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An application to a system of fractional hybrid differential equations of best proximity point (pair) theorems via measure of noncompactness

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Abstract

This paper establishes the best proximity point (BPP) theorem by using a newly developed contraction operator. Using the results, the optimal solution to a system of fractional hybrid differential equations is then investigated. To further illustrate the findings, an additional example is given.

Keywords: Measure of noncompactness (MNC), Hybrid differential equation, Best proximity point (BPP)
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1 Introduction

Finding the points that are closest to a given point or subgroup is one of the main issues in approximation theory. For $\breve{\mathcal{W}}$, a normed linear space (NLS), if $\breve{e} \cap \mathtt{T}(\breve{e}) \neq \phi$, where $\breve{e} \in \mathcal{E}$ and $\mathcal{E}$ is a non-empty subset of $\breve{\mathcal{W}}$, then the map $\mathtt{T} : \mathcal{E} \to \breve{\mathcal{W}}$ has a fixed point. If there isn't a fixed point for $\mathtt{T}$, then for any $\breve{e}$ in $\mathcal{E}$, the distance between $\breve{e}$ and $\mathtt{T}(\breve{e})$ is positive. If there is a minimum distance between $\breve{e}$ and $\mathtt{T}(\breve{e})$, then $\breve{e}$ is the best proximity point of $\mathtt{T}$. As a consequence, Ky Fan develops his best approximation theory. On the other hand, when $\mathtt{T}$ maps $\mathcal{E}$ into a distinct subset B of $\breve{\mathcal{W}}$, there is an issue. In this case, the extension of the problem is to find a point that approximates the distance between these two subsets. These points are known as best proximity points.

The structure of this paper will be as follows: We first review some basic terms and ideas related to best proximity theory. We then prove the best proximity point theorem for both cyclic and noncyclic contractive operators. We next go over their particular cases in the next section. Lastly, we use our findings to look at the best way to solve a system of fractional hybrid differential equations.

After Kuratowski and Hausdorff's generalization, numerous scholars investigated the concept of MNC in order to obtain significant extensions of the theory of compact operators. The main competence is the application of measures of noncompactness to ensure that the mappings satisfy the relevant inequalities. To help the reader understand our problem and objective, we thus provide a brief history. We review the standard fixed point problem in a Banach space $\breve{\mathcal{W}}$ by using certain regularity assumptions from Schauder [2].

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The authors of [10] first addressed the *BPP* results using *MNC*. After that, they used their findings to look into the existence of optimum solutions of a system of second-order differential equations. The application of *MNC* to find optimal solutions for ψ-Hilfer fractional differential equations with initial conditions was investigated by the authors of [11]. Using the Caputo fractional derivative of order $\check{r} \in (0,1]$, the authors of [5] defined both strong and mild solutions to the fractional hybrid boundary value problems in two types and investigated the presence of at least one mild solution for each type. Finally, they proved how crucial it is to consider the initial conditions. In [13], the authors used Petryshyn's fixed point theorem and *MNC* to investigate the solvability of functional-integral equations in Banach algebra. Additionally, they looked into the general class of functional equations, which includes a number of integral equations that arise in real-world and non-linear analysis situations. Using a Darbo-type fixed point theorem and *MNC*, the authors of [9] investigate the existence of solutions to infinite systems of second-order differential equations in the Banach sequence space l_p. Using a method related to *MNC*, the authors of [12] work on existence results for an infinite system of differential equations of order n with boundary conditions in the Banach spaces c_0 and l_1. In [7], the authors used a method related to *MNC* and the generalized Meir–Keeler fixed point theorem to demonstrate the existence of a solution for an infinite system of nonlinear integral equations in the Banach spaces l_p, $p > 1$. The authors examined if there are any solutions for nonlinear integral equations in [8]. Additionally, they presented an iteration approach to provide very accurate solutions for the nonlinear integral equations. At last, they determined the convergence condition and provided an upper bound of error.

Motivated by these works, in this paper we established a best proximity point theorem with the help of a newly defined contraction operator by using *MNC*, and studied the existence of optimum solutions of a system of fractional hybrid differential equations.

Theorem 1.1. [6] A continuous operator $\mathtt{J} : \mathtt{K} \to \mathtt{K}$ where $\mathtt{K}$ is a nonempty, convex and compact subset of a Banach space $\check{\mathcal{W}}$, then $\mathtt{J}$ admits at least a fixed point.

Clearly it is the generalization of Brouwer fixed point theorem. Consider a Banach space $\check{\mathcal{W}}$ and a closed ball $\mathtt{C}\left(\bar{f}, \bar{g}\right) = \{\bar{m} \in \check{\mathcal{W}} : \|\bar{m} - \bar{f}\| \ \leq \bar{g}\}$ in $\check{\mathcal{W}}$. Suppose $\bar{\mathtt{z}}$, for nonempty $(\mathtt{z})$ denotes the closure of $\mathtt{z}$ and $\overline{conv}\,(\mathtt{z})$ denotes the closed and convex hull of the non empty set $\mathtt{z}$ which is the smallest convex and closed set containing $\mathtt{z}$.

Also $M_{\check{\mathcal{W}}}$ and $N_{\check{\mathcal{W}}}$ represents the family of non empty bounded subsets of $\check{\mathcal{W}}$ and subfamily of $\check{\mathcal{W}}$ consisting all relatively compact sets respectively; R$= (-\infty, \infty)$, and $R_+ = [0, \infty)$. Measure of noncompactness (*MNC*) is defined axiomatically as follows:

Definition 1.2. [1] A map $\check{\mathcal{G}} : M_{\check{\mathcal{W}}} \to R_+$ is a *MNC* (measure of noncompactness) in a Banach space $\check{\mathcal{W}}$, if the following conditions are holds for $\check{\mathcal{G}}$:

1. $\ker \check{\mathcal{G}} = \{\tilde{\mathfrak{V}} \in M_{\check{\mathcal{W}}} : \check{\mathcal{G}}\left(\tilde{\mathfrak{V}}\right) = 0\} \neq \phi$,
2. $\tilde{\mathfrak{V}} \in \ker \check{\mathcal{G}}$ if and only if $\tilde{\mathfrak{V}}$ is relatively compact,
3. $\tilde{\mathfrak{V}}_1 \subseteq \tilde{\mathfrak{V}}_2 \Rightarrow \check{\mathcal{G}}\left(\tilde{\mathfrak{V}}_1\right) \leq \check{\mathcal{G}}\left(\tilde{\mathfrak{V}}_2\right)$,
4. $\check{\mathcal{G}}\left(\overline{\tilde{\mathfrak{V}}}\right) = \check{\mathcal{G}}\left(\tilde{\mathfrak{V}}\right)$,
5. $\check{\mathcal{G}}\left(\overline{conv}(\tilde{\mathfrak{V}})\right) = \check{\mathcal{G}}\left(\tilde{\mathfrak{V}}\right)$,
6. $\check{\mathcal{G}}\left(\acute{\delta}\tilde{\mathfrak{V}}_1 + \left(1 - \acute{\delta}\right)\tilde{\mathfrak{V}}_2\right) \leq \acute{\delta}\check{\mathcal{G}}\left(\tilde{\mathfrak{V}}_1\right) + \left(1 - \acute{\delta}\right)\check{\mathcal{G}}\left(\tilde{\mathfrak{V}}_2\right)$, for $\acute{\delta} \in [0,1]$,
7. $\max\left\{\check{\mathcal{G}}\left(\tilde{\mathfrak{V}}_1\right), \check{\mathcal{G}}\left(\tilde{\mathfrak{V}}_2\right)\right\} = \check{\mathcal{G}}\left(\tilde{\mathfrak{V}}_1 \cup \tilde{\mathfrak{V}}_2\right)$,
8. The set $\tilde{\mathfrak{V}}_\infty = \cap_{n=1}^{\infty}\tilde{\mathfrak{V}}_n$ is compact and non empty, if $\left(\tilde{\mathfrak{V}}_n\right)$ is a decreasing sequence of closed sets which are non empty in $M_{\check{\mathcal{W}}}$ and $\lim_{n\to\infty} \check{\mathcal{G}}\left(\tilde{\mathfrak{V}}_n\right) = 0$.

In particular, the space $\check{\mathcal{W}} = \mathtt{C}(\mathtt{I})$, where I is the closed and bounded interval, is the set of real valued continuous functions on I. Then $\check{\mathcal{W}}$ is a Banach space with the norm $\|\mathtt{D}\| = \sup\{|\mathtt{D}(\check{s})| : \check{s} \in \mathrm{I}\}, \mathtt{D} \in \check{\mathcal{W}}$. Assume that $\mathtt{U}(\neq \phi) \subseteq \check{\mathcal{W}}$ is bounded. For $\mathtt{D} \in \mathtt{U}$ and $\mathtt{r} > 0$, the modulus of continuity of $\mathtt{D}$, represented by $\beth(\mathtt{D}, \mathtt{r})$ i.e.,

$$\beth(\mathtt{D}, \mathtt{r}) = \sup\{|\mathtt{D}(\acute{s}_1) - \mathtt{D}(\acute{s}_2)| : \acute{s}_1, \acute{s}_2 \in \mathrm{I}, |\acute{s}_1 - \acute{s}_2| \leq \mathtt{r}\}.$$

Furthermore, we define

$$\beth(\mathtt{U}, \mathtt{r}) = \sup\{\beth(\mathtt{D}, \mathtt{r}) : \mathtt{D} \in \mathtt{U}\}; \beth_0(\mathtt{U}) = \lim_{\mathtt{r}\to 0} \beth(\mathtt{U}, \mathtt{r}).$$

A Hausdorff *MNC* $\daleth$ is given by $\daleth(\mathtt{U}) = \dfrac{1}{2}\beth_0(\mathtt{U})$, see [2]. It is widely known that the map $\beth_0$ is a *MNC* in $\check{\mathcal{W}}$.

2 Preliminaries

We collect some fundamental definitions and notations needed for the paper.

Definition 2.1. [6] Consider $\breve{\mathcal{W}}$ be a Banach space. Then

1. $\breve{\mathcal{W}}$ is uniformly convex Banach space if there exists a strictly increasing function $\bar{\mathcal{D}} : (0,2] \to [0,1]$ such that,

$$\begin{cases} \|\acute{r}_0 - \bar{a}_0\| \leq \bar{\varpi}, \\ \|\acute{\kappa}_0 - \bar{a}_0\| \leq \bar{\varpi}, \Longrightarrow \|\dfrac{\acute{r}_0 + \acute{\kappa}_0}{2} - \bar{a}_0\| \leq \left(1 - \bar{\mathcal{D}}\left(\dfrac{\bar{s}_1}{\bar{\varpi}}\right)\right)\bar{\varpi}; \\ \|\acute{r}_0 - \acute{\kappa}_0\| \geq \bar{s}_1 \end{cases}$$

for all $\acute{r}_0, \acute{\kappa}_0, \bar{a}_0 \in \breve{\mathcal{W}}$, $\bar{\varpi} > 0$ and $\bar{s}_1 \in [0, 2\bar{\varpi}]$.

