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On the capability, non-abelian tensor square and non-commuting graph of prime power groups

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Abstract

In this paper, we give necessary and sufficient conditions under which finite non-abelian metacyclic p-group (p an odd prime) G is capable. We also, determine the non-abelian tensor square $G \otimes G$ for the groups of order $p^{\alpha+\beta}$ for some $\alpha, \beta \in \mathbb{N}$. Finally, we obtain some conditions on the parameters of two prime power groups G_p and G_q in which the groups have isomorphic non-commuting graphs.

Keywords: prime power group, capable group, tensor square, isomorphic graphs
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1 Introduction

A group is metacyclic if there is a normal cyclic subgroup whose factor group is also cyclic. Meanwhile, a group is said to be capable if it is a central factor group. The term capability was introduced by Hall and Senior [6], where they defined a capable group as equal to its central factor group. Using the non-abelian tensor square, capability of 2-generator p-group of nilpotency class 2, where p is odd had already been computed by Bacon and Kappe [3]. Moreover, the Capability of 2-generator non-torsion groups of nilpotency class two determined by Kappe et al. [9]. On the other hand, Rashid et al. [12] used the Schur multiplier of a group of order $8q$ in determining whether a group of this type is capable. Beuerle and Kappe also had given the capability of finite metacyclic groups using tensor square [2].

The non-abelian tensor squares of a 2-Engel group and 2-generator 2-groups of nilpotency class two have been found [3, 10]. It is determined the non-abelian tensor square for groups of order p^2q by Jafari et al. where p, q are primes and $p < q$ [7]. Non-abelian tensor square of groups of order p^2q, special orthogonal groups and spin group are computed by some other researchers [13, 12]. In the recent studies, Beuerle and Kappe determined the non-abelian tensor squares and some homological functors of only infinite metacyclic groups [2]. Moreover, the non-commuting graph $\triangle(G)$ which is defined as the graph whose vertex set is $G \backslash Z(G)$ and edge set contains (x, y) such that $xy \neq yx$ have been studied by some researchers. Abdollahi and Shahverdi [1] studied the relation between some graph theoretical properties of $\triangle(G)$ and the group theory properties of G.

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In this research we deal with the capability and non-abelian tensor square of all finite non-abelian metacyclic p-groups where p is an odd prime. Moreover, necessary and sufficient conditions under which two non-abelian metacyclic prime power groups have isomorphic non-commuting graphs are given. To end this, the classification of finite metacyclic p-groups given by Beuerle [4] is used.

2 Preliminary results

In the case that p is an odd prime, Beuerle's classification [4] of non-abelian metacyclic p-group which can be given into three types, namely Type 1, 2 and 3 as follows:

Type 1:

$$G \cong \langle a, b | a^{p^\alpha} = b^{p^\beta} = 1, [a, b] = a^{p^{\alpha-\delta}} \rangle,$$

where p is an odd prime, $\alpha, \beta, \delta \in \mathbb{N}$, $\alpha \geq 2\delta$ and $\beta \geq \delta \geq 1$.

Type 2:

$$G \cong \langle a, b | a^{p^\alpha} = b^{p^\beta} = 1, [b, a] = a^{p^{\alpha-\delta}} \rangle,$$

where p is an odd prime, $\alpha, \beta, \delta \in \mathbb{N}$, $\delta \leq \alpha < 2\delta, \delta \leq \beta$ and $\delta \leq min\{\alpha - 1, \beta\}$.

Type 3:

$$G \cong \langle a, b | a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\epsilon}}, [b, a] = a^{p^{\alpha-\delta}} \rangle,$$

where p is an odd prime, $\alpha, \beta, \delta, \epsilon \in \mathbb{N}$, $\delta + \epsilon \leq \alpha < 2\delta, \delta \leq \beta$, $\alpha < \beta + \epsilon$ and $\delta \leq min\{\alpha - 1, \beta\}$.

