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Solving time-delay optimal control problems via artificial neural networks

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Abstract

This article presents a new approach for solving the Optimal Controls of linear time delay systems with a quadratic cost functional. In this study, the Artificial Neural Networks are employed for convert delay optimal control problem to a unconstrained optimization problem. Then by using an optimization algorithm, the optimal control law is obtained. Finally, Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Time-delay optimal control problems, Artificial neural networks, Unconstrained optimization problem
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1 Introduction

The control of systems with time-delay has been of considerable concern. Delays occur frequently in biological, chemical, electronic, engineering, transportation systems and so on. Therefore, there are many attempts available in the literature to approximately solve this problem.

Oğuztöreli [8], in1963, was one of the pioneers in the analytical-based approach for time-delay Optimal Control Problems (OCPs). For the first time, Kharatishvili [5] generalized the Pontryagin maximum principle for this type of problems. The system resulting his work is a two-point boundary value problem (TPBVP) involving both advance and delay terms whose exact solution, except in very special cases, is very difficult. Therefore, the main object of all computational aspects of optimal time-delay systems has been to devise a methodology to avoid the solution of the mentioned TPBVP.

Also, Artificial Neural networks (ANNs) are considerable as a effective tools for function approximation, recently. For the first time, in 1944, two researchers from Chicago University named McCullough and Walter Pitts presented the first model of neural networks. The perceptron was the first trainable neural network proposed by Cornell University psychologist Frank Rosenblatt in 1957. Mathematicians proved that continuous functions can be approximated by a multi-layer perceptron on the basis of a compact set of R^n. In a theorem in [4], Gybenko proves that a Neural Network approximation with a sigmoid active function can approximate continuous functions succesfully. For the first time, in order to solve PDEs and ODEs, Lagaris et al. proposed using neural networks [6]. Effati and Pakdaman in [3] used ANNs for approximating the state, co-state, and control functions for optimal control problems. Sabouri et al. in [9] can solve fractional optimal control problems with Neural Networks and etc. In another work Effati et al.

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Recently, Bhagya and Dash in [1] discussed a variety of applications of ANN to the modeling of nonlinear problems in food engineering.

Here, we attempt to implement the ability of neural networks to approximate time-delay OCPs. This paper is organized as follow:

The main results are discussed in sections 2 and 3, in these sections we design a new a new neural network for solving time-delay OCPs. In section 4, we give a numerical example to demonstrate the effectiveness and accuracy of the proposed technique. Finally, with the conclusion in 5, we end the article.

2 Delay Optimal Control Problem

Consider the linear system with delay in the state variable

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t-\tau) + Bu(t), & t_0 \leq t \leq t_f, \\ x(t) = \phi(t), & t_0 - \tau \leq t \leq t_0, \end{cases} \tag{2.1}$$

where $u(t)$ in $PC([t_0, t_f], \mathbb{R}^n)$ and $x(t)$ in $PC^1([t_0 - \tau, t_f], \mathbb{R}^n)$ are the control and state variables, respectively. In fact, the parameter $\tau > 0$ is nonnegative and indicates the time delay. Furthermore, the initial state function $\phi(t)$ is continuous in $C([t_0 - \tau, t_0], \mathbb{R}^n)$, and finally, the matrices A, B, and A_1 are real constants with appropriate dimensions. For $t \in [t_0, t_f]$, our aim is to obtain, $u^*(t)$, the optimal control law minimizing the quadratic cost function

$$J = \frac{1}{2} \int_{t_0}^{t_f} (u^T(t) R u(t) + x^T(t) Q x(t)) dt + \frac{1}{2} x^T(t_f) Q_f x(t_f), \tag{2.2}$$

in which $R \in \mathbb{R}^{m \times n}$ is a positive definite matrix and Q and $Q_f \in \mathbb{R}^{n \times n}$ are positive semi-definite matrices.

For time-delay OCPs(TDOCPs), it follows from [5] that the pontryagin maximum principle provides necessary conditions of optimality for the problem (2.1) and (2.2) as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t-\tau) - BR^{-1}B^T \lambda(t), & t_0 \leq t \leq t_f, \\ \dot{\lambda}(t) = \begin{cases} -Qx(t) - A^T\lambda(t) - A_1^T \lambda(t+\tau), & t_0 \leq t \leq t_f - \tau, \\ -Qx(t) - A^T \lambda(t), & t_f - \tau < t \leq t_f, \end{cases} \\ x(t) = \phi(t), \qquad t_0 - \tau \leq t \leq t_0, \\ \lambda(t_f) = Q_f x(t_f). \end{cases} \tag{2.3}$$