2. $\breve{\mathcal{W}}$ is strictly convex Banach space if for $\acute{r}_0, \acute{\kappa}_0, \bar{a}_0 \in \breve{\mathcal{W}}$ and $\bar{\varpi} > 0$, the following conditions are holds:

$$\begin{cases} \|\acute{r}_0 - \bar{a}_0\| \leq \bar{\varpi}, \\ \|\acute{\kappa}_0 - \bar{a}_0\| \leq \bar{\varpi}, \Longrightarrow \|\dfrac{\acute{r}_0 + \acute{\kappa}_0}{2} - \bar{a}_0\| < \bar{\varpi}. \\ \acute{r}_0 \neq \acute{\kappa}_0. \end{cases}$$

Consider a normed linear space (*NLS*) $\breve{\mathcal{W}}$. For any two non empty subset $\mathtt{N}_1, \mathtt{N}_2$ of $\breve{\mathcal{W}}$, the pair $(\mathtt{N}_1, \mathtt{N}_2)$ is closed if and only if both $\mathtt{N}_1$, $\mathtt{N}_2$ are closed; $(\mathtt{N}_1, \mathtt{N}_2) \subseteq (\mathsf{H}, \mathsf{C})$ if and only if $\mathtt{N}_1 \subseteq \mathsf{H}, \mathtt{N}_2 \subseteq \mathsf{C}$. In addition, we denote, $dist(\mathsf{H}, \mathsf{C}) = \inf\{\|\breve{\mu} - \breve{\nu}\| : (\breve{\mu}, \breve{\nu}) \in \mathsf{H} \times \mathsf{C}\}$,

$$\mathsf{H}_0 = \{\breve{\mu} \in \mathsf{H} : \text{there exists } \bar{\nu}_0 \in \mathsf{C}, \|\breve{\mu} - \bar{\nu}_0\| = dist(\mathsf{H}, \mathsf{C})\}$$

$$\mathsf{C}_0 = \{\breve{\nu} \in \mathsf{C} : \text{there exists } \breve{\mu}_1 \in \mathsf{H}, \|\breve{\mu}_1 - \breve{\nu}\| = dist(\mathsf{H}, \mathsf{C})\}.$$

Definition 2.2. [6] Consider $\mathfrak{Z}$ be a *NLS*. A non empty pair (H, C) of $\mathfrak{Z}$ is proximinal if $\mathsf{H} = \mathsf{H}_0$ and $\mathsf{C} = \mathsf{C}_0$. Also for a reflexive Banach space $\mathfrak{O}$, if the pair (H, C) be a closed, nonempty, convex and bounded in $\mathfrak{O}$, then $(\mathsf{H}_0, \mathsf{C}_0)$ is also convex, nonempty and closed pair. Consider a function $\mathtt{T} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$. We say that $\mathtt{T}$ is,

1. relatively nonexpansive, if $\|\mathtt{T}(\breve{\mu}) - \mathtt{T}(\breve{\nu})\| \leq \|\breve{\mu} - \breve{\nu}\|$ for any $(\breve{\mu}, \breve{\nu}) \in \mathsf{H} \times \mathsf{C}$,
2. cyclic, if $\mathtt{T}(\mathsf{H}) \subseteq \mathsf{C}$ and $\mathtt{T}(\mathsf{C}) \subseteq \mathsf{H}$,
3. non cyclic, if $\mathtt{T}(\mathsf{H}) \subseteq \mathsf{H}$ and $\mathtt{T}(\mathsf{C}) \subseteq \mathsf{C}$,
4. compact, if $\left(\overline{\mathtt{T}(\mathsf{H})}, \overline{\mathtt{T}(\mathsf{C})}\right)$ is compact.

Definition 2.3. [6] Consider (H, C) be a nonempty pair in a Banach space $\breve{\mathcal{W}}$ and $\bar{\mathtt{E}} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$ be a cyclic function, then $\tilde{\mathsf{q}} \in \mathsf{H} \cup \mathsf{C}$ is called a *BPP* of $\bar{\mathtt{E}}$ if $\|\tilde{\mathsf{q}} - \bar{\mathtt{E}}(\tilde{\mathsf{q}})\| = \mathrm{dist}(\mathsf{H}, \mathsf{C})$. If $\bar{\mathtt{E}}$ is non cyclic, then the pair $(\tilde{\mathsf{q}}, \tilde{\mathsf{w}}) \in \mathsf{H} \times \mathsf{C}$ is best proximity pair if $\|\tilde{\mathsf{q}} - \tilde{\mathsf{w}}\| = dist(\mathsf{H}, \mathsf{C})$, for $\tilde{\mathsf{q}} = \bar{\mathtt{E}}(\tilde{\mathsf{q}}), \tilde{\mathsf{w}} = \bar{\mathtt{E}}(\tilde{\mathsf{w}})$.

Corollary 2.4. [6] Suppose a Banach space $\breve{\mathcal{W}}$ and a nonempty, convex and compact pair $(\mathtt{N}_1, \mathtt{N}_2)$ in $\breve{\mathcal{W}}$. Let a cyclic and relatively nonexpansive mapping $\mathtt{T} : \mathtt{N}_1 \cup \mathtt{N}_2 \to \mathtt{N}_1 \cup \mathtt{N}_2$. Then $\mathtt{T}$ have a *BPP*.

Corollary 2.5. [6] Suppose a strictly convex Banach space $\breve{\mathcal{W}}$ and a compact, nonempty and convex pair $(\mathtt{N}_1, \mathtt{N}_2)$ in $\breve{\mathcal{W}}$. Let a relatively non-expansive and non-cyclic mapping $\mathtt{T} : \mathtt{N}_1 \cup \mathtt{N}_2 \to \mathtt{N}_1 \cup \mathtt{N}_2$. Then $\mathtt{T}$ have a best proximity pair.

Definition 2.6. [5] The left Caputo fractional derivative of order $\check{r} > 0$ is defined by

$$\mathtt{D}^{\check{r}} \tilde{F}(\tau_1) = \frac{1}{\tilde{\gamma}(m - \check{r})} \int_a^{\tau_1} (\tau_1 - \tilde{\varrho})^{m - \check{r} - 1} \mathtt{D}^m \tilde{F}(\tilde{\varrho})\, d\tilde{\varrho}, \tag{2.1}$$

where $m - 1 < \check{r} < m$, $\mathtt{D} = \dfrac{d}{d\tilde{\varrho}}$ and $\tilde{\gamma}(,)$ denotes Euler's gamma function.

3 Main result

Definition 3.1. [3] Consider the function $\mathbf{C} : \mathfrak{U} \subseteq \breve{\mathcal{W}} \to \breve{\mathcal{W}}$ and $\acute{\Phi} : M_{\breve{\mathcal{W}}} \to R^{+}$. $\mathbf{C}$ is said to be $\acute{\Phi}$-admissible if,

$$\acute{\Phi}(\breve{\varkappa}) \geq 1 \quad \Longrightarrow \quad \acute{\Phi}(Conv\mathbf{C}\breve{\varkappa}) \geq 1,$$

where $\breve{\varkappa} \subseteq \mathfrak{U}$ and $\breve{\varkappa}, \mathbf{C}\breve{\varkappa} \in M_{\breve{\mathcal{W}}}$.

Definition 3.2. [3] Assume the function $\breve{\varpi} : R^{+} \times R^{+} \to R^{+}$ with the following conditions:

(i) $\max\{\tilde{\mu}_0, \tilde{\nu}_0\} \leq \breve{\varpi}(\tilde{\mu}_0, \tilde{\nu}_0)$, for all $\tilde{\mu}_0, \tilde{\nu}_0 \geq 0$,
(ii) $\breve{\varpi}$ is non-decreasing and continuous,
(iii) $\breve{\varpi}\left(\tilde{u}_0 + \tilde{b}_0, \tilde{m}_0 + \tilde{t}_0\right) \leq \breve{\varpi}(\tilde{u}_0, \tilde{m}_0) + \breve{\varpi}\left(\tilde{b}_0, \tilde{t}_0\right)$.

We use $\mathbb{Q}$ to represent the collection of this functions. For example, $\breve{\varpi}(\bar{\mu}, \bar{\nu}) = \bar{\mu} + \bar{\nu}$.

Definition 3.3. [3] Assume a map $\tilde{\varrho} : R^{+} \to R^{+}$ with a non negative sequence $\{\tilde{a}_n\}$ such that

$$\lim_{n\to\infty} \tilde{\varrho}(\tilde{a}_n) = 0 \quad \text{if} \quad \lim_{n\to\infty} \tilde{a}_n = 0.$$

We use $\mathbb{K}$ to represent the collection of this functions. For example, $\tilde{\varrho}(\bar{\mu}) = \bar{\mu}$. Assume that a map $\acute{\Theta} : R^{+} \to R^{+}$ and $\tilde{\varrho} \in \mathbb{K}$ with the following conditions:

(i) $\acute{\Theta}$ is continuous with $\acute{\Theta}(\bar{\mu}) = 0$ if and only if $\bar{\mu} = 0$.
(ii) $\lim_{n\to\infty} \tilde{\varrho}(\bar{\mu}_n) < \acute{\Theta}(\bar{\mu})$ if $\lim_{n\to\infty} \bar{\mu}_n = \bar{\mu} > 0$.

We use $\mathbb{S}_{\tilde{\varrho}}$ to represent the collection of this class of functions. H and C will be nonempty convex subsets of a Banach space $\breve{\mathcal{W}}$ in this section.