In the following, we give some notations which are used in the next sections. For some integers m, n, r and s we consider the notation

$$G(m, n, r, s) = \langle x, y \mid x^m = 1, y^{-1}xy = x^r, y^n = x^s \rangle,$$

to show a metacyclic two generators group. From now till end, all groups are considered as finite non abelian metacyclic p- group of Types 1, 2 or 3. For a non-negative integer m, let C_m be the cyclic group of order m and υ_m the set of integers relatively prime to m. For x and y not both zero, let $(x, y) = \gcd(x, y)$. We use the notation $\mathrm{Deg}\triangle(G)$ for the set of degrees of vertices of non-commuting graph $\triangle(G)$. The function $\top_n$ in the following definition will be used in the subsequent sections [4]. It considerably facilitates our exposition.

Definition 2.1. Let n, k be integers with $n \geq 0$, $k \in \upsilon_n$ and $y \in \mathbb{N}$. Then define a function $\top_n : \upsilon_n \times \mathbb{N} \longrightarrow \mathbb{N}$ by

$$\top_n(k, y) = 1 + k + \cdots + k^{y-1}.$$

The next four lemmas enable us to determine the exact prime powers dividing certain expressions. We first begin with a definition mentioned in [4].

Definition 2.2. Let p be a prime and x a non-negative integer. If $x \neq 0$ then $[x]_p$ be the largest integer such that $p^{[x]_p}$ divides x.

The following lemma is an application of the binomial formula and is used to prove the subsequent theorem [4].

Lemma 2.3. Let x, m and n be non-negative integers and p a prime with $(p, x) = 1$. Then

$$\left[(p^n x + 1)^{p^m} - 1\right]_p = m + n \text{ if } p \neq 2; \tag{2.1}$$

$$\left[(2^n x + 1)^{2^m} - 1\right]_2 = \begin{cases} m + n, & \text{if } n > 1 \text{ or } m = 0, \\ [x + 1]_2 + m + 1, & \text{if } n = 1 \text{ and } m > 0, \end{cases} \tag{2.2}$$

$$\left[(2^n x - 1)^{2^m} - 1\right]_2 = \begin{cases} 1, & \text{if } n > 1 \text{ and } m = 0, \\ m + n, & \text{if } n > 1 \text{ or } m > 0, \\ [x - 1]_2 + m + 1, & \text{if } n = 1 \text{ and } x \neq 0, \\ 0, & \text{if } n = 1 \text{ and } x = 1. \end{cases} \tag{2.3}$$

The result of the next theorem is needed to prove some theorems in the next section for determining the capability of group G.

Theorem 2.4. Let p, c, d, e be positive integers with p a prime, $c = p^d + 1$ or $c = 2^d - 1$, $d \geq 2$ and f a non-negative integer. Then

$$\left[(c^{p^f e} - 1)\right]_p \geq \begin{cases} 1 & \text{if } c = 2^d - 1 \text{ and } e \text{ is odd and } f = 0, \\ f + d & \text{otherwise,} \end{cases}$$

with equality if $(e, p) = 1$.

Proof . Let p, c, d, e and f be as given in the assumption. Using the binomial formula we expand c^e as follows:

$$\begin{aligned} c^e &= \sum_{i=0}^{e} \binom{e}{i} p^{di} (\pm 1)^{e-i} = \sum_{i=1}^{e} \binom{e}{i} p^{di} (\pm 1)^{e-i} + (\pm 1)^e \\ &= p^d \left(\sum_{i=1}^{e} \binom{e}{i} p^{d(i-1)} (\pm 1)^{e-i} \right) + (\pm 1)^e. \end{aligned}$$

Thus $c^e = p^{d'} x + (\pm 1)^e$ for some positive integers d' and x such that $d' \geq d$, $x \leq \sum_{i=1}^{e} \binom{e}{i} p^{d(i-1)} (\pm 1)^{e-i}$ and $(p, x) = 1$.

Now, if $c = p^d + 1$ where p is odd, we have $c^e = p^{d'} x + 1$. Using Lemma 2.3 with n replaced by d', we get $\left[(c^{p^f e} - 1)\right]_p = f + d'$. In addition, assume $c = p^d + 1$ and $p = 2$. So, we get $c^e = 2^{d'} x + 1$. Therefore, we obtain $\left[(c^{p^f e} - 1)\right]_p = f + d'$. Moreover, suppose $c = 2^d - 1$ and e is even. Hence, we get $c^e = 2^{d'} x + 1$. Therefore, from Lemma implies $\left[(c^{p^f e} - 1)\right]_p = f + d'$. Finally, let $c = 2^d - 1$ and e is odd. It follows that $c^e = 2^{d'} x - 1$. If $f = 0$, we get $\left[(c^{p^f e} - 1)\right]_p = 1$. If $f > 0$, we have $\left[(c^{p^f e} - 1)\right]_p = f + d'$. Thus, the desired results are followed.