The Hamiltonian function from which the above conditions are derived is

$$H(x, u, \lambda, t) = \lambda^T(t)[Ax(t) + Bu(t) + A_1 x(t-\tau) + \frac{1}{2} x^T(t) Q x(t) + \frac{1}{2} u^T(t) R u(t)], \tag{2.4}$$

where $\lambda(t) \in PC^1([t_0, t_f], \mathbb{R}^n)$ is called co-state vector. Moreover,

$$u^*(t) = -R^{-1} B^T \lambda(t), \tag{2.5}$$

for $t_0 \leq t \leq t_f$, is the optimal control law. We recall that the system (2.3) is a TPBVP with both time-advance and time-delay terms. Unfortunately, in general, this problem does not have any analytical solution. Therefore, providing an efficient method for solving this difficult problem numerically is very important.

3 Design of a Neural Network for TDOCP

For solving TPBVP (2.3), we suggested following approximations based on ANN for state and co-state variables:

$$\begin{aligned} x_N(t, W_x) &= \begin{cases} \phi(t) + (\psi(t) - \psi(t_0)) N_x(t, W_x), & t \geq t_0 \\ \phi(t), \quad t \leq t_0 \end{cases} \\ \lambda_N(t, W_\lambda) &= \begin{cases} N_\lambda(t, W_\lambda), \quad \text{If} \quad Q_f = 0 \\ (t - t_f) N_\lambda(t, W_\lambda) + Q_f x(t_f), \quad \text{If} \quad Q_f \neq 0 \end{cases} \end{aligned} \tag{3.1}$$

where x_N and λ_N are satisfying in initial and final conditions. Also,

$$N(t, W) = \sum_{i=1}^{k} v^i \sigma(\theta^i), \qquad \theta^i = w^i t + b^i, \tag{3.2}$$

is a perceptron ANN with two layers. W_x and W_λ are weight vectors according to input, output and bias weightes for $x(t)$ and $\lambda(t)$. σ is considered as an arbitrary activation function, here we implemented the sigmoid function in the numerical examples, as follows:

$$\sigma(x) = \frac{1}{1+e^{-x}}, \tag{3.3}$$

with proposing this approximation functions for $x(t)$ and $\lambda(t)$ and substituting them in TPBVP (2.3), we have

$$\begin{cases} \dot{x}_N(t, W_x) = Ax_N(t, W_x) + A_1 x_N(t-\tau, W_x) - S\lambda_N(t, W_\lambda), \qquad t_0 \leqslant t \leqslant t_f, \\ \dot{\lambda}_N(t, W_\lambda) = \begin{cases} -Qx_N(t, W_x) - A^T\lambda_N(t, W_\lambda) - A_1^T\lambda_N(t+\tau, W_\lambda), & t_0 \leqslant t < t_f - \tau, \\ -Qx_N(t, W_x) - A^T\lambda_N(t, W_\lambda), & t_f - \tau \leqslant t \leqslant t_f, \end{cases} \\ x_N(t) = \phi(t), \qquad t_0 - \tau \leq t \leqslant t_0, \\ \lambda_N(t_f, W_\lambda) = Q_f x_N(t_f, W_x), \end{cases} \tag{3.4}$$

For solving (3.4), we introduce the following error function

$$\begin{cases} E_x(t, W) = \Big(\dot{x}_N(t, W_x) - \big(Ax_N(t, W_x) + A_1 x_N(t-\tau, W_x) - S\lambda_N(t, W_\lambda)\big)\Big)^2, \quad t_0 \leqslant t \leqslant t_f, \\ E_\lambda(t, W) = \begin{cases} \Big(\dot{\lambda}_N(t, W_\lambda) - \big(-Qx_N(t, W_x) - A^T\lambda_N(t, W_\lambda) - A_1^T\lambda_N(t+\tau, W_\lambda)\big)\Big)^2, \\ t_0 \leqslant t < t_f - \tau, \\ \Big(\dot{\lambda}_N(t, W_\lambda) - \big(-Qx_N(t, W_x) - A^T\lambda_N(t, W_\lambda)\big)\Big)^2, \qquad t_f - \tau \leqslant t \leqslant t_f, \end{cases} \end{cases} \tag{3.5}$$

where $W = (W_x, W_\lambda)$ contains all the weights of the approximate functions. Finally, we write the neural network error function as

$$R(t, W) = E_x(t, W) + E_\lambda(t, W) \tag{3.6}$$

Now, in order to minimize the weights of the neural network, discretize the interval $[t_0, t_f]$ with m points $t_k, k = 1, 2, \ldots, m$, then we are solving the following unconstrained optimization problem

$$\min R(W) = \sum_{k=1}^{m} E_x(t_k, W) + E_\lambda(t_k, W) \tag{3.7}$$

Any classical mathematical optimization algorithm such as the fastest reduction, Newton, conjugate gradient,... and heuristic approaches such as Genetic or Ant algorithms, can be used to solve this problem. We have used of matlab optimization packages.