Definition 3.4. Consider (H, C) be convex and nonempty pair in a Banach space $\breve{\mathcal{W}}$ with a *MNC* $\breve{\mathcal{G}}$ on $\breve{\mathcal{W}}$. A mapping $\mathtt{T} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$ which is cyclic (*noncyclic*), is said to be a $(\acute{\Theta}, \breve{\varpi}, \breve{F}, \tilde{\varrho})$-contractive operator such that for any convex, nonempty, proximinal, closed, bounded and $\mathtt{T}$-invariant pair (ι_1, ι_2) such that dist $(\iota_1, \iota_2) = \text{dist}(\mathsf{H}, \mathsf{C})$, we have,

$$\acute{\Phi}(\breve{\varkappa})\, \acute{\Theta}\left[\breve{\varpi}\left(\breve{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2)), \breve{F}\left(\breve{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2))\right)\right)\right] \leq \tilde{\varrho}\left[\breve{\varpi}\left(\breve{\mathcal{G}}(\iota_1 \cup \iota_2), \breve{F}\left(\breve{\mathcal{G}}(\iota_1 \cup \iota_2)\right)\right)\right], \tag{3.1}$$

where $\breve{\varkappa}$ is a non empty subset of $\breve{\Pi} = \mathsf{H} \cup \mathsf{C}$, $\acute{\Theta} \in \mathbb{S}_{\tilde{\varrho}}$, $\breve{\varpi} \in \mathbb{Q}$, $\tilde{\varrho} \in \mathbb{K}$ and $\breve{F} : R^{+} \to R^{+}$ is an non decreasing and continuous function. Also the function $\mathtt{T}$ is $\acute{\Phi}$-admissible with $\acute{\Phi}\left(\breve{\Pi}\right) \geq 1$.

Theorem 3.5. Consider a relatively non expansive, cyclic and $(\acute{\Theta}, \breve{\varpi}, \breve{F}, \tilde{\varrho})$-contractive operator $\mathtt{T} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$, then $\mathtt{T}$ has a *BPP*, if $\mathsf{H}_0 \neq \phi$.

Proof . Since $\mathsf{H}_0 \neq \phi$, $(\mathsf{H}_0, \mathsf{C}_0) \neq \phi$. By the given conditions on $\mathtt{T}$, clearly $(\mathsf{H}_0, \mathsf{C}_0)$ is a closed, convex, proximinal and $\mathtt{T}$-invariant pair. For each $\bar{p}_0 \in \mathsf{H}_0$, there is a $\bar{q}_0 \in \mathsf{C}_0$ satisfying $\|\bar{p}_0 - \bar{q}_0\| = dist(\mathsf{H}, \mathsf{C})$. Since $\mathtt{T}$ is relatively non expansive, we get, $\|\mathtt{T}\bar{p}_0 - \mathtt{T}\bar{q}_0\| \leq \|\bar{p}_0 - \bar{q}_0\| = dist(\mathsf{H}, \mathsf{C})$, which implies $\mathtt{T}\bar{p}_0 \in \mathsf{C}_0$, that is $\mathtt{T}(\mathsf{H}_0) \subseteq \mathsf{C}_0$. Similarly, $\mathtt{T}(\mathsf{C}_0) \subseteq \mathsf{H}_0$, hence we get $\mathtt{T}$ is cyclic on $\mathsf{H}_0 \cup \mathsf{C}_0$.

Let us assume that $\tilde{\mathfrak{X}}_0 = \mathsf{H}_0$, $\mathfrak{W}_0 = \mathsf{C}_0$ and $\left\{\left(\tilde{\mathfrak{X}}_n, \mathfrak{W}_n\right)\right\}$ be a sequence of pairs with $\tilde{\mathfrak{X}}_n = \overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{n-1}\right)\right)$ and $\mathfrak{W}_n = \overline{conv}(\mathtt{T}(\mathfrak{W}_{n-1}))$, for all $n \in \mathbb{N}$. Now our claim is, $\tilde{\mathfrak{X}}_{n+1} \subseteq \mathfrak{W}_n$ and $\mathfrak{W}_n \subseteq \tilde{\mathfrak{X}}_{n-1}$ for all $n \in \mathbb{N}$. In fact $\mathfrak{W}_1 = \overline{conv}(\mathtt{T}(\mathfrak{W}_0)) = \overline{conv}(\mathtt{T}(\mathsf{C}_0)) \subseteq \overline{conv}(\mathsf{H}_0) = \mathsf{H}_0 = \tilde{\mathfrak{X}}_0$. Hence we can write,

$$\mathtt{T}(\mathfrak{W}_1) \subseteq \mathtt{T}\left(\tilde{\mathfrak{X}}_0\right) \;\; and \; \mathfrak{W}_2 = \overline{conv}(\mathtt{T}(\mathfrak{W}_1)) \subseteq \overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_0\right)\right) = \tilde{\mathfrak{X}}_1.$$

With the similar argument, we get by using induction, $\mathfrak{W}_n \subseteq \tilde{\mathfrak{X}}_{n-1}$. Similarly, we get $\tilde{\mathfrak{X}}_{n+1} \subseteq \mathfrak{W}_n$, for all $n \in \mathbb{N}$. Hence we can write $\tilde{\mathfrak{X}}_{n+2} \subseteq \mathfrak{W}_{n+1} \subseteq \tilde{\mathfrak{X}}_n \subseteq \mathfrak{W}_{n-1}$, for all $n \in \mathbb{N}$. So in $\mathsf{H}_0 \times \mathsf{C}_0$, the decreasing sequence of *NBCC* pairs is $\left\{\left(\tilde{\mathfrak{X}}_{2n}, \mathfrak{W}_{2n}\right)\right\}$. Moreover,

$$\mathtt{T}(\mathfrak{W}_{2n}) \subseteq \mathtt{T}\left(\tilde{\mathfrak{X}}_{2n-1}\right) \subseteq \overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n-1}\right)\right) = \tilde{\mathfrak{X}}_{2n}, \tag{3.2}$$

$$\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right) \subseteq \mathtt{T}(\mathfrak{W}_{2n-1}) \subseteq \overline{conv}(\mathtt{T}(\mathfrak{W}_{2n-1})) = \mathfrak{W}_{2n}. \tag{3.3}$$

Hence we get $\left(\tilde{\mathfrak{X}}_{2n}, \mathfrak{W}_{2n}\right)$ is $\mathtt{T}-$*invariant* pair, for all n $\in \mathbb{N}$. Now, if the pair $(\breve{x}, \breve{y}) \in \mathsf{H}_0 \times \mathsf{C}_0$ is proximinal, we have,

$$dist\left(\tilde{\mathfrak{X}}_{2n}, \mathfrak{W}_{2n}\right) \leq \|\mathtt{T}^{2n}\breve{x} - \mathtt{T}^{2n}\breve{y}\| \leq \|\breve{x} - \breve{y}\| = dist\left(\mathsf{H}, \mathsf{C}\right).$$

Now we are to show that, for all $n \in \mathbb{N}$, the pair $\left(\tilde{\mathfrak{X}}_n, \mathfrak{W}_n\right)$ is proximinal. For n=0 we have $\left(\tilde{\mathfrak{X}}_0, \mathfrak{W}_0\right)$ is proximinal pair. Let us assume that $\left(\tilde{\mathfrak{X}}_k, \mathfrak{W}_k\right)$ is proximinal and an arbitrary $\tilde{\pi}$ such that $\tilde{\pi} \in \tilde{\mathfrak{X}}_{k+1} = \overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_k\right)\right)$. So $\tilde{\pi} = \sum_{\acute{k}=1}^{\acute{n}} \Psi_{\acute{k}}\mathtt{T}\left(J_{\acute{k}}\right)$ with $J_{\acute{k}} \in \tilde{\mathfrak{X}}_k, \acute{n} \in [1,\infty), \Psi_{\acute{k}} \geq 0$ and $\sum_{\acute{k}=1}^{\acute{n}} \Psi_{\acute{k}} = 1$. By assumption, we have $\left(\tilde{\mathfrak{X}}_k, \mathfrak{W}_k\right)$ is proximity pair, there exists $\tilde{\rho}_{\acute{k}} \in \mathfrak{W}_k$ for $1 \leq \acute{k} \leq \acute{n}$ such that $\|J_{\acute{k}} - \tilde{\rho}_{\acute{k}}\| = dist\left(\tilde{\mathfrak{X}}_k, \mathfrak{W}_k\right) = dist\left(\mathsf{H}, \mathsf{C}\right)$. Consider $\tilde{\rho} = \sum_{\acute{k}=1}^{\acute{n}} \Psi_{\acute{k}}\mathtt{T}\left(\tilde{\rho}_{\acute{k}}\right)$. Then $\tilde{\rho} \in \overline{conv}\left(\mathtt{T}\left(\mathfrak{W}_k\right)\right) = \mathfrak{W}_{k+1}$, and

$$\|\tilde{\pi} - \tilde{\rho}\| = \|\sum_{\acute{k}=1}^{\acute{n}} \Psi_{\acute{k}}\mathtt{T}\left(J_{\acute{k}}\right) - \sum_{\acute{k}=1}^{\acute{n}} \Psi_{\acute{k}}\mathtt{T}\left(\tilde{\rho}_{\acute{k}}\right)\| \leq \sum_{\acute{k}=1}^{\acute{n}} \Psi_{\acute{k}}\|J_{\acute{k}} - \tilde{\rho}_{\acute{k}}\| = dist\left(\mathsf{H}, \mathsf{C}\right). \tag{3.4}$$

Hence $\left(\tilde{\mathfrak{X}}_{k+1}, \mathfrak{W}_{k+1}\right)$ is the proximinal pair and by induction hypothesis our claim is proved.