Now, assume that $(e, p) = 1$. It is observed that

$$\sum_{i=1}^{e} \binom{e}{i} p^{d(i-1)} (\pm 1)^{e-i} = p^d \left(\sum_{i=2}^{e} \binom{e}{i} p^{d(i-2)} (\pm 1)^{e-i} \right) + (\pm 1)^{e-1} e.$$

Hence, $\sum_{i=1}^{e} \binom{e}{i} p^{d(i-1)} (\pm 1)^{e-i}$ is relatively prime to p and $d = d'$. Therefore, it is concluded that equality is obtained for $(e, p) = 1$. $\square$

The following theorem is needed to prove some results in the next section for determining the capability of p-group G.

Theorem 2.5. Let a, p, c, d, e be positive integers with p a prime,$c = p^d + 1$ or $c = 2^d - 1$, $d \geq 2$ and f a non-negative integer. Then

$$\left[\top_{p^a}(c^e, p^f)\right]_p \geq \begin{cases} f, & \text{if } c = p^d + 1 \text{ or } e \text{ an even}, \\ 0, & \text{if } c = 2^d - 1 \text{ and } e \text{ an odd and } f = 0, \\ f + d - 1, & \text{if } c = 2^d - 1 \text{ and } e \text{ an odd and } f > 0. \end{cases}$$

with equality if $(e, p) = 1$.

Proof . Let a, p, c, d, e and f be as given in the assumption. By direct computation, we get $c^{p^f e} - 1 = (c^e - 1) \top_{p^a}(c^e, p^f)$ and thus $[\top_{p^a}(c^e, p^f)]_p = [c^{p^f e} - 1]_p - [c^e - 1]_p$. First, let d' and x be as stated in the proof of Lemma 2.4. Now, suppose that $c = p^d + 1$ and p is odd. It follows that $c^e = p^{d'} x + 1$. Hence, $[\top_{p^a}(c^e, p^f)]_p = f + d' - d' = f$. However, if $c = p^d + 1$ and $p = 2$, we get $c^e = 2^{d'} x + 1$. Therefore, we obtain $[\top_{p^a}(c^e, p^f)]_p = f + d' - d' = f$. Moreover, suppose $c = 2^d - 1$ and e is even. It follows that $c^e = 2^{d'} x + 1$. Hence, we have $[\top_{p^a}(c^e, p^f)]_p = f + d' - d' = f$. Finally, when $c = 2^d - 1$ and e is odd, we have $c^e = 2^{d'} x - 1$. If $f = 0$, we get $[\top_{p^a}(c^e, p^f)]_p = 0 = f$. If $f > 0$, then we obtain $[\top_{p^a}(c^e, p^f)]_p = f + d' - 1$. Now the claim follows. The equality for $(e, p) = 1$ is obtained from the equality in Lemma 2.4. $\square$

Finite metacyclic groups which presented by Beyl *et al.* are given as follows [5]:

Lemma 2.6. Every finite group G having a cyclic normal subgroup of order m with cyclic factor group of order n has a presentation

$$G(m,n,r,s) = \langle x, y \mid x^m = 1, y^{-1}xy = x^r, y^n = x^s \rangle,$$

where r and s are positive integers satisfying $r^n \equiv 1 \pmod m$ and $(m, 1 + r + r^2 + \cdots + r^{n-1}) \equiv 0 \pmod s$.

Next proposition computes the epicenter $Z^*(G)$ for the finite metacyclic group G.

Proposition 2.7. [5] Let $G = G(m,n,r,s)$ and k be the smallest positive divisor of n satisfying $1+r+r^2+\cdots+r^{k-1} \equiv 0 \pmod s$. Then $Z^*(G)$ for the group $G(m,n,r,s)$ is the cyclic group of order mn/ks generated by y^k.

Using Proposition 2.7, Beyl *et al.* [5] established necessary and sufficient conditions for every finite metacyclic group to be capable.

