gradient Ricci almost soliton

$$\mathrm{Ric} + Hessh \geq -\lambda g,$$

and more over we had two condition for potential function h as follows

$$|h(y) - h(z)| \leq K_1 d(y,z)^{\alpha}, \text{ and } \sup_{x\in M, 0\leq r\leq 1} \left(r^{\beta}\|\nabla h\|^{*}_{q,B(x,r)}\right) \leq K_2.$$

Then there is a constant $r_0 = r_0(n, N, K_1, K_2, \alpha, \beta, l_1, l_2, l_3)$, such that by the same conditions as last theorem, the solution of (1.1) satisfies

$$\sup_{B(x,\frac{r}{2})} |\nabla u|^2 \leq C(n, N, K_1, K_2, \alpha, \beta, l_1, l_2, l_3)\left[r^{-2}(\|u\|^{*}_{2,B(x,r)})^2 + (\|h\|^{*}_{2q,B(x,r)})^2\right],$$

for any $q > \frac{n}{2}$.

Remark 2.6. Note that if in (1.1), $H = 0$, and λ be a constant then we obtain similar results as [19], and if only $H = 0$, then we have the same result for $\lambda \leq N$ with constant $C = C(n, N, K, \alpha, \rho)$. Also, for the case that λ be a constant, for example $\lambda = b$, then just C changes as $C = C(n, b, K, \alpha, \rho, l_1, l_2, l_3)$. Moreover, when $\lambda = 0$, then we have $C = C(n, K, \alpha, \rho, l_1, l_2, l_3)$ such that

$$\sup_{B(x,\frac{1}{2}r)} |\nabla u|^2 \leq C(n, K, \alpha, \rho, l_1, l_2, l_3)\left[(\|f\|^{*}_{2q,B(x,r)})^2 + r^{-2}(\|u\|^{*}_{2,B(x,r)})^2\right].$$

By this fact coefficient C for the Corollary 2.5 changes as follows:

(i) if $H = 0$, then $C = C(n, N, K_1, K_2, \alpha, \beta)$,

(ii) if $\lambda = b$, b is a constant, then $C = C(n, b, K_1, K_2, \alpha, \beta, l_1, l_2, l_3)$,

(iii) if $\lambda = 0$, then $C = C(n, K_1, K_2, \alpha, \beta, l_1, l_2, l_3)$.

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