If $\max\left\{\breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n_0}\right), \breve{\mathcal{G}}\left(\mathfrak{W}_{2n_0}\right)\right\} = 0$, for some $n_0 \in [1,\infty) \cup \{0\}$, then we have $\mathtt{T} : \tilde{\mathfrak{X}}_{2n_0} \cup \mathfrak{W}_{2n_0} \to \tilde{\mathfrak{X}}_{2n_0} \cup \mathfrak{W}_{2n_0}$ is compact. By Corollary 2.4, $\mathtt{T}$ have a *BPP*. Hence we consider that $\max\left\{\breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_n\right), \breve{\mathcal{G}}\left(\mathfrak{W}_n\right)\right\} > 0$, for all $n \in [1,\infty)$. Since $\tilde{\mathfrak{X}}_{2n+1} \subseteq \mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right)$ and $\mathfrak{W}_{2n+1} \subseteq \mathtt{T}\left(\mathfrak{W}_{2n}\right)$, we have,

$$\begin{aligned}
&\acute{\Theta}\left[\breve{\varpi}\left(\breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n+1} \cup \mathfrak{W}_{2n+1}\right), \breve{F}\left(\breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n+1} \cup \mathfrak{W}_{2n+1}\right)\right)\right)\right] \\
=&\acute{\Theta}\left[\breve{\varpi}\left(\max\left\{\breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n+1}\right), \breve{\mathcal{G}}\left(\mathfrak{W}_{2n+1}\right)\right\}, \breve{F}\left(\max\left\{\breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n+1}\right), \breve{\mathcal{G}}\left(\mathfrak{W}_{2n+1}\right)\right\}\right)\right)\right] \\
=&\acute{\Theta}\left[\breve{\varpi}\left(\max\left\{\breve{\mathcal{G}}\left(\overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right)\right)\right), \breve{\mathcal{G}}\left(\overline{conv}\left(\mathtt{T}\left(\mathfrak{W}_{2n}\right)\right)\right)\right\}, \breve{F}\left(\max\left\{\breve{\mathcal{G}}\left(\overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right)\right)\right), \breve{\mathcal{G}}\left(\overline{conv}\left(\mathtt{T}\left(\mathfrak{W}_{2n}\right)\right)\right)\right\}\right)\right)\right] \\
=&\acute{\Theta}\left[\breve{\varpi}\left(\max\left\{\breve{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right)\right), \breve{\mathcal{G}}\left(\mathtt{T}\left(\mathfrak{W}_{2n}\right)\right)\right\}, \breve{F}\left(\max\left\{\breve{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right)\right), \breve{\mathcal{G}}\left(\mathtt{T}\left(\mathfrak{W}_{2n}\right)\right)\right\}\right)\right)\right] \\
=&\acute{\Theta}\left[\breve{\varpi}\left(\breve{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right) \cup \mathtt{T}\left(\mathfrak{W}_{2n}\right)\right), \breve{F}\left(\breve{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right) \cup \mathtt{T}\left(\mathfrak{W}_{2n}\right)\right)\right)\right)\right] \\
\leq&\acute{\Phi}\left(\breve{\varkappa}\right)\acute{\Theta}\left[\breve{\varpi}\left(\breve{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right) \cup \mathtt{T}\left(\mathfrak{W}_{2n}\right)\right), \breve{F}\left(\breve{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{2n}\right) \cup \mathtt{T}\left(\mathfrak{W}_{2n}\right)\right)\right)\right)\right] \\
\leq&\tilde{\varrho}\left[\breve{\varpi}\left(\breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n} \cup \mathfrak{W}_{2n}\right), \breve{F}\left(\breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n} \cup \mathfrak{W}_{2n}\right)\right)\right)\right].
\end{aligned}$$

Clearly, $\left\{\breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n} \cup \mathfrak{W}_{2n}\right)\right\}_{n=1}^{\infty}$ is a decreasing and non-negative sequence. Thus, there exists $\mathcal{V} \geq 0$ such that $\lim_{n\to\infty} \breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n} \cup \mathfrak{W}_{2n}\right) = \mathcal{V}$. If possible, consider $\mathcal{V} > 0$. Then we have as $n \to \infty$

$$\begin{aligned}
\acute{\Theta}\left[\breve{\varpi}\left(\mathcal{V}, \breve{F}\left(\mathcal{V}\right)\right)\right] &\leq \tilde{\varrho}\left[\breve{\varpi}\left(\mathcal{V}, \breve{F}\left(\mathcal{V}\right)\right)\right] \\
&< \acute{\Theta}\left[\breve{\varpi}\left(\mathcal{V}, \breve{F}\left(\mathcal{V}\right)\right)\right],
\end{aligned}$$

a contradiction. Thus $\mathcal{V}$=0, that is $\lim_{n\to\infty} \breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n} \cup \mathfrak{W}_{2n}\right) = 0$. Thus

$$\lim_{n\to\infty} \breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n} \cup \mathfrak{W}_{2n}\right) = \max\left\{\lim_{n\to\infty} \breve{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{2n}\right), \lim_{n\to\infty} \breve{\mathcal{G}}\left(\mathfrak{W}_{2n}\right)\right\} = 0.$$

If $\tilde{\mathfrak{X}}_\infty = \cap_{n=0}^{\infty}\tilde{\mathfrak{X}}_{2n}$ and $\mathfrak{W}_\infty = \cap_{n=0}^{\infty}\mathfrak{W}_{2n}$, We get the non empty, compact, convex pair $\left(\tilde{\mathfrak{X}}_\infty, \mathfrak{W}_\infty\right)$ which is $\mathtt{T}$-invariant with dist$\left(\tilde{\mathfrak{X}}_\infty, \mathfrak{W}_\infty\right)$ = dist(H, C). Thus $\mathtt{T}$ has a *BPP*. □

Theorem 3.6. Consider a relatively nonexpansive, noncyclic and $(\acute{\Theta}, \breve{\varpi}, \check{F}, \tilde{\varrho})$-contractive operator $\mathtt{T} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$ on a strictly convex Banach space $\breve{\mathcal{W}}$, then $\mathtt{T}$ has a best proximity pair, if $\mathsf{H}_0 \neq \phi$.

Proof . Following the proof of the Theorem 3.5, define a pair $\left(\tilde{\mathfrak{X}}_n, \mathfrak{W}_n\right)$ as $\tilde{\mathfrak{X}}_n = \overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{n-1}\right)\right)$ and $\mathfrak{W}_n = \overline{conv}\left(\mathtt{T}\left(\mathfrak{W}_{n-1}\right)\right)$, $n \in \mathbb{N}$ with $\tilde{\mathfrak{X}}_0 = \mathsf{H}_0$ and $\mathfrak{W}_0 = \mathsf{C}_0$. We get a *NBCC* and decreasing sequence of pair $\left\{\left(\tilde{\mathfrak{X}}_n, \mathfrak{W}_n\right)\right\}$ in $\mathsf{H}_0 \times \mathsf{C}_0$. Also,

$$\mathtt{T}\left(\tilde{\mathfrak{X}}_n\right) \subseteq \mathtt{T}\left(\tilde{\mathfrak{X}}_{n-1}\right) \subseteq \overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_{n-1}\right)\right) = \tilde{\mathfrak{X}}_n, \tag{3.5}$$

$$\mathtt{T}\left(\mathfrak{W}_n\right) \subseteq \mathtt{T}\left(\mathfrak{W}_{n-1}\right) \subseteq \overline{conv}\left(\mathtt{T}\left(\mathfrak{W}_{n-1}\right)\right) = \mathfrak{W}_n. \tag{3.6}$$

Thus, the pair $\left(\tilde{\mathfrak{X}}_n, \mathfrak{W}_n\right)$ is $\mathtt{T}$-invariant for all n ≥ 1. Following the proof of the Theorem 3.5, we obtain a proximinal pair $\left(\tilde{\mathfrak{X}}_n, \mathfrak{W}_n\right)$, for all non negative integer n such that dist$\left(\tilde{\mathfrak{X}}_n, \mathfrak{W}_n\right) = \text{dist}(\mathsf{H}, \mathsf{C})$. Now if $\max\left\{\check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{n_0}\right), \check{\mathcal{G}}\left(\mathfrak{W}_{n_0}\right)\right\} = 0$, for some positive integer n_0, then $\mathtt{T} : \tilde{\mathfrak{X}}_{n_0} \cup \mathfrak{W}_{n_0} \to \tilde{\mathfrak{X}}_{n_0} \cup \mathfrak{W}_{n_0}$ is compact. Hence from Corollary 2.5, we get the desired result. Thus we consider that, $max\left\{\check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_n\right), \check{\mathcal{G}}\left(\mathfrak{W}_n\right)\right\} > 0$. Since $\tilde{\mathfrak{X}}_{n+1} \subseteq \mathtt{T}\left(\tilde{\mathfrak{X}}_n\right)$ and $\mathfrak{W}_{n+1} \subseteq \mathtt{T}\left(\mathfrak{W}_n\right)$, we have,

$$\begin{aligned}
&\acute{\Theta}\left[\breve{\varpi}\left(\check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{n+1} \cup \mathfrak{W}_{n+1}\right), \check{F}\left(\check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{n+1} \cup \mathfrak{W}_{n+1}\right)\right)\right)\right] \\
=&\acute{\Theta}\left[\breve{\varpi}\left(\max\left\{\check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{n+1}\right), \check{\mathcal{G}}\left(\mathfrak{W}_{n+1}\right)\right\}, \check{F}\left(\max\left\{\check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_{n+1}\right), \check{\mathcal{G}}\left(\mathfrak{W}_{n+1}\right)\right\}\right)\right)\right] \\
=&\acute{\Theta}\left[\breve{\varpi}\left(\max\left\{\check{\mathcal{G}}\left(\overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_n\right)\right)\right), \check{\mathcal{G}}\left(\overline{conv}\left(\mathtt{T}\left(\mathfrak{W}_n\right)\right)\right)\right\}, \check{F}\left(\max\left\{\check{\mathcal{G}}\left(\overline{conv}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_n\right)\right)\right), \check{\mathcal{G}}\left(\overline{conv}\left(\mathtt{T}\left(\mathfrak{W}_n\right)\right)\right)\right\}\right)\right)\right] \\
=&\acute{\Theta}\left[\breve{\varpi}\left(\max\left\{\check{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_n\right)\right), \check{\mathcal{G}}\left(\mathtt{T}\left(\mathfrak{W}_n\right)\right)\right\}, \check{F}\left(\max\left\{\check{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_n\right)\right), \check{\mathcal{G}}\left(\mathtt{T}\left(\mathfrak{W}_n\right)\right)\right\}\right)\right)\right] \\
=&\acute{\Theta}\left[\breve{\varpi}\left(\check{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_n\right) \cup \mathtt{T}\left(\mathfrak{W}_n\right)\right), \check{F}\left(\check{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_n\right) \cup \mathtt{T}\left(\mathfrak{W}_n\right)\right)\right)\right)\right] \\
\leq&\acute{\Phi}\left(\check{\varkappa}\right)\acute{\Theta}\left[\breve{\varpi}\left(\check{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_n\right) \cup \mathtt{T}\left(\mathfrak{W}_n\right)\right), \check{F}\left(\check{\mathcal{G}}\left(\mathtt{T}\left(\tilde{\mathfrak{X}}_n\right) \cup \mathtt{T}\left(\mathfrak{W}_n\right)\right)\right)\right)\right] \\
\leq&\tilde{\varrho}\left[\breve{\varpi}\left(\check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_n \cup \mathfrak{W}_n\right), \check{F}\left(\check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_n \cup \mathfrak{W}_n\right)\right)\right)\right].
\end{aligned}$$