Theorem 2.8. [5] The group $G(m,n,r,s)$ is capable if and only if $s = m$ and n is the smallest positive integer satisfying $1 + r + r^2 + \cdots + r^{n-1} \equiv 0 \pmod m$.

The following criterion now characterizes the capability of the groups.

Theorem 2.9. [5] A group G is capable if and only if $Z^*(G) = 1$.

Definition 2.10. For a group G the non-abelian tensor square, $G \otimes G$, is the group generated by the symbols $g \otimes h$ where $g, h \in G$, subject to the relations;

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h), \tag{2.4}$$

$$g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h'), \tag{2.5}$$

for all $g, g', h, h' \in G$, where ${}^h g = hgh^{-1}$ denotes the conjugate of g by h.

3 Main results

In this section we determine all capable groups of finite non-abelian metacyclic p-groups where p is an odd prime.

Theorem 3.1. Let G be a finite non-abelian metacyclic p-group, where p is an odd prime. Then G is capable if and only if G is of Type (1) or Type (2) with $\beta = \alpha$.

Proof . Suppose that G is a finite non-abelian metacyclic p-group of Type (1). We represent G so that the presentation matches the one for all finite metacyclic groups given in Lemma 2.6. The presentation of G can be rewritten as follows:

$$G \cong \langle a, b | a^{p^\alpha} = 1, aba^{-1}b^{-1} = a^{p^{\alpha-\delta}}, b^{p^\beta} = a^{p^\alpha} \rangle,$$
$$\textit{where } \alpha, \beta, \delta \in \mathbb{N}, \alpha \geq 2\delta \textit{ and } \beta \geq \delta \geq 1.$$

From the group presentation, it is easy to see that $b^{-1}ab = a^{p^{\alpha-\delta}+1}$. Thus,

$$G(p^\alpha, p^\beta, p^{\alpha-\delta}+1, p^\alpha) \cong \langle a, b | a^{p^\alpha} = 1, b^{-1}ab = a^{p^{\alpha-\delta}+1}, b^{p^\beta} = a^{p^\alpha} \rangle,$$

where $m = p^\alpha, n = p^\beta, r = p^{\alpha-\delta}+1$ and $s = p^\alpha$. From the considerations above and using Theorem 2.4, we have $\left[(p^{\alpha-\delta}+1)^{p^\beta} - 1\right]_p = \beta + \alpha - \delta$. So, $p^{\beta+\alpha-\delta}$ divides $(p^{\alpha-\delta}+1)^{p^\beta} - 1$. Since p^α divides $p^{\beta+\alpha-\delta}$ for all α, β and δ satisfying the conditions of Type (1), p^α divides $(p^{\alpha-\delta}+1)^{p^\beta} - 1$. Thus, $(p^{\alpha-\delta}+1)^{p^\beta} \equiv 1 \pmod{p^\alpha}$ for all α, β and δ satisfying the conditions of Type (1). Using Definition 2.1, we have

$$1 + (p^{\alpha-\delta}+1) + (p^{\alpha-\delta}+1)^2 + \cdots + (p^{\alpha-\delta}+1)^{p^\beta-1} = \top_{p^\alpha}(p^{\alpha-\delta}+1, p^\beta).$$

By Theorem 2.5, $\left[\top_{p^\alpha}(p^{\alpha-\delta}+1, p^\beta)\right]_p = \beta$. So, β is the greatest integer such that p^β divides $\top_{p^\alpha}(p^{\alpha-\delta}+1, p^\beta)$. Thus, $\left(p^\alpha, \top_{p^\alpha}(p^{\alpha-\delta}+1, p^\beta)\right) \equiv 0 \pmod{p^\alpha}$ for $\beta \geq \alpha$. Now, Theorem 2.8, Proposition 2.7 and Theorem 2.9 are applied to determine the capability. It is obvious that $s = m = p^\alpha$. For the second condition, three cases need to be considered:

Case 1: if $\beta > \alpha$ then $n = p^\beta$ satisfying $\top_{p^\alpha}(p^{\alpha-\delta}+1, p^\beta) \equiv 0 \pmod{p^\alpha}$. But there exists a positive integer, $n' = p^{\beta-1} < p^\beta = n$ satisfying $\top_{p^\alpha}(p^{\alpha-\delta}+1, p^{\beta-1}) \equiv 0 \pmod{p^\alpha}$. Therefore, G is not capable.