4 Numerical results

Example 4.1. Consider the time-delay system

$$\begin{cases} \dot{x} = u(t) - x(t-1), & 0 \leq t \leq 1, \\ x(t) = 1, & -1 \leq t \leq 0, \end{cases} \tag{4.1}$$

to minimize this quadratic cost functional

$$J = \int_0^1 [\frac{1}{2}x^2(t) + \frac{1}{2}u^2(t)]dt. \tag{4.2}$$

Now, our aim is to obtain the optimal control, $u(t)$, subject to (4.1) that minimizes (4.2). The necessary conditions of optimality for the problem (4.1) and (4.2) are as follow

$$\begin{cases} \dot{x}(t) = -x(t-1) + u(t) \\ \dot{\lambda}(t) = \begin{cases} -x(t) + \lambda(t+1), & t = 0, \\ -x(t) & 0 < t \leq 1, \end{cases} \\ x(t) = 1, \quad -1 \leq t \leq 0, \\ \lambda(1) = 0, \end{cases}$$

and the optimal control law is

$$u^*(t) = -\lambda(t)$$

with choosing $\psi(t) = t$. we have following approximation functions for $x(t)$ and $\lambda(t)$,

$$x_N(t, W_x) = \begin{cases} 1 + tN_x(t, W_x), & t \geq 0 \\ 1, & t \leq 0, \end{cases}$$

$$\lambda_N(t, W_\lambda) = N_\lambda(t, W_\lambda), \quad 0 \leq t \leq 1$$

To solve this problem, we trained the neural network in the interval $[0, 1]$ with 100 training points and $k = 10$ neurons. The exact solutions for $u(t)$ and $x(t)$ are, respectively, obtained as follows:

$$u^*(t) = 1 + \frac{1}{\cosh(1)}(\sinh(t-1) - \cosh(t)), \tag{4.3}$$

$$x^*(t) = \frac{1}{\cosh(1)}(\cosh(t-1) - \sinh(t)). \tag{4.4}$$

Moreover, it follows from [7] that the optimal value of cost functional is $J^* = 0.1480542786$. It can be shown that the approximate value of the cost functional calculated by the proposed ANN method is equal to $J = 0.1480543001$. It is clear that the approximate value of J is very close to the optimal value. Also, we depict the simulation curves of the trajectory of $x(t)$, control variable $u(t)$, and their exact values in Figs. 1 and 2.

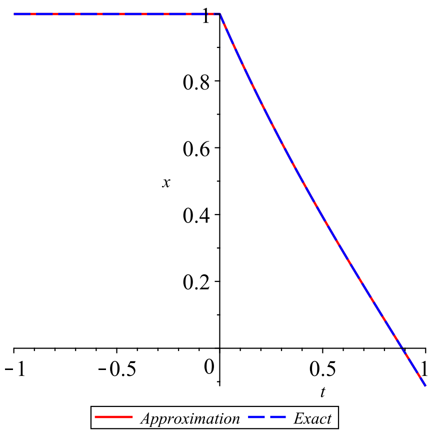

Figure 1: Approximation and exact values of state variable for Example 4.1

5 Conclusion

In this paper, we propose approximation functions based on neural network model for stste and co-state variables, to solve time-delay OCPs. This technique convert time-delay OCPs to a unconstrained optimization problem that can be easily solved using an optimization algorithm. The numerical results were presented to illustrate the high accuracy and efficiency of our proposed approach. Further research can be done on the extension of using Neural networks for solving time-delay OCPs with time dependent delays in the control and state.

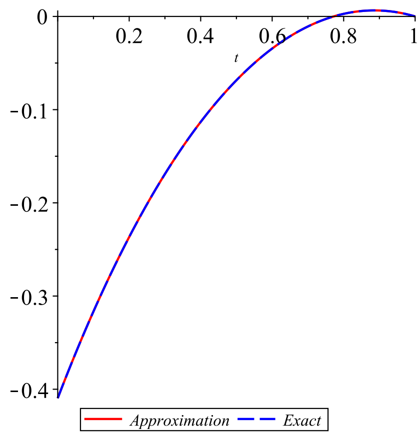

Figure 2: Approximation and exact values of control variable for Example 4.1

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