Clearly, $\left\{\check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_n \cup \mathfrak{W}_n\right)\right\}_{n=1}^{\infty}$ is a decreasing and non-negative sequence. Thus, there exists $\mathcal{V} \geq 0$ such that $\lim_{n\to\infty} \check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_n \cup \mathfrak{W}_n\right) = \mathcal{V}$. If possible, consider $\mathcal{V} > 0$. Then we have

$$\begin{aligned}
\acute{\Theta}\left[\breve{\varpi}\left(\mathcal{V}, \check{F}\left(\mathcal{V}\right)\right)\right] &\leq \tilde{\varrho}\left[\breve{\varpi}\left(\mathcal{V}, \check{F}\left(\mathcal{V}\right)\right)\right]; \quad \text{as} \quad n \to \infty \\
&< \acute{\Theta}\left[\breve{\varpi}\left(\mathcal{V}, \check{F}\left(\mathcal{V}\right)\right)\right],
\end{aligned}$$

a contradiction. Thus $\mathcal{V} = 0$, that is $\lim_{n\to\infty} \check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_n \cup \mathfrak{W}_n\right) = 0$. Thus

$$\lim_{n\to\infty} \check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_n \cup \mathfrak{W}_n\right) = \max\left\{\lim_{n\to\infty} \check{\mathcal{G}}\left(\tilde{\mathfrak{X}}_n\right), \lim_{n\to\infty} \check{\mathcal{G}}\left(\mathfrak{W}_n\right)\right\} = 0.$$

If $\tilde{\mathfrak{X}}_\infty = \cap_{n=0}^{\infty} \tilde{\mathfrak{X}}_n$ and $\mathfrak{W}_\infty = \cap_{n=0}^{\infty} \mathfrak{W}_n$, We get the non empty, compact, convex pair $\left(\tilde{\mathfrak{X}}_\infty, \mathfrak{W}_\infty\right)$ which is $\mathtt{T}$-invariant with dist$\left(\tilde{\mathfrak{X}}_\infty, \mathfrak{W}_\infty\right) = \text{dist}(\mathsf{H}, \mathsf{C})$. Thus $\mathtt{T}$ has a best proximity pair. □

The next results are special cases of Theorem 3.5. The noncyclic version of the following corollaries are satisfied in strictly convex Banach spaces.

Corollary 3.7. Consider a cyclic relatively nonexpansive mapping $\mathtt{T} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$ such that for any nonempty, bounded, closed, convex, proximinal and $\mathtt{T}$-invariant pair (ι_1, ι_2) with dist $(\iota_1, \iota_2) = \text{dist}(\mathsf{H}, \mathsf{C})$,

$$2^{\grave{w}}\acute{\Theta}\left[\breve{\varpi}\left(\check{\mathcal{G}}\left(\mathtt{T}\left(\iota_1\right) \cup \mathtt{T}\left(\iota_2\right)\right), \check{F}\left(\check{\mathcal{G}}\left(\mathtt{T}\left(\iota_1\right) \cup \mathtt{T}\left(\iota_2\right)\right)\right)\right)\right] \leq \tilde{\varrho}\left[\breve{\varpi}\left(\check{\mathcal{G}}\left(\iota_1 \cup \iota_2\right), \check{F}\left(\check{\mathcal{G}}\left(\iota_1 \cup \iota_2\right)\right)\right)\right], \tag{3.7}$$

for $\grave{w} \geq 0$. Then $\mathtt{T}$ has a *BPP* if $\mathsf{H}_0 \neq \emptyset$.

Proof . Putting $\acute{\Phi}(\grave{w}) = 2^{\grave{w}}$, $\grave{w} \geq 0$ in equation (3.1) of Definition 3.4 and using Theorem 3.5, the result follows. □

Corollary 3.8. Consider a cyclic relatively nonexpansive mapping $\mathtt{T} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$ such that for any nonempty, bounded, closed, convex, proximinal and T-invariant pair (ι_1, ι_2) with $\text{dist}(\iota_1, \iota_2) = \text{dist}(\mathsf{H}, \mathsf{C})$,

$$2^{\grave{w}}\acute{\Theta}\left[\check{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2)) + \check{F}\left(\check{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2))\right)\right] \leq \tilde{\varrho}\left[\check{\mathcal{G}}(\iota_1 \cup \iota_2) + \check{F}\left(\check{\mathcal{G}}(\iota_1 \cup \iota_2)\right)\right]. \tag{3.8}$$

Then T has a *BPP* if $\mathsf{H}_0 \neq \emptyset$.

Proof . Putting $\breve{\varpi}(\bar{\mu}, \bar{\nu}) = \bar{\mu} + \bar{\nu}$ in equation (3.7) of Corollary 3.7 and using Theorem 3.5, the result follows. □

Corollary 3.9. Consider a cyclic relatively nonexpansive mapping $\mathtt{T} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$ such that for any nonempty, bounded, closed, convex, proximinal and T-invariant pair (ι_1, ι_2) with $\text{dist}(\iota_1, \iota_2) = \text{dist}(\mathsf{H}, \mathsf{C})$,

$$2^{\grave{w}}\acute{\Theta}\left[\check{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2)) + \check{F}\left(\check{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2))\right)\right] \leq \check{\mathcal{G}}(\iota_1 \cup \iota_2) + \check{F}\left(\check{\mathcal{G}}(\iota_1 \cup \iota_2)\right). \tag{3.9}$$

Then T has a *BPP* if $\mathsf{H}_0 \neq \emptyset$.

Proof . Putting $\tilde{\varrho}(\bar{\mu}) = \bar{\mu}$ in equation (3.8) of Corollary 3.8 and using Theorem 3.5, the result follows. □

Corollary 3.10. Consider a cyclic relatively nonexpansive mapping $\mathtt{T} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$ such that for any nonempty, bounded, closed, convex, proximinal and T-invariant pair (ι_1, ι_2) with $\text{dist}(\iota_1, \iota_2) = \text{dist}(\mathsf{H}, \mathsf{C})$,

$$2^{\grave{w}}\left[\check{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2)) + \check{F}\left(\check{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2))\right)\right] \leq \check{\mathcal{G}}(\iota_1 \cup \iota_2) + \check{F}\left(\check{\mathcal{G}}(\iota_1 \cup \iota_2)\right). \tag{3.10}$$

Then T has a *BPP* if $\mathsf{H}_0 \neq \emptyset$.

Proof . Putting $\acute{\Theta}(\bar{\mu}) = \bar{\mu}$ in equation (3.9) of Corollary 3.9 and using Theorem 3.5, the result follows. □

Corollary 3.11. Consider a cyclic relatively nonexpansive mapping $\mathtt{T} : \mathsf{H} \cup \mathsf{C} \to \mathsf{H} \cup \mathsf{C}$ such that for any nonempty, bounded, closed, convex, proximinal and T-invariant pair (ι_1, ι_2) with $\text{dist}(\iota_1, \iota_2) = \text{dist}(\mathsf{H}, \mathsf{C})$,

$$\check{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2)) \leq \check{\mathcal{G}}(\iota_1 \cup \iota_2). \tag{3.11}$$

Then T has a *BPP* if $\mathsf{H}_0 \neq \emptyset$.

Proof . Putting $\check{F}(\bar{\mu}) = 0$ in equation (3.10) of Corollary 3.10, we get

$$\check{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2)) \leq 2^{\grave{w}}\left[\check{\mathcal{G}}(\mathtt{T}(\iota_1) \cup \mathtt{T}(\iota_2))\right] \leq \check{\mathcal{G}}(\iota_1 \cup \iota_2),$$

and using Theorem 3.5, the result follows. □

4 Applications

We use our conclusions to look into the optimum solution to a system of fractional hybrid differential equation. Consider the following system of fractional hybrid differential equations:

$$\mathtt{D}^{\check{r}}\left[\tilde{\varepsilon}(\tau_1) - \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1))\right] = \mathcal{H}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho})), \tag{4.1}$$

$$\mathtt{D}^{\check{r}}\left[\tilde{\varkappa}(\tau_1) - \tilde{F}(\tau_1, \tilde{\varkappa}(\tau_1))\right] = \mathcal{Z}(\tilde{\varrho}, \tilde{\varkappa}(\tilde{\varrho})). \tag{4.2}$$

with $\tilde{\varepsilon}(0) = \tilde{\pi}_0$, $\tilde{\varkappa}(0) = \tilde{\beta}_0$, $\tilde{\pi}_0$, $\tilde{\beta}_0 \in R$, $\check{r} \in (0,1]$, $\tau_1 \in [0,1] = \mathsf{I}$ and $\mathcal{H}, \mathcal{Z}, \tilde{F}$ are continuous functions with $\|\mathcal{H}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\| \leq \mathsf{I}_1$, $\|\mathcal{Z}(\tilde{\varrho}, \tilde{\varkappa}(\tilde{\varrho}))\| \leq \mathsf{I}_2$. Now, we consider the following system of integral equations equivalent to equations (4.1) and (4.2):

$$\begin{cases} \tilde{\varepsilon}(\tau_1) = \tilde{\pi}_0 - \tilde{F}(0, \tilde{\pi}_0) + \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) + \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{H}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}, \\ \tilde{\varkappa}(\tau_1) = \tilde{\beta}_0 - \tilde{F}\left(0, \tilde{\beta}_0\right) + \tilde{F}(\tau_1, \tilde{\varkappa}(\tau_1)) + \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varkappa}(\tilde{\varrho}))\, d\tilde{\varrho}, \end{cases} \tag{4.3}$$

for $\tau_1 \in [0,1] = \mathtt{I}$. Also, assume that $(R, \|.\|)$ is a Banach space and two closed ball $\tilde{P}_1 = \mathtt{E}_1\left(\tilde{\pi}_0, \tilde{b}\right)$, $\tilde{P}_2 = \mathtt{E}_2\left(\tilde{\beta}_0, \tilde{b}\right)$ in R, with $\tilde{b} \in [0, \infty)$. Consider a standard Banach space $\mathtt{N} = \mathtt{C}(\mathfrak{F}, R)$ of continuous function with supremum norm for $\mathfrak{F} \subseteq \mathtt{I}$. Let

$$\mathtt{N}_1 = \mathtt{C}\left(\mathfrak{F}, \tilde{P}_1\right) = \{\tilde{\varepsilon} : \mathfrak{F} \to \tilde{P}_1 : \tilde{\varepsilon} \in \mathtt{N}\}, \quad \text{and} \quad \mathtt{N}_2 = \mathtt{C}\left(\mathfrak{F}, \tilde{P}_2\right) = \{\tilde{\varkappa} : \mathfrak{F} \to \tilde{P}_2 : \tilde{\varkappa} \in \mathtt{N}\}.$$