Case 2: if $\beta < \alpha$ then $\top_{p^\alpha}(p^{\alpha-\delta}+1, p^\beta) \not\equiv 0 \pmod{p^\alpha}$. Thus, G is not capable in this case as well.

Case 3: if $\beta = \alpha$ then $n = p^\beta$ satisfies $\top_{p^\alpha}(p^{\alpha-\delta}+1, p^\beta) \equiv 0 \pmod{p^\alpha}$. Now, let p^i, $0 \le i < \beta$ be a divisor of $n = p^\beta$. Thus, $\left[\top_{p^\alpha}(p^{\alpha-\delta}+1, p^i)\right]_p = i$. Since $\beta = \alpha$ and $i < \beta$, $p^\alpha \nmid p^i$. So, $\top_{p^\alpha}(p^{\alpha-\delta}+1, p^i) \not\equiv 0 \pmod{p^\alpha}$. Hence, the smallest positive divisor of $n = p^\beta$ satisfying $\top_{p^\alpha}(p^{\alpha-\delta}+1, p^\beta) \equiv 0 \pmod{p^\alpha}$ is n itself. Proposition 2.7, shows that G has a trivial $Z^*(G)$, so using Theorem 2.9, the group G is capable.

Now, if G is the group of Type (2), using Lemma 2.2 the presentation of G can be rewritten as follows:

$$G(p^\alpha, p^\beta, p^{\alpha-\delta}+1, p^\alpha) \cong \langle x, y | x^{p^\alpha} = 1, y^{-1}xy = x^{p^{\alpha-\delta}+1}, y^{p^\beta} = x^{p^\alpha} \rangle,$$

where $m = p^\alpha, n = p^\beta, r = p^{\alpha-\delta}+1$ and $s = p^\alpha$. Similar as for Type (1), we get that G is capable if and only if $\beta = \alpha$. □

Theorem 3.2. Let G be a p-group of Type (3). Then G is not capable for all α, β, δ and ϵ.

Proof . Suppose that G is a group of Type (3), the presentation of G can be rewritten as follows:

$$G(p^\alpha, p^\beta, p^{\alpha-\delta}+1, p^{\alpha-\epsilon}) \cong \langle x, y | x^{p^\alpha} = 1, y^{-1}xy = x^{p^{\alpha-\delta}+1}, y^{p^\beta} = x^{p^{\alpha-\epsilon}} \rangle,$$

where $m = p^\alpha, n = p^\beta, r = p^{\alpha-\delta}+1$ and $s = p^{\alpha-\epsilon}$. Using Theorem 2.4, we have $(p^{\alpha-\delta}+1)^{p^\beta} \equiv 1 \pmod{p^\alpha}$ for all α, β and δ satisfying the conditions of the group G. Since $\alpha < \beta + \epsilon$, we conclude that $\left(p^\alpha, \top_{p^\alpha}(p^{\alpha-\delta}+1, p^\beta)\right) \equiv 0 \pmod{p^{\alpha-\epsilon}}$ for all α, β, δ and ϵ satisfying the conditions of the group. Since $s \neq m$, by Theorem 2.8, we get that G is not capable for all α, β, δ and ϵ. □

Now, we compute the non-abelian tensor square of the groups of Types 1, 2 and 3 . We start by computing the non-abelian tensor square of Type (1). It is shown that $G \otimes G$ is a direct product of cyclic groups.

Theorem 3.3. Let G be a finite non-abelian metacyclic p-group of Type (1). Then

$$G \otimes G \cong \begin{cases} C_{p^{\alpha-\delta}} \times C^3_{p^\beta}, & \text{if } \beta \le \alpha - \delta, \\ C_{p^{\alpha-\delta}} \times C_{p^\beta} \times C_{p^{\alpha-\delta}} \times C_{p^\beta}, & \text{if } \alpha - \delta \le \beta \le \alpha, \\ C_{p^{\alpha-\delta}} \times C_{p^\alpha} \times C_{p^{\alpha-\delta}} \times C_{p^\beta}, & \text{if } \alpha \le \beta. \end{cases}$$

and is generated by $a \otimes a$, $b \otimes a$, $(b \otimes a)(a \otimes b)$, $b \otimes b$ of orders $p^{\alpha-\delta}$, $p^{\min\{\alpha,\beta\}}$, $p^{\min\{\alpha-\delta,\beta\}}$ and p^β respectively.