Then $(\mathtt{N}_1, \mathtt{N}_2)$ is *NBCC* pair in $\mathtt{N}$. Now for every $\tilde{\varepsilon} \in \mathtt{N}_1, \tilde{\varkappa} \in \mathtt{N}_2$,

$$\|\tilde{\varepsilon} - \tilde{\varkappa}\| = \sup_{\tau_1 \in \mathtt{I}} \|\tilde{\varepsilon}(\tau_1) - \tilde{\varkappa}(\tau_1)\| \geq \|\tilde{\pi}_0 - \tilde{\beta}_0\|.$$

Thus $\text{dist}(\mathtt{N}_1, \mathtt{N}_2) = \|\tilde{\pi}_0 - \tilde{\beta}_0\|$. Now, we define $\mathtt{T} : \mathtt{N}_1 \cup \mathtt{N}_2 \to \mathtt{N}$ such that,

$$\mathtt{T}(\tilde{\varepsilon}(\tau_1)) = \begin{cases} \tilde{\beta}_0 - \tilde{F}\left(0, \tilde{\beta}_0\right) + \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) + \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}, & \tilde{\varepsilon} \in \mathtt{N}_1, \\ \tilde{\pi}_0 - \tilde{F}(0, \tilde{\pi}_0) + \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) + \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{H}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}, & \tilde{\varepsilon} \in \mathtt{N}_2. \end{cases}$$

Clearly, $\mathtt{T}$ is cyclic, and if $\|\tilde{r}_0 - \mathtt{T}(\tilde{r}_0)\| = dist(\mathtt{N}_1, \mathtt{N}_2)$, for $\tilde{r}_0 \in \mathtt{N}_1 \cup \mathtt{N}_2$, then $\tilde{r}_0$ is the optimum solution of the system (4.3). Thus $\tilde{r}_0$ is the *BPP* for the operator $\mathtt{T}$.

Theorem 4.1. [4] Consider $\acute{\mathtt{U}} \in \mathtt{C}\left[\grave{\vartheta}_1, \grave{\vartheta}_2\right]$ with $\grave{\vartheta}_1 < \grave{\vartheta}_2$. Let $\acute{\mathtt{V}}$ is Lebesgue integrable on $\left[\grave{\vartheta}_1, \grave{\vartheta}_2\right]$ and $\acute{\mathtt{V}}$ does not change its sign in $\left[\grave{\vartheta}_1, \grave{\vartheta}_2\right]$ with $\breve{a} \in \left(\grave{\vartheta}_1, \grave{\vartheta}_2\right)$. Then the generalized mean value theorem of integral calculus gives,

$$\int_{\grave{\vartheta}_1}^{\grave{\vartheta}_2} \acute{\mathtt{U}}(\bar{\iota})\, \acute{\mathtt{V}}(\bar{\iota})\, d\bar{\iota} = \acute{\mathtt{U}}(\breve{a}) \int_{\grave{\vartheta}_1}^{\grave{\vartheta}_2} \acute{\mathtt{V}}(\bar{\iota})\, d\bar{\iota}.$$

Theorem 4.2. Consider $\acute{\mathtt{U}} \in \mathtt{C}\left[\grave{\vartheta}_1, \grave{\vartheta}_2\right]$ with $\grave{\vartheta}_1 < \grave{\vartheta}_2$ and $\breve{r} > 0$. Let $\acute{\mathtt{V}}$ is Lebesgue integrable on $\left[\grave{\vartheta}_1, \grave{\vartheta}_2\right]$ and $\acute{\mathtt{V}}$ does not change its sign in $\left[\grave{\vartheta}_1, \grave{\vartheta}_2\right]$. Then there exists $\breve{a} \in \left(\grave{\vartheta}_1, \tilde{\varepsilon}\right) \subset \left(\grave{\vartheta}_1, \grave{\vartheta}_2\right)$ such that $\mathcal{I}^{\breve{r}}_{\grave{\vartheta}_1}\left(\acute{\mathtt{U}}\acute{\mathtt{V}}\right)(\tilde{\varepsilon}) = \acute{\mathtt{U}}(\breve{a})\, \mathcal{I}^{\breve{r}}_{\grave{\vartheta}_1} \acute{\mathtt{V}}(\tilde{\varepsilon})$, for almost every $\tilde{\varepsilon} \in \left(\grave{\vartheta}_1, \grave{\vartheta}_2\right]$. For $\breve{r} \geq 1$ or $\acute{\mathtt{V}} \in \mathtt{C}\left[\grave{\vartheta}_1, \grave{\vartheta}_2\right]$, the above result holds for every $\tilde{\varepsilon} \in \left(\grave{\vartheta}_1, \grave{\vartheta}_2\right]$.

Proof . Here

$$\mathcal{I}^{\breve{r}}_{\grave{\vartheta}_1}\left(\acute{\mathtt{U}}\acute{\mathtt{V}}\right)(\tilde{\varepsilon}) = \frac{1}{\tilde{\gamma}(\breve{r})} \int_{\grave{\vartheta}_1}^{\tau_1} (\tau_1 - \tilde{\varrho})^{\breve{r}-1} \acute{\mathtt{U}}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, \acute{\mathtt{V}}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}.$$

Consider $\tilde{\acute{\mathtt{V}}}(\tilde{\varepsilon}) = \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \acute{\mathtt{V}}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))$. Now we have,

$$\begin{aligned} \mathcal{I}^{\breve{r}}_{\grave{\vartheta}_1}\left(\acute{\mathtt{U}}\acute{\mathtt{V}}\right)(\tilde{\varepsilon}) &= \int_{\grave{\vartheta}_1}^{\tau_1} \acute{\mathtt{U}}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, \tilde{\acute{\mathtt{V}}}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} \\ &= \acute{\mathtt{U}}(\breve{a}) \int_{\grave{\vartheta}_1}^{\tau_1} \tilde{\acute{\mathtt{V}}}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} \\ &= \acute{\mathtt{U}}(\breve{a}) \int_{\grave{\vartheta}_1}^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \acute{\mathtt{V}}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} \\ &= \acute{\mathtt{U}}(\breve{a})\, \mathcal{I}^{\breve{r}}_{\grave{\vartheta}_1} \acute{\mathtt{V}}(\tilde{\varepsilon}). \end{aligned}$$

□

Theorem 4.3. Assume that $\beth_0$ is a *MNC* on $\mathtt{N}$ with $0 < \breve{r} \leq 1$, $\tilde{F}(\breve{\mu}, \breve{\nu}) = \breve{\mu}\breve{\nu}$, $\breve{\mu}, \breve{\nu} \in R$, $|\tau_1 \tilde{\varepsilon}(\tau_1) - \tau_2 \tilde{\varepsilon}(\tau_2)| \leq |\tau_2 - \tau_1|$, $\left\{ |\tilde{\varepsilon}(\tau_1)| + \frac{1}{\tilde{\gamma}(\breve{r}+1)} \mathsf{l}_2 \tau_1^{\breve{r}} \right\} \leq \mathtt{L}$, for $\mathtt{L} > 0$. Then an optimal solution exists for the system of equations (4.3) if:

For all $\tilde{\varepsilon} \in \mathtt{N}_1, \tilde{\varkappa} \in \mathtt{N}_2, \tau_1 \in \mathfrak{F}$ there exists $l \geq 0$ and $\check{s} \geq 0$ such that,

(i) $|\tilde{\varepsilon}(\tau_1) - \tilde{\varkappa}(\tau_1)| \leq l$,

(ii) $|\tilde{\beta}_0 - \tilde{\pi}_0| \leq \check{s}$,

(iii) $|\mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho})) - \mathcal{H}(\tilde{\varrho}, \tilde{\varkappa}(\tilde{\varrho}))| \leq \tilde{\gamma}(\breve{r}+1) \left\{ |\tilde{\varepsilon} - \tilde{\varkappa}| - \check{s} - l \right\}$.

Proof . First, we show that the operator $\mathtt{T}$ is cyclic. For $\tilde{\varepsilon} \in \mathtt{N}_1$,

$$\begin{aligned}
\|\mathtt{T}(\tilde{\varepsilon}(\tau_1)) - \tilde{\beta}_0\| &= \left\| \tilde{\beta}_0 - \tilde{F}\left(0, \tilde{\beta}_0\right) + \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) + \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} - \tilde{\beta}_0 \right\| \\
&= \left\| \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) - \tilde{F}\left(0, \tilde{\beta}_0\right) + \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} \right\| \\
&\leq \left| \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) \right| + \left| \tilde{F}\left(0, \tilde{\beta}_0\right) \right| + \left| \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} \right| \\
&\leq |\tau_1 \tilde{\varepsilon}(\tau_1)| + \mathsf{l}_2 \left| \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})}\, d\tilde{\varrho} \right| \\
&\leq \tau_1 |\tilde{\varepsilon}(\tau_1)| + \frac{1}{\tilde{\gamma}(\breve{r})} \mathsf{l}_2 \left| \int_0^{\tau_1} (\tau_1 - \tilde{\varrho})^{\breve{r}-1}\, d\tilde{\varrho} \right| \\
&\leq |\tilde{\varepsilon}(\tau_1)| + \frac{1}{\tilde{\gamma}(\breve{r})} \mathsf{l}_2 \left| \left[\frac{-(\tau_1 - \tilde{\varrho})^{\breve{r}}}{\breve{r}} \right]_0^{\tau_1} \right| \\
&\leq |\tilde{\varepsilon}(\tau_1)| + \frac{1}{\tilde{\gamma}(\breve{r}+1)} \mathsf{l}_2 \tau_1^{\breve{r}} \\
&\leq \mathtt{L}.
\end{aligned}$$

Hence, $\mathtt{T}(\tilde{\varepsilon}(\tau_1)) \in \mathtt{N}_2$. By using a similar method, we can show that $\mathtt{T}(\tilde{\varepsilon}(\tau_1)) \in \mathtt{N}_1$ for $\tilde{\varepsilon} \in \mathtt{N}_2$. Thus, $\mathtt{T}$ is cyclic. We now show that $\mathtt{T}(\mathtt{N}_1)$ is an equicontinuous and bounded subset of $\mathtt{N}_2$. Consider $\tilde{\varepsilon} \in \mathtt{N}_1$ and $\tau_1 \in \mathfrak{F}$ with $0 < \breve{r} \leq 1$. We have,

$$\begin{aligned}
\|\mathtt{T}(\tilde{\varepsilon}(\tau_1))\| &= \left\| \tilde{\beta}_0 - \tilde{F}\left(0, \tilde{\beta}_0\right) + \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) + \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} \right\| \\
&\leq \left| \tilde{\beta}_0 - \tilde{F}\left(0, \tilde{\beta}_0\right) \right| + \left| \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) \right| + \left| \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} \right| \\
&\leq |\tilde{\beta}_0| + |\tau_1 \tilde{\varepsilon}(\tau_1)| + \mathsf{l}_2 \left| \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})}\, d\tilde{\varrho} \right| \\
&\leq |\tilde{\beta}_0| + \tau_1 |\tilde{\varepsilon}(\tau_1)| + \frac{1}{\tilde{\gamma}(\breve{r})} \mathsf{l}_2 \left| \int_0^{\tau_1} (\tau_1 - \tilde{\varrho})^{\breve{r}-1}\, d\tilde{\varrho} \right| \\
&\leq |\tilde{\beta}_0| + |\tilde{\varepsilon}(\tau_1)| + \frac{1}{\tilde{\gamma}(\breve{r})} \mathsf{l}_2 \left| \left[\frac{-(\tau_1 - \tilde{\varrho})^{\breve{r}}}{\breve{r}} \right]_0^{\tau_1} \right| \\
&\leq |\tilde{\beta}_0| + |\tilde{\varepsilon}(\tau_1)| + \frac{1}{\tilde{\gamma}(\breve{r}+1)} \mathsf{l}_2 |\tau_1^{\breve{r}}| \\
&\leq |\tilde{\beta}_0| + \mathtt{L}.
\end{aligned}$$

Therefore $\mathtt{T}(\mathtt{N}_1)$ is bounded. Suppose $\tau_1, \tau_2 \in \mathfrak{F}$, $\tau_1 > \tau_2$ and $\tilde{\varepsilon} \in \mathtt{N}_1$. Then

$$\begin{aligned}
&\|\mathtt{T}(\tilde{\varepsilon}(\tau_1)) - \mathtt{T}(\tilde{\varepsilon}(\tau_2))\| \\
=&\left\|\tilde{\beta}_0 - \tilde{F}\left(0, \tilde{\beta}_0\right) + \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) + \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} - \tilde{\beta}_0 + \tilde{F}\left(0, \tilde{\beta}_0\right) - \tilde{F}(\tau_2, \tilde{\varepsilon}(\tau_2)) \right.\\
&\left. - \int_0^{\tau_2} \frac{(\tau_2 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}\right\| \\
\leq&\left|\tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) - \tilde{F}(\tau_2, \tilde{\varepsilon}(\tau_2))\right| + \left|\int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} - \int_0^{\tau_2} \frac{(\tau_2 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}\right| \\
\leq&|\tau_1 \tilde{\varepsilon}(\tau_1) - \tau_2 \tilde{\varepsilon}(\tau_2)| + \left|\int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} - \int_0^{\tau_1} \frac{(\tau_2 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} + \right.\\
&\left. + \int_0^{\tau_1} \frac{(\tau_2 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} - \int_0^{\tau_2} \frac{(\tau_2 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}\right| \\
\leq&|\tau_1 \tilde{\varepsilon}(\tau_1) - \tau_2 \tilde{\varepsilon}(\tau_2)| + \left|\int_0^{\tau_1} \frac{\{(\tau_1 - \tilde{\varrho})^{\breve{r}-1} - (\tau_2 - \tilde{\varrho})^{\breve{r}-1}\}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} + \int_{\tau_2}^{\tau_1} \frac{(\tau_2 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}\right| \\
\leq&|\tau_1 \tilde{\varepsilon}(\tau_1) - \tau_2 \tilde{\varepsilon}(\tau_2)| + \left|\int_0^{\tau_1} \frac{\{(\tau_1 - \tilde{\varrho})^{\breve{r}-1} - (\tau_2 - \tilde{\varrho})^{\breve{r}-1}\}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}\right| + \left|\int_{\tau_2}^{\tau_1} \frac{(\tau_2 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho}\right| \\
\leq&|\tau_1 \tilde{\varepsilon}(\tau_1) - \tau_2 \tilde{\varepsilon}(\tau_2)| + \frac{1}{\tilde{\gamma}(\breve{r})} \mathsf{l}_2 \left|\int_0^{\tau_1} \{(\tau_1 - \tilde{\varrho})^{\breve{r}-1} - (\tau_2 - \tilde{\varrho})^{\breve{r}-1}\}\, d\tilde{\varrho}\right| + \frac{1}{\tilde{\gamma}(\breve{r})} \mathsf{l}_2 \left|\int_{\tau_2}^{\tau_1} (\tau_2 - \tilde{\varrho})^{\breve{r}-1}\, d\tilde{\varrho}\right| \\
\leq&|\tau_2 - \tau_1| + \frac{1}{\tilde{\gamma}(\breve{r})} \mathsf{l}_2 \left|\left\{\frac{-(\tau_1 - \tilde{\varrho})^{\breve{r}}}{\breve{r}} + \frac{(\tau_2 - \tilde{\varrho})^{\breve{r}}}{\breve{r}}\right\}_0^{\tau_1}\right| + \frac{1}{\tilde{\gamma}(\breve{r})} \mathsf{l}_2 \left|\left\{\frac{-(\tau_2 - \tilde{\varrho})^{\breve{r}}}{\breve{r}}\right\}_{\tau_2}^{\tau_1}\right| \\
\leq&|\tau_2 - \tau_1| + \frac{1}{\tilde{\gamma}(\breve{r}+1)} \mathsf{l}_2 \left|\left\{-(\tau_1 - \tilde{\varrho})^{\breve{r}} + (\tau_2 - \tilde{\varrho})^{\breve{r}}\right\}_0^{\tau_1}\right| + \frac{1}{\tilde{\gamma}(\breve{r}+1)} \mathsf{l}_2 \left|\left\{-(\tau_2 - \tilde{\varrho})^{\breve{r}}\right\}_{\tau_2}^{\tau_1}\right| \\
=&|\tau_2 - \tau_1| + \frac{1}{\tilde{\gamma}(\breve{r}+1)} \mathsf{l}_2 \left\{\left|\left\{(\tau_2 - \tau_1)^{\breve{r}} + (\tau_1)^{\breve{r}} - (\tau_2)^{\breve{r}}\right\} - (\tau_2 - \tau_1)^{\breve{r}}\right|\right\} \\
=&|\tau_2 - \tau_1| + \frac{1}{\tilde{\gamma}(\breve{r}+1)} \mathsf{l}_2 \left\{\left|(\tau_1)^{\breve{r}} - (\tau_2)^{\breve{r}}\right|\right\}.
\end{aligned}$$

As $\tau_2 \to \tau_1$, $\|\mathtt{T}(\tilde{\varepsilon}(\tau_1)) - \mathtt{T}(\tilde{\varepsilon}(\tau_2))\| \to 0$. That is $\mathtt{T}(\mathtt{N}_1)$ is equicontinuous. We can show that $\mathtt{T}(\mathtt{N}_2)$ is equicontinuious and bounded in $\mathtt{N}_1$ with the similar manner. Thus by Arzela-Ascoli theorem, we conclude that $(\mathtt{N}_1, \mathtt{N}_2)$ is a relatively compact pair. Now for each $(\tilde{\varepsilon}, \tilde{\varkappa}) \in \mathtt{N}_1 \times \mathtt{N}_2$,

$$\begin{aligned}
&\|\mathtt{T}(\tilde{\varepsilon}(\tau_1)) - \mathtt{T}(\tilde{\varkappa}(\tau_1))\| \\
=&\left\|\tilde{\beta}_0 - \tilde{F}\left(0, \tilde{\beta}_0\right) + \tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) + \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} - \tilde{\pi}_0 + \tilde{F}(0, \tilde{\pi}_0) - \tilde{F}(\tau_1, \tilde{\varkappa}(\tau_1)) \right.\\
&\left. - \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{H}(\tilde{\varrho}, \tilde{\varkappa}(\tilde{\varrho}))\, d\tilde{\varrho}\right\| \\
\leq&\left|\tilde{\beta}_0 - \tilde{\pi}_0\right| + \left|\tilde{F}(\tau_1, \tilde{\varepsilon}(\tau_1)) - \tilde{F}(\tau_1, \tilde{\varkappa}(\tau_1))\right| + \left|\int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho}))\, d\tilde{\varrho} - \int_0^{\tau_1} \frac{(\tau_1 - \tilde{\varrho})^{\breve{r}-1}}{\tilde{\gamma}(\breve{r})} \mathcal{H}(\tilde{\varrho}, \tilde{\varkappa}(\tilde{\varrho}))\, d\tilde{\varrho}\right| \\
\leq&\breve{s} + \left|\tau_1 \tilde{\varepsilon}(\tau_1) - \tau_1 \tilde{\varkappa}(\tau_1)\right| + \frac{1}{\tilde{\gamma}(\breve{r})} \left|\int_0^{\tau_1} (\tau_1 - \tilde{\varrho})^{\breve{r}-1} \left\{\mathcal{Z}(\tilde{\varrho}, \tilde{\varepsilon}(\tilde{\varrho})) - \mathcal{H}(\tilde{\varrho}, \tilde{\varkappa}(\tilde{\varrho}))\right\} d\tilde{\varrho}\right| \\
\leq&\breve{s} + \left|\tilde{\varepsilon}(\tau_1) - \tilde{\varkappa}(\tau_1)\right| + \frac{1}{\tilde{\gamma}(\breve{r})} \tilde{\gamma}(\breve{r}+1) \left|\int_0^{\tau_1} (\tau_1 - \tilde{\varrho})^{\breve{r}-1} \left\{|\tilde{\varepsilon} - \tilde{\varkappa}| - \breve{s} - l\right\} d\tilde{\varrho}\right| \\
\leq&\breve{s} + \left|\tilde{\varepsilon}(\tau_1) - \tilde{\varkappa}(\tau_1)\right| + \frac{1}{\tilde{\gamma}(\breve{r})} \tilde{\gamma}(\breve{r}+1) \left\{|\tilde{\varepsilon} - \tilde{\varkappa}| - \breve{s} - l\right\} \left|\left\{\frac{-(\tau_1 - \tilde{\varrho})^{\breve{r}}}{\breve{r}}\right\}_0^{\tau_1}\right| \\
\leq&\breve{s} + l + \frac{1}{\tilde{\gamma}(\breve{r}+1)} \tilde{\gamma}(\breve{r}+1) \left\{|\tilde{\varepsilon} - \tilde{\varkappa}| - \breve{s} - l\right\} |(\tau_1)^{\breve{r}}| \\
\leq&\breve{s} + l + \frac{1}{\tilde{\gamma}(\breve{r}+1)} \tilde{\gamma}(\breve{r}+1) \left\{|\tilde{\varepsilon} - \tilde{\varkappa}| - \breve{s} - l\right\} \\
=&|\tilde{\varepsilon} - \tilde{\varkappa}| \\
\leq&\|\tilde{\varepsilon} - \tilde{\varkappa}\|.
\end{aligned}$$