Proof . Let G be a metacyclic group of Type (1) with p an odd prime. From the presentation of G, we consider $m = p^\alpha$, $n = p^\beta$ and $r = p^{\alpha-\delta}+1$. Thus from [8], $G \otimes G$ can be written as a direct product of four cyclic groups with generators $a \otimes a$, $b \otimes a$, $(b \otimes a)(a \otimes b)$ and $b \otimes b$ of orders $(m, r-1)$, $(m, \top_m(r, n))$, $(m, r-1, \top_m(r, n))$ and n respectively.

All that remains to be done is to calculate the orders of the factors explicitly. The order of the first factor is $(m, r-1) = \left(p^\alpha, p^{\alpha-\delta}\right) = p^{\alpha-\delta}$. The order of the second factor is $(m, \top_m(r, n)) = \left(p^\alpha, \top_{p^\alpha}\left(p^{\alpha-\delta}+1, p^\beta\right)\right)$. The order of the third factor is $(m, r-1, \top_m(r, n)) = \left(p^\alpha, p^{\alpha-\delta}, \top_{p^\alpha}\left(p^{\alpha-\delta}+1, p^\beta\right)\right)$. The order of the fourth factor is p^β. Lemma 2.5 leads to $\left(p^\alpha, \top_{p^\alpha}\left(p^{\alpha-\delta}+1, p^\beta\right)\right) = p^{\min\{\alpha,\beta\}}$ and $\left(p^\alpha, p^{\alpha-\delta}, \top_{p^\alpha}\left(p^{\alpha-\delta}+1, p^\beta\right)\right) = p^{\min\{\alpha-\delta,\beta\}}$. Thus, we conclude that if $\beta \le \alpha - \delta$ then $G \otimes G \cong C_{p^{\alpha-\delta}} \times C^3_{p^\beta}$, if $\alpha - \delta \le \beta \le \alpha$ then $G \otimes G \cong C_{p^{\alpha-\delta}} \times C_{p^\beta} \times C_{p^{\alpha-\delta}} \times C_{p^\beta}$ and if $\alpha \le \beta$ then $G \otimes G \cong C_{p^{\alpha-\delta}} \times C_{p^\alpha} \times C_{p^{\alpha-\delta}} \times C_{p^\beta}$, the desired result. □

Using a similar method and observing the condition $\alpha - \delta < \beta$ on the parameters of Type (2), the following theorem for computing the non-abelian tensor square of groups of Type (2) can be proved.

Theorem 3.4. Let G be a finite non-abelian metacyclic p-group of Type (2). Then

$$G \otimes G \cong \begin{cases} C_{p^{\alpha-\delta}} \times C_{p^\beta} \times C_{p^{\alpha-\delta}} \times C_{p^\beta}, & \text{if } \beta \le \alpha, \\ C_{p^{\alpha-\delta}} \times C_{p^\alpha} \times C_{p^{\alpha-\delta}} \times C_{p^\beta}, & \text{if } \alpha \le \beta. \end{cases}$$

generated by $a \otimes a$, $b \otimes a$, $(b \otimes a)(a \otimes b)$, $b \otimes b$ of orders $p^{\alpha-\delta}$, $p^{\min\{\alpha,\beta\}}$, $p^{\alpha-\delta}$ and p^β respectively.

The non-abelian tensor square of groups of Type (3) is given in the following theorem.

Theorem 3.5. Let G be a finite non-abelian metacyclic p-group of Type (3). Then

$$G \otimes G \cong C_{p^{\alpha-\delta}} \times C_{p^{\alpha-\epsilon}} \times C_{p^{\alpha-\delta}} \times C_{p^{\beta}}.$$

Proof . Suppose that G is a p-group of Type (3) and p is an odd prime

$$G \cong \langle a, b | a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\epsilon}}, [b, a] = a^{p^{\alpha-\delta}} \rangle.$$

Observe that $[b, a] = a^{p^{\alpha-\delta}} = bab^{-1} = a^{p^{\alpha-\delta}+1}$ and we consider $r = p^{\alpha-\delta} + 1$, then G is of type $G_p(\alpha, \beta, \epsilon, \delta, +)$. From [4] and observing the conditions on the parameters of groups of Type (3) that $\alpha - \epsilon < \beta$ and $\alpha - \delta < \beta$, the result now follows. □