Thus $\mathtt{T}$ is relatively nonexpansive. Now, we assume that the pair $(\iota_1, \iota_2) \subseteq (\mathtt{N}_1, \mathtt{N}_2)$ is a *NBCC*, $\mathtt{T}$-invariant, proximinal pair and $dist(\iota_1,\iota_2) = dist(\mathtt{N}_1,\mathtt{N}_2)$. Using theorem 4.2, we get,

$$\begin{aligned}
\beth_0(\mathtt{T}(\iota_1)\cup\mathtt{T}(\iota_2)) &= \max\left\{\beth_0(\mathtt{T}(\iota_1)),\beth_0(\mathtt{T}(\iota_2))\right\}\\
&\le \max\left\{\sup_{\tau_1\in\mathsf{I}}\left\{\beth_0(\{\mathtt{T}\tilde{\varepsilon}(\tau_1):\tilde{\varepsilon}\in\iota_1\})\right\},\sup_{\tau_1\in\mathsf{I}}\left\{\beth_0(\{\mathtt{T}\tilde{\varkappa}(\tau_1):\tilde{\varkappa}\in\iota_2\})\right\}\right\}\\
&= \max\left\{\sup_{\tau_1\in\mathsf{I}}\left\{\beth_0\left(\left\{\tilde{\beta}_0-\tilde{F}\left(0,\tilde{\beta}_0\right)+\tilde{F}(\tau_1,\tilde{\varepsilon}(\tau_1))+\int_0^{\tau_1}\frac{(\tau_1-\tilde{\varrho})^{\check{r}-1}}{\tilde{\gamma}(\check{r})}\mathcal{Z}(\tilde{\varrho},\tilde{\varepsilon}(\tilde{\varrho}))\,d\tilde{\varrho}\right\}\right)\right\},\right.\\
&\quad\left.\sup_{\tau_1\in\mathsf{I}}\left\{\beth_0\left(\left\{\tilde{\pi}_0-\tilde{F}(0,\tilde{\pi}_0)+\tilde{F}(\tau_1,\tilde{\varkappa}(\tau_1))+\int_0^{\tau_1}\frac{(\tau_1-\tilde{\varrho})^{\check{r}-1}}{\tilde{\gamma}(\check{r})}\mathcal{H}(\tilde{\varrho},\tilde{\varkappa}(\tilde{\varrho}))\,d\tilde{\varrho}\right\}\right)\right\}\right\}\\
&= \max\left\{\sup_{\tau_1\in\mathsf{I}}\left\{\beth_0\left(\left\{\tilde{\beta}_0-\tilde{F}\left(0,\tilde{\beta}_0\right)+\tilde{F}(\tau_1,\tilde{\varepsilon}(\tau_1))+\frac{1}{\tilde{\gamma}(\check{r}+1)}\mathcal{Z}\left(\tilde{i},\tilde{\varepsilon}\left(\tilde{i}\right)\right)(\tau_1)^{\check{r}}:\tilde{i}\in(0,\tau_1)\right\}\right)\right\},\right.\\
&\quad\left.\sup_{\tau_1\in\mathsf{I}}\left\{\beth_0\left(\left\{\tilde{\pi}_0-\tilde{F}(0,\tilde{\pi}_0)+\tilde{F}(\tau_1,\tilde{\varkappa}(\tau_1))+\frac{1}{\tilde{\gamma}(\check{r}+1)}\mathcal{H}\left(\tilde{i},\tilde{\varkappa}\left(\tilde{i}\right)\right)(\tau_1)^{\check{r}}:\tilde{i}\in(0,\tau_1)\right\}\right)\right\}\right\}\\
&\le \max\left\{\sup_{\tau_1\in\mathsf{I}}\left\{\beth_0\left(\left\{\tau_1\tilde{\varepsilon}(\tau_1)+\frac{1}{\tilde{\gamma}(\check{r}+1)}\mathcal{Z}\left(\tilde{i},\tilde{\varepsilon}\left(\tilde{i}\right)\right)(\tau_1)^{\check{r}}:\tilde{i}\in(0,\tau_1)\right\}\right)\right\},\right.\\
&\quad\left.\sup_{\tau_1\in\mathsf{I}}\left\{\beth_0\left(\left\{\tau_1\tilde{\varkappa}(\tau_1)+\frac{1}{\tilde{\gamma}(\check{r}+1)}\mathcal{H}\left(\tilde{i},\tilde{\varkappa}\left(\tilde{i}\right)\right)(\tau_1)^{\check{r}}:\tilde{i}\in(0,\tau_1)\right\}\right)\right\}\right\}\\
&\le \max\left\{\beth_0\left(\left\{\iota_1+\mathcal{Z}\left(\tilde{i},\tilde{\varepsilon}\left(\tilde{i}\right)\right)\right\}\right),\beth_0\left(\left\{\iota_2+\mathcal{H}\left(\tilde{i},\tilde{\varkappa}\left(\tilde{i}\right)\right)\right\}\right)\right\}\le\beth_0(\iota_1\cup\iota_2).
\end{aligned}$$

Thus by using Corollary 3.11, $\mathtt{T}$ has a *BPP*. Hence, the system (4.3) has $s\in\mathtt{N}_1\cup\mathtt{N}_2$ as an optimal solution. □

Example 4.4. Assume the following system of equations with $\|\mathcal{Z}(\tilde{\varrho},\tilde{\varepsilon}(\tilde{\varrho}))\|\le 1$, $\|\mathcal{H}(\tilde{\varrho},\tilde{\varkappa}(\tilde{\varrho}))\|\le 1$, $1=1$, $\check{s}=1$, $\check{r}=1$, $\frac{1}{2}\|\tilde{\varepsilon}-\tilde{\varkappa}\|\le\|\tilde{\varepsilon}-\tilde{\varkappa}\|-2$ and $\tilde{\tau}_1\in\mathfrak{F}=[0,1]$ as

$$\begin{aligned}
\tilde{\varepsilon}(\tau_1) &= \tilde{F}(\tau_1,\tilde{\varepsilon}(\tau_1))-\tilde{F}(0,0)+\int_0^{\tau_1}\frac{1}{\tilde{\gamma}(1)}\sin\left(\frac{\tilde{\varepsilon}}{2}\right)d\tilde{\varrho},\\
\tilde{\varkappa}(\tau_1) &= 1-\tilde{F}(0,1)+\tilde{F}(\tau_1,\tilde{\varkappa}(\tau_1))+\int_0^{\tau_1}\frac{1}{\tilde{\gamma}(1)}\sin\left(\frac{\tilde{\varkappa}}{2}\right)d\tilde{\varrho}.
\end{aligned}$$

Consider $\mathtt{N}_1=\{\tau_1\}$ and $\mathtt{N}_2=\{\tau_1+1\}$ on $\mathfrak{F}=[0,1]$. Define an operator $\mathtt{T}:\mathtt{N}_1\cup\mathtt{N}_2\to\mathtt{N}$ such that,

$$\mathtt{T}(\tilde{\varepsilon}(\tau_1))=\begin{cases}1-\tilde{F}(0,1)+\tilde{F}(\tau_1,\tilde{\varepsilon}(\tau_1))+\int_0^{\tau_1}\frac{1}{\tilde{\gamma}(1)}\sin\left(\frac{\tilde{\varepsilon}}{2}\right)d\tilde{\varrho}, & \tilde{\varepsilon}\in\mathtt{N}_1,\\ \tilde{F}(\tau_1,\tilde{\varepsilon}(\tau_1))-\tilde{F}(0,0)+\int_0^{\tau_1}\frac{1}{\tilde{\gamma}(1)}\sin\left(\frac{\tilde{\varepsilon}}{2}\right)d\tilde{\varrho}, & \tilde{\varepsilon}\in\mathtt{N}_2.\end{cases}$$

Here $\mathcal{H}:\mathfrak{F}\times\iota_2\longrightarrow\iota_2$, and $\mathcal{Z}:\mathfrak{F}\times\iota_1\longrightarrow\iota_1$ with $\iota_1=\{\tau_1\}$, $\iota_2=\{\tau_1+1\}$ on $\mathfrak{F}=[0,1]$. Now for $\tilde{\varepsilon}\in\mathtt{N}_1,\tilde{\varkappa}\in\mathtt{N}_2$, we have,

$$\begin{aligned}
\|\mathcal{Z}(\tilde{\varrho},\tilde{\varepsilon}(\tilde{\varrho}))-\mathcal{H}(\tilde{\varrho},\tilde{\varkappa}(\tilde{\varrho}))\| &= \left\|\sin\left(\frac{\tilde{\varepsilon}}{2}\right)-\sin\left(\frac{\tilde{\varkappa}}{2}\right)\right\|\\
&\le\left\|2\cos\left(\frac{\frac{\tilde{\varepsilon}}{2}+\frac{\tilde{\varkappa}}{2}}{2}\right)\sin\left(\frac{\frac{\tilde{\varepsilon}}{2}-\frac{\tilde{\varkappa}}{2}}{2}\right)\right\|\\
&\le 2\left\|\cos\left(\frac{\frac{\tilde{\varepsilon}}{2}+\frac{\tilde{\varkappa}}{2}}{2}\right)\right\|\left\|\sin\left(\frac{\frac{\tilde{\varepsilon}}{2}-\frac{\tilde{\varkappa}}{2}}{2}\right)\right\|\\
&\le 2\left\|\frac{\frac{\tilde{\varepsilon}}{2}-\frac{\tilde{\varkappa}}{2}}{2}\right\|\le\frac{1}{2}\|\tilde{\varepsilon}-\tilde{\varkappa}\|\\
&\le\|\tilde{\varepsilon}-\tilde{\varkappa}\|-2.
\end{aligned}$$

Since the above system of equations satisfies all the conditions of theorem (4.3). Therefore $r \in \{0,1\} = \mathtt{N}_1 \cup \mathtt{N}_2$ is the optimal solution for the above system of equations at $\tau_1 = 0$ as $\|r - \mathtt{T}(r)\| = 1 = dist(\mathtt{N}_1, \mathtt{N}_2)$.

5 Conclusion

This paper established a best proximity point theorem with the help of newly defined contraction operator by using *MNC*, and we used it to determine the existence of optimum solutions of the system of fractional hybrid differential equations. The approach used in this article can therefore be used to obtain the existence of optimum solution of other fractional differential equations.

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