We now investigate the non- commuting graphs of the groups of types 1, 2 and 3. These groups can be fallen in the following presentation:

$$\mathcal{G} \cong \langle a, b | a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\epsilon}}, a^{b} = a^{p^{\alpha-\delta}+1} \rangle, \tag{3.1}$$

for some $\alpha, \beta \in \mathbb{N}$ and $\delta, \epsilon \geq 0$ are integers, where $\beta \geq \gamma \geq 1$ and p is an odd prime. In what follows, $\mathrm{Deg}(\Delta(\mathcal{G}))$ stands for the set of all degrees of vertices of graph $\triangle(\mathcal{G})$. Also, we call $\mathrm{ccz}(\mathcal{G})$ for the set of all conjugacy class sizes of group $\mathcal{G}$.

The following theorems gives necessary and sufficient conditions under which two non-abelian prime power groups G_p and G_q of $\mathcal{G}$ type, have isomorphic non-commuting graphs.

Theorem 3.6. Consider p-group $G_p = G(\alpha_1, \beta_1, \delta_1, \epsilon_1)$ and q-group $G_q = G_q(\alpha_2, \beta_2, \delta_2, \epsilon_2)$ in the $\mathcal{G}$ group Type. Then $\triangle(G_p) \simeq \triangle(G_q)$ if and only if $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$ and $\delta_1 = \delta_2$.

Proof . Let $f : \triangle(G_p) \longrightarrow \triangle(G_q)$ be an isomorphism from $\triangle(G_p)$ to $\triangle(G_q)$. Suppose that $x_1, x_2 \in G_p \backslash Z(G_p)$, such that $\deg(x_1)$ and $\deg(x_2)$ are two minimum elements of the set $\mathrm{Deg}\triangle(G_p)$ such that $\deg(x_1) \leq \deg\,(x_2)$. Moreover, if $y_i = f(x_i)$ for $i = 1, 2$, then $\deg(y_1)$ is the minimum of $\mathrm{Deg}\triangle(G_q)$ and $\deg(y_2) = \min\, \mathrm{Deg}\triangle(G_q) - \{\deg(y_1)\}$. Since $\deg(x_1){=}\deg(y_i)$, it follows by Theorem 3.1 [11] that:

$$p^{\alpha_1+\beta_1} - p^{\alpha_1+\beta_1-1} = q^{\alpha_2+\beta_2} - q^{\alpha_2+\beta_2-1}$$

$$p^{\alpha_1+\beta_1} - p^{\alpha_1+\beta_1-2} = q^{\alpha_2+\beta_2} - q^{\alpha_2+\beta_2-2}.$$

By dividing the first sides of the above equations by their second sides, we obtain $p/p + 1 = q/q + 1$. It follows that $p = q$. Hence, $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$. Form which it follows that $|G_p| = |G_q|$, as well as $|Z(G_p)| = |Z(G_q)|$. Thus, clearly we arrive at $\delta_1 = \delta_2$. Conversely, suppose that $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$ and $\delta_1 = \delta_2$ and $\mu : Z(G_p) \longrightarrow Z(G_q)$ is a bijection, then the map $\varphi : G_p \backslash Z(G_p) \longrightarrow G_q \backslash Z(G_q)$ defined by $\varphi(a^i b^j z) = a_1^i b_1^j \mu(z)$ for $1 \leq i, j \leq p^{\delta}$ is an isomorphism between $\triangle(G_p)$ and $\triangle(G_q)$. Therefore, $\triangle(G_p) \simeq \triangle(G_q)$. □

To illustrate the above theorem, we consider the following example.

Example 3.7. By taking $p = 7$, $q = 5$, $\alpha_1 = 3$, $\alpha_2 = 4$, $\beta_1 = 2$, $\beta_2 = 1$, $\delta_1 = \delta_2 = 1$ and $\epsilon_1 = \epsilon_2 = 0$ in Theorem 3.6, two non-commuting graphs $\triangle G_7(3, 2, 1, 0)$ and $\triangle G_5(4, 1, 1, 0)$ are isomorphic.

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